Decentralized and distributed control

Multivariable centralized control

M. Farina\textsuperscript{1}  G. Ferrari Trecate\textsuperscript{2}

\textsuperscript{1}Dipartimento di Elettronica, Informazione e Bioingegneria (DEIB)
Politecnico di Milano, Italy
farina@elet.polimi.it

\textsuperscript{2}Dipartimento di Ingegneria Industriale e dell’Informazione (DIII)
Università degli Studi di Pavia, Italy
giancarlo.ferrari@unipv.it

EECI-HYCON2 Graduate School on Control 2015
Supélec, France
Outline

1. LTI systems and controllers
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2. Controllability, observability and centralized fixed modes
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2. Controllability, observability and centralized fixed modes

3. Controllers for stability and eigenvalue assignment
LTI Systems and centralized controllers

MIMO LTI system

\[ \Sigma : \begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx \\
x(0) &= x_0
\end{align*} \]

- \( x(t) \in \mathbb{R}^n \) state
- \( u(t) \in \mathbb{R}^m \) input
- \( y(t) \in \mathbb{R}^p \) output
LTI Systems and centralized controllers

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Taxonomy of feedback controllers

Centralized Dynamical Output Feedback (CeDOf)

\[ \begin{align*}
\dot{x}_r &= Fx_r + Gy + Hv \\
u &= -K_xx_r - K_yy + K_vv
\end{align*} \]

\[ x_r(t) \in \mathbb{R}^{nr} \text{ controller state} \]

\[ v(t) \in \mathbb{R}^r \text{ setpoint} \]
Centralized Static Output Feedback (CeSOf)

\[ u = -K_y y + K_v v \]

Centralized Static State Feedback (CeSSf)

\[ u = -K_x x + K_v v \]
Systems and centralized controllers

Centralized Static Output Feedback (CeSOf)

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Centralized Static State Feedback (CeSSf)

\[ u = -K_x x + K_v v \]

In the sequel

Mainly regulation problems, i.e. the control goal is to stabilize the origin of the closed-loop system \( \Rightarrow K_v = 0 \)
Controllability, observability and centralized fixed modes
Controllability and observability

MIMO LTI system

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \\ x(0) = x_0 \end{cases}$$

Definition

- $\Sigma$ is controllable if $\forall x(0), \exists T > 0, u(t)|_{t \in (0, T]}$ such that $x(T) = 0$
- $\Sigma$ is observable if, for $u(t) = 0$, every state $x(0)$ can be reconstructed from $y(t)|_{t \in (0, T]}$ for some $T > 0$
Controllability and observability

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Remarks

- Controllability: property of the pair \((A, B)\). Observability: property of the pair \((A, C)\)
Controllability and observability

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- \( \Sigma \) is controllable if \( \forall x(0), \exists T > 0, u(t)|_{t \in (0, T]} \) such that \( x(T) = 0 \)
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Remarks

- Controllability: property of the pair \( (A, B) \). Observability: property of the pair \( (A, C) \)
- \( (A, C) \) observable iff \( (A', C') \) controllable \( \Rightarrow \) in the sequel, only controllability!
Controllability tests

Definition

The eigenvalue $\lambda_i \in \sigma(A)^a$ is controllable if $\text{rank} \left( \begin{bmatrix} A - \lambda_i I & B \end{bmatrix} \right) = n$

$^a\sigma(A)$ is the spectrum of matrix $A$. 
Controllability tests

**Definition**

The eigenvalue $\lambda_i \in \sigma(A)$ is controllable if \[ \text{rank} \left( [A - \lambda_i I \ B] \right) = n \]

$\sigma(A)$ is the spectrum of matrix $A$.

**Connection with system modes**

Assume $A$ diagonalizable, i.e. $V^{-1}AV = \tilde{A} = \text{diag}(\lambda_1, \ldots, \lambda_n)$, $V = [v_1, \ldots, v_n]$.

Perform the change of coordinates $\tilde{x} = V^{-1}x$ and obtain the following system that is equivalent to $\Sigma$

\[ \tilde{\Sigma} : \begin{cases} \dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u \\ y = \tilde{C}\tilde{x} \end{cases} \]

\[ \tilde{B} = \text{col}(\tilde{b}_1', \ldots, \tilde{b}_n') \]

State-decoupled dynamics

\[ \tilde{x}_i(t) = e^{\lambda_i t}\tilde{x}_{0,i} + \int_0^t e^{\lambda_i (t-\tau)}\tilde{b}_i' u(\tau) d\tau \]

\[ e^{\lambda_i t}\tilde{b}_i' u \]
Controllability tests

Then

\[ x(t) = \sum_{i=1}^{n} v_i \left( e^{\lambda_i t} \tilde{x}_{0,i} + e^{\lambda_i t} \tilde{b}_i' \ast u \right) \]

and one can show that the mode\(^a\) \( \lambda_i \) is controllable iff \( \tilde{b}_i \neq 0 \).

**Remark:** similar conclusions if \( A \) is not diagonalizable

\(^a\)Sometimes modes are defined as \( v_i e^{\lambda_i t} \)
Controllability tests

Then

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\[ ^a \text{Sometimes modes are defined as } v_i e^{\lambda_i t} \]

**Proposition**

The following conditions are equivalent:

a) \( \Sigma \) is controllable

b) \( \text{rank} \left( S_c \right) = n, \quad S_c = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \)

c) all \( \lambda_i \in \sigma(A) \) are controllable
Centralized fixed modes

Closed-loop system with CeSOF

\[ \Sigma : \begin{cases} 
\dot{x} = Ax + Bu \\
y = Cx \\
u = -K_y y 
\end{cases} \Rightarrow \Sigma_{cl} : \dot{x} = (A - BK_y C)x \]
### Centralized fixed modes

#### Closed-loop system with CeSOIf

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#### Definition

The Centralized Fixed Modes (CFMs) are the elements of the set

\[
\Lambda_f = \bigcap_{K_y \in \mathbb{R}^{m \times p}} \sigma(A - BK_y C)
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Centralized fixed modes

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- CFMs are eigenvalues of \( A \) (pick \( K_y = 0 \))
- Eigenvalues that are not CFMs can be moved by CeSOF ... but not in arbitrary positions
Centralized fixed modes

Example

\[
\Sigma: \begin{cases}
\dot{x}_1 &= \begin{bmatrix} 0 & 0 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\
\dot{x}_2 &= \begin{bmatrix} 1 & 0 \end{bmatrix} x_2
\end{cases}
\]

\[
y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\]

\(\Sigma\) is controllable, observable and it has two null eigenvalues.
Centralized fixed modes

Example

\[ \Sigma : \begin{cases} 
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\
y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} 
\end{cases} \]

\( \Sigma \) is controllable, observable and it has two null eigenvalues.
Closing the CeSO\( f \) \( u = -K_y y \) one gets the closed-loop system

\[
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -K_y \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\]

whose eigenvalues are \( \lambda_{1,2} = \pm \sqrt{-K_y} \Rightarrow \) on either the real or the imaginary axis.
Theorem
A CFM is an eigenvalue of $A$ that is uncontrollable or unobservable.

Remark
For LTI SISO systems, CFM are eigenvalues that do not appear among the poles of the transfer function.
Controllers for stability and eigenvalue assignment
Theorem (eigenvalue assignment)

Let \( \Lambda_f \) be the set of CFMs of system \( \Sigma \). Then

(i) For arbitrary CeDOf, the eigenvalues of the closed-loop system include the CFMs

(ii) There is a CeDOf such that eigenvalues of the closed-loop system that are not in \( \Lambda_f \) are in prescribed positions

Corollary (stabilization)

There is a CeDOf that stabilizes \( \Sigma \) iff all CFMs have strictly negative real part

Theorem (state feedback)

If \( y = x \), stabilization and eigenvalue assignment can be performed through the CeSSf controller \( u = -Kx \).
Stabilization and eigenvalue assignment

**Theorem (eigenvalue assignment)**

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Stabilization and eigenvalue assignment

**Theorem (eigenvalue assignment)**
Let $\Lambda_f$ be the set of CFMs of system $\Sigma$. Then

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There is a CeDOf that stabilizes $\Sigma$ iff all CFMs have strictly negative real part

**Theorem (state feedback)**
If $y = x$, stabilization and eigenvalue assignment can be performed through the CeSSf controller $u = -Kx$. 
Design of stabilizing CeDOf

Luenberger observer

\[ \hat{\Sigma} : \begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) & e(t) = x(t) - \hat{x}(t) \text{ error} \\ \hat{x}(0) = \hat{x}_0 & \end{cases} \]

\[ \begin{cases} \dot{e} = (A - LC)e \\ e(0) = x(0) - \hat{x}(0) \end{cases} \]
Design of stabilizing CeDOf

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Theorem

There exists \( L \) [resp. \( K \) in \( u = -K\hat{x} \)] such that all observable eigenvalues [resp. controllable eigenvalues] of \( A \) are moved to prescribed positions in \( A - LC \) [resp. \( A - BK \)]

Remarks

There are constructive procedures for designing the gains \( L \) and \( K \).

If \( A - LC \) is Hurwitz, then \( \hat{x}(t) - \hat{x}(0) \to 0 \) as \( t \to +\infty \).
Design of stabilizing CeDOF

Luenberger observer

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- There are constructive procedures for designing the gains \( L \) and \( K \)
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Design of stabilizing CeDOf

Theorem (separation principle)

The spectrum $\sigma_{cl}$ of $\Sigma$ controlled with the CeDOf given by $\hat{\Sigma}$ and $u = -K\hat{x}$ is

$$\sigma_{cl} = \sigma(A - LC) \cup \sigma(A - BK)$$

Remark

The design of the CeSSf and the Luenberger Observer can be done independently.