Decentralized and distributed control
Stability analysis for large-scale systems

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Outline

1. Introduction
2. Connective stability and vector Lyapunov functions
3. Application to cascaded systems
4. Examples
   - Temperature control
   - Three-tank system
5. Conclusions
6. Suggested readings
Introduction

Consider an **autonomous plant** description \( \mathbf{u} = 0 \) given by the interaction-oriented model:

### Subsystem \( S_i \) model

\[
\begin{align*}
\dot{x}_i &= A_{ii}x_i + E_is_i \\
 z_i &= C_{zi}x_i
\end{align*}
\]
Consider an autonomous plant description \((u = 0)\) given by the interaction-oriented model:

**Subsystem \(S_i\) model**
\[
\begin{align*}
\dot{x}_i &= A_{ii}x_i + E_i s_i \\
z_i &= C_{zi}x_i
\end{align*}
\]

**Interaction model**
\[
s_i = \sum_{j=1}^{M} L_{ij} z_j
\]

\(L_{ij}\) are interconnection matrices. We also define the interconnection gain
\[
l_{ij} = \|L_{ij}\|
\]
such that \(\|s_i\| \leq \sum_{j=1}^{M} l_{ij} \|z_j\|\)
Introduction

The analysis of the stability of the local subsystems

$$\dot{x}_i = A_{ii}x_i + E_is_i$$
$$z_i = C_{zi}x_i$$

does not reveal the properties of the overall interconnected system

$$\dot{x}_o = A_ox_o + B_ou$$

Bottom up approach

The overall system properties have to be studied as *emerging properties* of the whole plant.

One must consider both:

- the properties of the single subsystems,
- the characterization of their interactions.
Introduction

The composite system method defines the guidelines for the overall system stability tests.

Two milestone ideas

I) If all the subsystems are stable then, if $L = 0$, the stability of the overall system follows. In view of the continuous dependence of the eigenvalues of the overall system matrix upon $L$ the main question is: under what conditions on $L$ the stability for the interconnected system is guaranteed?

II) Sufficient conditions can be derived using some key aggregate parameters (i.e., aggregate models) of the single subsystems and of their mutual interconnections (e.g., the scalars $l_{ij}$ instead of the matrices $L_{ij}$).
Introduction

Steps

1. verify the stability of the subsystems

\[
\dot{x}_i = A_{ii}x_i + E_i s_i \\
 z_i = C_{zi} x_i
\]

2. identify the aggregate models (i.e., the key *aggregate parameters*) describing the subsystems’ dynamics and their mutual interactions;

3. verify the stability of the aggregate overall system using the aggregate parameters;

4. infer the stability of the unstructured model

\[
\dot{x}_o = A_o x_o + B_o u
\]

from the stability of the aggregate overall model.
Introduction

The previously discussed milestone ideas I) and II) are used in connection with the following major observation.

Third milestone idea: connective stability

In large-scale systems modelling, there is seldom uncertain knowledge of the interconnection model between different parts of the complete system. Therefore, the stability of the overall system must be guaranteed for a wide range of (possibly non-linear and time varying) uncertainties on the terms $L_{ij}$.

The connective stability issue is, indeed, to establish stability conditions that are valid for all interaction models with

$$\|L_{ij}\| \leq \bar{t}_{ij}$$
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Connective stability and vector Lyapunov functions

Remark that the network of systems

\[
\dot{x}_i = A_{ii}x_i + E_is_i \\
z_i = C_{zi}x_i
\]

has a unique equilibrium condition in \( x_i = 0 \) for all \( i = 1, \ldots, M \).

Stability of the isolated subsystems

The *isolated subsystems*

\[
\dot{x}_i = A_{ii}x_i
\]

are asymptotically stable.

For all \( i = 1, \ldots, M \), for any \( Q_i = Q_i^T > 0 \) the exists \( P_i = P_i^T > 0 \), satisfying the Lyapunov equation

\[
P_iA_{ii} + A_{ii}^TP_i = -Q_i
\]
Connective stability and vector Lyapunov functions

A "non-quadratic" Lyapunov function

A Lyapunov function for

\[ \dot{x}_i = A_{ii} x_i \]

is

\[ v_i(x_i) = \sqrt{x_i^T P_i x_i} \]

In fact

\[ \dot{v}_i^{ISOL}(x_i) = \frac{d v_i}{d x_i} \dot{x}_i = (x_i^T P_i x_i)^{-\frac{1}{2}} x_i^T P_i \dot{x}_i \]
A "non-quadratic" Lyapunov function

A Lyapunov function for

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\[ v_i(x_i) = \sqrt{x_i^T P_i x_i} \]

In fact

\[ \dot{v}_i^{SOL}(x_i) = \frac{dv_i}{dx_i} \dot{x}_i = (x_i^T P_i x_i)^{-\frac{1}{2}} x_i^T P_i \dot{x}_i \]

\[ = \frac{1}{2} (x_i^T P_i x_i)^{-\frac{1}{2}} x_i^T (P_i A_{ii} + A_{ii}^T P_i) x_i \]
Connective stability and vector Lyapunov functions

A "non-quadratic" Lyapunov function

A Lyapunov function for

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\[ \dot{v}_i^{ISOL}(x_i) = \frac{dv_i}{dx_i} \dot{x}_i = (x_i^T P_i x_i)^{-\frac{1}{2}} x_i^T P_i \dot{x}_i \]

\[ = \frac{1}{2} (x_i^T P_i x_i)^{-\frac{1}{2}} x_i^T (P_i A_{ii} + A_{ii}^T P_i) x_i \]

\[ = -\frac{1}{2} (x_i^T P_i x_i)^{-\frac{1}{2}} x_i^T Q_i x_i \leq -\frac{\lambda_{\min}(Q_i)}{2\sqrt{\lambda_{\max}(P_i)}} \| x_i \| \]
Connective stability and vector Lyapunov functions

Furthermore

\[ v_i(x_i) \leq \sqrt{\lambda_{\text{MAX}}(P_i)} \|x_i\| \]
Connective stability and vector Lyapunov functions

Furthermore

\[ v_i(x_i) \leq \sqrt{\lambda_{\text{MAX}}(P_i)} \| x_i \| \]

\[ v_i(x_i) \geq \sqrt{\lambda_{\text{MIN}}(P_i)} \| x_i \| \]

Note that the following property also holds:

\[ \| dv_i \| = \| (x_i^T P_i x_i) \| \leq \lambda_{\text{MAX}}(P_i) \sqrt{\lambda_{\text{MIN}}(P_i)} \| x_i \| \]
Connective stability and vector Lyapunov functions

Furthermore

\[ v_i(x_i) \leq \sqrt{\lambda_{\text{MAX}}(P_i)} \| x_i \| \]
\[ v_i(x_i) \geq \sqrt{\lambda_{\text{MIN}}(P_i)} \| x_i \| \]

Note that the following property also holds:

\[ \left\| \frac{dv_i}{dx_i} \right\| = \| (x_i^T P_i x_i)^{-\frac{1}{2}} x_i^T P_i \| \leq \frac{\lambda_{\text{MAX}}(P_i)}{\sqrt{\lambda_{\text{MIN}}(P_i)}} \]
Connective stability and vector Lyapunov functions

We define parameters $c_{i1}$, $c_{i2}$, $c_{i3}$, $c_{i4}$ according to:

\[ v_i(x_i) \geq \sqrt{\lambda_{\text{min}}(P_i)} \| x_i \| \]

\[ v_i(x_i) \leq \sqrt{\lambda_{\text{MAX}}(P_i)} \| x_i \| \]

\[ \dot{v}_i^{\text{ISOL}}(x_i) \leq -\frac{\lambda_{\text{min}}(Q_i)}{2\sqrt{\lambda_{\text{MAX}}(P_i)}} \| x_i \| \]

\[ \left\| \frac{dv_i}{dx_i} \right\| \leq \frac{\lambda_{\text{MAX}}(P_i)}{\sqrt{\lambda_{\text{min}}(P_i)}} \]
Connective stability and vector Lyapunov functions

We define parameters $c_{i1}$, $c_{i2}$, $c_{i3}$, $c_{i4}$ according to:

\[
    v_i(x_i) \geq \sqrt{\lambda_{\text{min}}(P_i)} \| x_i \| = c_{i1} \| x_i \|
\]

\[
    v_i(x_i) \leq \sqrt{\lambda_{\text{MAX}}(P_i)} \| x_i \| = c_{i2} \| x_i \|
\]

\[
    \dot{v}_i^{SOL}(x_i) \leq -\frac{\lambda_{\text{min}}(Q_i)}{2\sqrt{\lambda_{\text{MAX}}(P_i)}} \| x_i \| = -c_{i3} \| x_i \|
\]

\[
    \left\| \frac{dv_i}{dx_i} \right\| \leq \frac{\lambda_{\text{MAX}}(P_i)}{\sqrt{\lambda_{\text{min}}(P_i)}} = c_{i4}
\]
Connective stability and vector Lyapunov functions

We define parameters \( c_{i1}, c_{i2}, c_{i3}, c_{i4} \) according to:

\[
\begin{align*}
  v_i(x_i) &\geq c_{i1}\|x_i\| \\
  v_i(x_i) &\leq c_{i2}\|x_i\| \\
  \dot{v_i}^{ISOL}(x_i) &\leq -c_{i3}\|x_i\| \\
  \left\| \frac{dv_i}{dx_i} \right\| &\leq c_{i4}
\end{align*}
\]
Connective stability and vector Lyapunov functions

Considering also the interconnection model

\[ \dot{x}_i = A_{ii}x_i + E_is_i \]

\[ z_i = C_{zi}x_i \]

we finally define the following positive real numbers:

\[ b_{i1} = \| E_i \| \]

\[ b_{i2} = \| C_{zi} \| \]
Connective stability and vector Lyapunov functions

Key parameters and models

\[ \dot{x}_i = A_{ii}x_i + E_i s_i \]

\[ z_i = C_{zi} x_i \]

\[ b_{i1} = \|E_i\| \]

\[ b_{i2} = \|C_{zi}\| \]

\[ \|s_i\| \leq \sum_{j \neq i} \bar{l}_{ij} \|z_j\| \]

\[ v_i(x_i) = \sqrt{x_i^T P_i x_i} \]

\[ v_i(x_i) \geq c_{i1} \|x_i\| \]

\[ v_i(x_i) \leq c_{i2} \|x_i\| \]

\[ \dot{v}_i^{ISOL}(x_i) \leq -c_{i3} \|x_i\| \]

\[ \left\| \frac{dv_i}{dx_i} \right\| \leq c_{i4} \]
Connective stability and vector Lyapunov functions

Key parameters and models

\[ \dot{x}_i = A_{ii} x_i + E_i s_i \]
\[ z_i = C z_i x_i \]
\[ b_{i1} = \| E_i \| \]
\[ b_{i2} = \| C z_i \| \]
\[ \| s_i \| \leq \sum_{j \neq i} \bar{l}_{ij} \| z_j \| \]
\[ v_i(x_i) = \sqrt{x_i^T P_i x_i} \]
\[ v_i(x_i) \geq c_{i1} \| x_i \| \]
\[ v_i(x_i) \leq c_{i2} \| x_i \| \]
\[ \dot{v}_{ISOL}^i(x_i) \leq -c_{i3} \| x_i \| \]
\[ \| \frac{dv_i}{dx_i} \| \leq c_{i4} \]

Considering the non-isolated system:

\[ \dot{v}_i(x_i) = \frac{dv_i}{dx_i} \dot{x}_i \]
Connective stability and vector Lyapunov functions

Key parameters and models

\[
\begin{align*}
\dot{x}_i &= A_{ii}x_i + E_i s_i \\
z_i &= C_{zi}x_i \\
b_{i1} &= \|E_i\| \\
b_{i2} &= \|C_{zi}\| \\
\|s_i\| &= \sum_{j \neq i} \bar{l}_{ij} \|z_j\| \\
v_i(x_i) &= \sqrt{x_i^T P_i x_i} \\
v_i(x_i) &\geq c_{i1} \|x_i\| \\
v_i(x_i) &\leq c_{i2} \|x_i\| \\
\dot{v}_i^{ISOL}(x_i) &\leq -c_{i3} \|x_i\| \\
\|\frac{dv_i}{dx_i}\| &\leq c_{i4}
\end{align*}
\]

Considering the non-isolated system:

\[
\dot{v}_i(x_i) = \frac{dv_i}{dx_i} \dot{x}_i = \frac{dv_i}{dx_i} (A_{ii}x_i + E_i s_i)
\]

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Connective stability and vector Lyapunov functions

Key parameters and models

\[
\begin{align*}
\dot{x}_i &= A_{ii}x_i + E_is_i \\
z_i &= C_{zi}x_i \\
b_{i1} &= \|E_i\| \\
b_{i2} &= \|C_{zi}\| \\
\|s_i\| &\leq \sum_{j \neq i} \bar{l}_{ij} \|z_j\| \\
v_i(x_i) &= \sqrt{x_i^TP_ix_i} \\
v_i(x_i) &\geq c_{i1} \|x_i\| \\
v_i(x_i) &\leq c_{i2} \|x_i\| \\
\dot{v}_i^{ISOL}(x_i) &\leq -c_{i3} \|x_i\| \\
\|\frac{dv_i}{dx_i}\| &\leq c_{i4}
\end{align*}
\]

Considering the non-isolated system:

\[
\begin{align*}
\dot{v}_i(x_i) &= \frac{dv_i}{dx_i} \dot{x}_i \\
&= \frac{dv_i}{dx_i} (A_{ii}x_i + E_is_i) \\
&= \dot{v}_i^{ISOL}(x_i) + \left\{ \frac{dv_i}{dx_i} E_is_i \right\}
\end{align*}
\]
Connective stability and vector Lyapunov functions

Key parameters and models

\[
\begin{align*}
\dot{x}_i &= A_{ii}x_i + E_i s_i \\
\dot{z}_i &= C_{zi}x_i \\
b_{i1} &= \|E_i\| \\
b_{i2} &= \|C_{zi}\| \\
\|s_i\| &\leq \sum_{j \neq i} \bar{l}_{ij} \|z_j\| \\
v_i(x_i) &= \sqrt{x_i^T P_i x_i} \\
v_i(x_i) &\geq c_{i1} \|x_i\| \\
v_i(x_i) &\leq c_{i2} \|x_i\| \\
\dot{v}_i^{SOL}(x_i) &\leq -c_{i3} \|x_i\| \\
\left\| \frac{dv_i}{dx_i} \right\| &\leq c_{i4}
\end{align*}
\]

Considering the non-isolated system:

\[
\begin{align*}
\dot{v}_i(x_i) &= \frac{dv_i}{dx_i} \dot{x}_i \\
&= \frac{dv_i}{dx_i} (A_{ii}x_i + E_i s_i) \\
&= \dot{v}_i^{SOL}(x_i) + \left\{ \frac{dv_i}{dx_i} E_i s_i \right\} \\
&\leq -c_{i3} \|x_i\| + \left\| \frac{dv_i}{dx_i} \right\| \|E_i\| \|s_i\|
\end{align*}
\]
Connective stability and vector Lyapunov functions

### Key parameters and models

- $\dot{x}_i = A_{ii} x_i + E_i s_i$
- $z_i = C z_i x_i$
- $b_{i1} = \|E_i\|$
- $b_{i2} = \|C z_i\|$
- $\|s_i\| \leq \sum_{j \neq i} \bar{l}_{ij} \|z_j\|$
- $\nu_i(x_i) = \sqrt{x_i^T P_i x_i}$
- $\nu_i(x_i) \geq c_{i1} \|x_i\|$
- $\nu_i(x_i) \leq c_{i2} \|x_i\|$
- $\dot{\nu}_i^{\text{SOL}}(x_i) \leq -c_{i3} \|x_i\|$
- $\left\|\frac{d\nu_i}{dx_i}\right\| \leq c_{i4}$

Considering the non-isolated system:

$$\dot{v}_i(x_i) = \frac{d\nu_i}{dx_i} \dot{x}_i$$

$$= \frac{d\nu_i}{dx_i} (A_{ii} x_i + E_i s_i)$$

$$= \dot{\nu}_i^{\text{SOL}}(x_i) + \left\{ \frac{d\nu_i}{dx_i} E_i s_i \right\}$$

$$\leq -c_{i3} \|x_i\| + \left\|\frac{d\nu_i}{dx_i}\right\| \|E_i\| \|s_i\|$$

$$\leq -c_{i3} \|x_i\| + \left\|\frac{d\nu_i}{dx_i}\right\| \|E_i\| \sum_{j=1}^M \bar{l}_{ij} \|z_j\|$$
Connective stability and vector Lyapunov functions

Key parameters and models

\[
\begin{align*}
\dot{x}_i &= A_{ii} x_i + E_i s_i \\
\dot{z}_i &= C_{zi} x_i \\
b_{i1} &= \|E_i\| \\
b_{i2} &= \|C_{zi}\| \\
\|s_i\| &\leq \sum_{j \neq i} \bar{l}_{ij} \|z_j\| \\
v_i(x_i) &= \sqrt{x_i^T P_i x_i} \\
v_i(x_i) &\geq c_{i1} \|x_i\| \\
v_i(x_i) &\leq c_{i2} \|x_i\| \\
\dot{v}_i^{SOL}(x_i) &\leq -c_{i3} \|x_i\| \\
\|\frac{dv_i}{dx_i}\| &\leq c_{i4}
\end{align*}
\]

Considering the non-isolated system:

\[
\begin{align*}
\dot{v}_i(x_i) &= \frac{dv_i}{dx_i} \dot{x}_i \\
&= \frac{dv_i}{dx_i} (A_{ii} x_i + E_i s_i) \\
&= \dot{v}_i^{SOL}(x_i) + \left\{ \frac{dv_i}{dx_i} E_i s_i \right\} \\
&\leq -c_{i3} \|x_i\| + \|\frac{dv_i}{dx_i}\| \|E_i\| \|s_i\| \\
&\leq -c_{i3} \|x_i\| + \|\frac{dv_i}{dx_i}\| \|E_i\| \sum_{j=1}^{M} \bar{l}_{ij} \|z_j\| \\
&\leq -c_{i3} \|x_i\| + c_{i4} \|E_i\| \sum_{j=1}^{M} \bar{l}_{ij} \|C_{zj}\| \|x_j\|
\end{align*}
\]
Connective stability and vector Lyapunov functions

Considering the non-isolated system:

\[
\dot{v}_i(x_i) = \frac{dv_i}{dx_i} \dot{x}_i \\
= \frac{dv_i}{dx_i} (A_{ii} x_i + E_i s_i) \\
= \dot{v}_i^{\text{ISOL}}(x_i) + \left\{ \frac{dv_i}{dx_i} E_i s_i \right\} \\
\leq -c_{i3} \| x_i \| + \| \frac{dv_i}{dx_i} \| \| E_i \| \sum_{j=1}^{M} \bar{l}_{ij} \| z_j \|
\]

\[
\leq -c_{i3} \| x_i \| + c_{i4} \| E_i \| \sum_{j=1}^{M} \bar{l}_{ij} \| C z_j \| \| x_j \| \\
\leq -c_{i3} \frac{1}{c_{i2}} v_i(x_i) + c_{i4} b_i \sum_{j=1}^{M} \bar{l}_{ij} \frac{b_{ij}}{c_{j1}} v_j(x_j)
\]
Connective stability and vector Lyapunov functions

Therefore, for all $i = 1, \ldots, M$, the aggregate dynamical model is

$$
\dot{v}_i(x_i) \leq -\frac{c_{i3}}{c_{i2}} v_i(x_i) + c_{i4} b_{i1} \sum_{j=1}^{M} \frac{\bar{L}_{ij} b_{j2}}{c_{j1}} v_j(x_j)
$$

which is valid for all possible $L_{ij}$s satisfying the bound

$$
\|L_{ij}\| \leq \bar{L}_{ij}
$$

The stability properties of the overall model ($M$ equations with dynamic variables $v_i(x_i)$, $i = 1, \ldots, M$) can be established collectively, defining the vector Lyapunov function

$$
v(x) = \begin{bmatrix} v_1(x_1) \\ \vdots \\ v_M(x_M) \end{bmatrix}
$$
Connective stability and vector Lyapunov functions

Overall aggregate model

The overall aggregate model is

\[ \dot{v} \leq Mv \]

where, since \( l_{ii} = 0 \) for all \( i \), the elements of matrix \( M \) are \( \mu_{ij}, i, j = 1, \ldots, M \), defined as

\[
\mu_{ij} = \begin{cases} 
-\frac{c_i c_j}{c_i} & \text{if } j = i \\
\frac{b_i l_{ij} b_j c_i}{c_j} & \text{if } j \neq i 
\end{cases}
\]

Stability results

- the stability of the aggregate model is proved if \( M \) is asymptotically stable,
- the stability of the aggregate model implies the stability of the overall autonomous model

\[ \dot{x}_o = A_o x_o \]
Remarks:

- the vector Lyapunov function method provides a **sufficient** (therefore **conservative** \(\text{condition}\) for stability of the overall system;
- the verification of the stability properties of \(M\) is less demanding than the verification of the stability properties of \(A_o\), since \(M << n\);
- by resorting to the Gershgorin Circle Theorem, it can be proved that the stability of the overall system holds if, for all \(i = 1, \ldots, M\)

\[
\frac{c_{i3}}{c_{i2}} > \sum_{j=1}^{M} c_{i4} b_{i1} \frac{l_{ij}}{c_{j1}} b_{j2}
\]

which can be verified, in a **distributed fashion**, independently by each of the subsystems.
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Application to cascaded systems

Assume that the autonomous subsystems have a cascaded hierarchical structure, i.e.,

\[ \dot{x}_i = A_{ii}x_i + \sum_{j=1}^{i-1} A_{ij}x_j \]

Main result

If \( A_{ii} \) are stable for all \( i = 1, \ldots, M \), then the overall system is stable.

We have proved (see the lecture "Models of large-scale systems") the equivalence of this representation with the interaction-oriented one

\[ \begin{align*}
\dot{x}_i &= A_{ii}x_i + E_i s_i \\
z_i &= C_{zi}x_i
\end{align*} \]
Application to cascaded systems

We have that

\[ C_{zi} = \begin{bmatrix} I_{n_i} \\ 0_{m_i \times n_i} \end{bmatrix} \]

\[ E_i = \begin{bmatrix} I_{n_i} & 0_{n_i \times p_i} \end{bmatrix} \]

and

\[ L_{ij} = \begin{bmatrix} A_{ij} & B_{ij} \\ C_{ij} & 0_{p_i \times m_j} \end{bmatrix} \]

In view of the cascaded structure, if \( j > i \) it holds that

\[ A_{ij} = 0, B_{ij} = 0, C_{ij} = 0 \Rightarrow L_{ij} = 0 \]

and, consequently, \( l_{ij} = 0 \).

Recall that the entries of matrix \( M \) are \( \mu_{ij}, i,j=1,\ldots,M \), defined as

\[ \mu_{ij} = \begin{cases} -\frac{c_{i3}}{c_{i2}} & \text{if } j = i \\ c_{i4} b_{i1} \bar{l}_{ij} \frac{b_{i2}}{c_{j1}} & \text{if } j < i \\ 0 & \text{if } j > i \end{cases} \]

Then \( M \) has a lower block diagonal structure, with negative elements \(-\frac{c_{i3}}{c_{i2}}\) on the diagonal. Therefore it is stable.
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Examples
Temperature control

Consider the problem of controlling the temperature of the building
Examples
Temperature control

Model

According to the $\varepsilon$-decomposition, with allowed interaction level $\varepsilon = \gamma_1$, we obtain $M = 2$ non-overlapping sub-models

$$
\begin{bmatrix}
\dot{\delta T}_A \\
\dot{\delta T}_C
\end{bmatrix}
= 
\begin{bmatrix}
-(\gamma_1 + \gamma_2 + \gamma_e) & \gamma_1 \\
\gamma_1 & -(\gamma_1 + \gamma_2 + \gamma_e)
\end{bmatrix}
\begin{bmatrix}
\delta T_A \\
\delta T_C
\end{bmatrix}
+ 
\begin{bmatrix}
\delta q_A \\
\delta q_C
\end{bmatrix}
+ \gamma_2
\begin{bmatrix}
\delta T_B \\
\delta T_D
\end{bmatrix}
$$

$$
\begin{bmatrix}
\dot{\delta T}_B \\
\dot{\delta T}_D
\end{bmatrix}
= 
\begin{bmatrix}
-(\gamma_1 + \gamma_2 + \gamma_e) & \gamma_1 \\
\gamma_1 & -(\gamma_1 + \gamma_2 + \gamma_e)
\end{bmatrix}
\begin{bmatrix}
\delta T_B \\
\delta T_D
\end{bmatrix}
+ 
\begin{bmatrix}
\delta q_B \\
\delta q_D
\end{bmatrix}
+ \gamma_2
\begin{bmatrix}
\delta T_A \\
\delta T_C
\end{bmatrix}
$$

To obtain the autonomous form, we apply decentralized controllers to each of the subsystems:

$$
\begin{bmatrix}
\delta q_A \\
\delta q_C
\end{bmatrix}
= 
\begin{bmatrix}
-\gamma_{control} & -\gamma_1 \\
-\gamma_1 & -\gamma_{control}
\end{bmatrix}
\begin{bmatrix}
\delta T_A \\
\delta T_C
\end{bmatrix}
$$

$$
\begin{bmatrix}
\delta q_B \\
\delta q_D
\end{bmatrix}
= 
\begin{bmatrix}
-\gamma_{control} & -\gamma_1 \\
-\gamma_1 & -\gamma_{control}
\end{bmatrix}
\begin{bmatrix}
\delta T_B \\
\delta T_D
\end{bmatrix}
$$

where $\gamma_{control} > 0$
Autonomous model

Denoting $\gamma_{cl} = \gamma_1 + \gamma_2 + \gamma_e + \gamma_{control}$ we obtain the closed loop (autonomous) model

$$
\begin{bmatrix}
\dot{\delta T}_A \\
\dot{\delta T}_C
\end{bmatrix} =
\begin{bmatrix}
-\gamma_{cl} & 0 \\
0 & -\gamma_{cl}
\end{bmatrix}
\begin{bmatrix}
\delta T_A \\
\delta T_C
\end{bmatrix} +
\begin{bmatrix}
\gamma_2 & 0 \\
0 & \gamma_2
\end{bmatrix}
\begin{bmatrix}
\delta T_B \\
\delta T_D
\end{bmatrix}
$$

$$
\begin{bmatrix}
\dot{\delta T}_B \\
\dot{\delta T}_D
\end{bmatrix} =
\begin{bmatrix}
-\gamma_{cl} & 0 \\
0 & -\gamma_{cl}
\end{bmatrix}
\begin{bmatrix}
\delta T_B \\
\delta T_D
\end{bmatrix} +
\begin{bmatrix}
\gamma_2 & 0 \\
0 & \gamma_2
\end{bmatrix}
\begin{bmatrix}
\delta T_A \\
\delta T_C
\end{bmatrix}
$$

Note that the two submodels are identical, which in turn contain identical equations. Now defining $Q_i = l_2$ for both subsystems, the matrix $P_i, i = 1, 2$, verifying the Lyapunov equation

$$
P_i A_{ii} + A_{ii}^T P_i = -Q_i$$

is equal to $P_i = \frac{1}{2\gamma_{cl}} l_2, i = 1, 2$. 
Examples

Temperature control

With $Q_i = l_2$ and $P_i = \frac{1}{2\gamma_{cl}} l_2$,

\[
\begin{align*}
    c_{i1} &= \sqrt{\lambda_{\text{min}}(P_i)} \\
    c_{i2} &= \sqrt{\lambda_{\text{MAX}}(P_i)} \\
    c_{i3} &= \frac{\lambda_{\text{min}}(Q_i)}{2\sqrt{\lambda_{\text{MAX}}(P_i)}} \\
    c_{i4} &= \frac{\lambda_{\text{MAX}}(P_i)}{\sqrt{\lambda_{\text{min}}(P_i)}} \\
    b_{i1} &= \|E_i\| \\
    b_{i2} &= \|C_{zi}\| \\
    l_{12} &= \|A_{12}\| \\
    l_{21} &= \|A_{21}\|
\end{align*}
\]
Examples

Temperature control

With $Q_i = l_2$ and $P_i = \frac{1}{2\gamma_{cl}} l_2$,

\[
    c_{i1} = \sqrt{\lambda_{\text{min}}(P_i)} = \sqrt{\frac{1}{2\gamma_{cl}}}
    \\
    c_{i2} = \sqrt{\lambda_{\text{MAX}}(P_i)} = \sqrt{\frac{1}{2\gamma_{cl}}}
    \\
    c_{i3} = \frac{\lambda_{\text{min}}(Q_i)}{2\sqrt{\lambda_{\text{MAX}}(P_i)}} = \sqrt{\frac{\gamma_{cl}}{2}}
    \\
    c_{i4} = \frac{\lambda_{\text{MAX}}(P_i)}{\sqrt{\lambda_{\text{min}}(P_i)}} = \sqrt{\frac{1}{2\gamma_{cl}}}
    \\
    b_{i1} = \|E_i\| = 1
    \\
    b_{i2} = \|C_{zi}\| = 1
    \\
    l_{12} = \|A_{12}\| = \gamma_2
    \\
    l_{21} = \|A_{21}\| = \gamma_2
\]
Examples

Temperature control

With $Q_i = I_2$ and $P_i = \frac{1}{2\gamma_{cl}} I_2$,

\[
\begin{align*}
    c_{i1} &= \sqrt{\lambda_{\text{min}}(P_i)} = \sqrt{\frac{1}{2\gamma_{cl}}} \\
    c_{i2} &= \sqrt{\lambda_{\text{MAX}}(P_i)} = \sqrt{\frac{1}{2\gamma_{cl}}} \\
    c_{i3} &= \frac{\lambda_{\text{min}}(Q_i)}{2\sqrt{\lambda_{\text{MAX}}(P_i)}} = \sqrt{\frac{\gamma_{cl}}{2}} \\
    c_{i4} &= \frac{\lambda_{\text{MAX}}(P_i)}{\sqrt{\lambda_{\text{min}}(P_i)}} = \sqrt{2\gamma_{cl}} \\
    b_{i1} &= \|E_i\| = 1 \\
    b_{i2} &= \|C_{zi}\| = 1 \\
    l_{12} &= \|A_{12}\| = \gamma_2 \\
    l_{21} &= \|A_{21}\| = \gamma_2
\end{align*}
\]

\[
M = \begin{bmatrix}
    -\frac{c_{13}}{c_{12}} & c_{14} b_{11} l_{12} & \frac{b_{22}}{c_{21}} \\
    c_{24} b_{21} l_{21} & \frac{b_{12}}{c_{11}} & -\frac{c_{23}}{c_{22}} \\
\end{bmatrix} = \begin{bmatrix}
    -\gamma_{cl} & \gamma_2 \\
    \gamma_2 & -\gamma_{cl}
\end{bmatrix}
\]
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Examples

Three-tank system

Consider the system illustrated in the following Figure, consisting in a cascade interconnection of three tanks.
Examples
Three-tank system

Model

We apply the LBT decomposition and obtain the two submodels

\[
\dot{\delta x}_2 = -\delta x_2 + \delta u_2 \\
\begin{bmatrix}
\dot{\delta x}_1 \\
\dot{\delta x}_3
\end{bmatrix} = \begin{bmatrix}
-1 & 1 \\
0 & -1
\end{bmatrix} \begin{bmatrix}
\delta x_1 \\
\delta x_3
\end{bmatrix} + \begin{bmatrix}
0 \\
1
\end{bmatrix} \delta u_1 + \begin{bmatrix}
0 \\
1
\end{bmatrix} \delta x_2
\]

To obtain the autonomous form, we apply decentralized controllers to each of the subsystems:

\[
\delta u_2 = -k_2 \delta x_2 \\
\delta u_1 = -\begin{bmatrix}
k_2^2 \\
\frac{k_{(1,3)}}{4}
\end{bmatrix} \begin{bmatrix}
k_{(1,3)} \\
k_1,3
\end{bmatrix} \begin{bmatrix}
\delta x_1 \\
\delta x_3
\end{bmatrix}
\]

where \(k_2, k_{(1,3)} > 0\).
Examples
Three-tank system

Autonomous model

\[
\begin{align*}
\dot{\delta x}_2 &= f_2 \delta x_2 \\
\begin{bmatrix}
\dot{\delta x}_1 \\
\dot{\delta x}_3
\end{bmatrix} &= F_{(1,3)} \begin{bmatrix}
\delta x_1 \\
\delta x_3
\end{bmatrix} + \begin{bmatrix}
0 \\
1
\end{bmatrix} \delta x_2
\end{align*}
\]

where \( f_2 = -(k_2 + 1) \) and

\[
F_{(1,3)} = \begin{bmatrix}
-1 & 1 \\
-\frac{k_{(1,3)}^2}{4} & -(k_{(1,3)} + 1)
\end{bmatrix}
\]

\( F_{(1,3)} \) has both eigenvalues equal to \( -(\frac{k_{(1,3)}}{2} + 1) \).

For \( k_{(1,3)} = 2 \) and \( k_2 = 1 \), then it is possible to prove that \( P_{(1,3)} = l_2 \) and \( P_2 = 1 \) are solutions to the Lyapunov equation

\[
P_i A_{ii} + A_{ii}^T P_i = -Q_i
\]

with \( Q_{(1,3)} = -(F_{(1,3)} + F_{(1,3)}^T) = \text{diag}(2, 6) > 0 \) and \( Q_2 = 2(k_2 + 1) = 6 \).
Examples

Three-tank system

With $Q_{(1,3)} = -(F_{(1,3)} + F^T_{(1,3)}) = \text{diag}(2, 6)$, $P_{(1,3)} = l_2$, $Q_2 = 2(k_2 + 1) = 6$, and $P_2 = 1$

$$c_{i1} = \sqrt{\lambda_{\text{min}}(P_i)}$$
$$c_{i2} = \frac{\sqrt{\lambda_{\text{MAX}}(P_i)}}{\lambda_{\text{min}}(Q_i)}$$
$$c_{i3} = \frac{2\sqrt{\lambda_{\text{MAX}}(P_i)}}{\lambda_{\text{MAX}}(P_i)}$$
$$c_{i4} = \sqrt{\lambda_{\text{min}}(P_i)}$$
$$b_{i1} = \|E_i\|$$
$$b_{i2} = \|C_{zi}\|$$
$$l_{12} = \|A_{12}\|$$
$$l_{21} = \|A_{21}\|$$
Examples

Three-tank system

With \( Q_{(1,3)} = -(F_{(1,3)} + F_{(1,3)}^T) = \text{diag}(2, 6) \), \( P_{(1,3)} = I_2 \),
\( Q_2 = 2(k_2 + 1) = 6 \), and \( P_2 = 1 \)

\[
\begin{align*}
c_{i1} &= \sqrt{\lambda_{\text{min}}(P_i)} \\
c_{i2} &= \sqrt{\lambda_{\text{MAX}}(P_i)} \\
c_{i3} &= \frac{\lambda_{\text{min}}(Q_i)}{2\sqrt{\lambda_{\text{MAX}}(P_i)}} \\
c_{i4} &= \frac{\lambda_{\text{MAX}}(P_i)}{\sqrt{\lambda_{\text{min}}(P_i)}}
\end{align*}
\]

\[
\begin{align*}
c_{(1,3)1} &= 1 \\
c_{(1,3)2} &= 1 \\
c_{(1,3)3} &= 1 \\
c_{(1,3)4} &= 1
\end{align*}
\]
Examples

Three-tank system

With \( Q_{(1,3)} = -(F_{(1,3)} + F^T_{(1,3)}) = \text{diag}(2, 6) \), \( P_{(1,3)} = I_2 \), \( Q_2 = 2(k_2 + 1) = 6 \), and \( P_2 = 1 \)

\[
\begin{align*}
 c_{i1} &= \sqrt{\lambda_{\min}(P_i)} \\
 c_{i2} &= \sqrt{\lambda_{\max}(P_i)} \\
 c_{i3} &= \frac{\lambda_{\min}(Q_i)}{2 \sqrt{\lambda_{\max}(P_i)}} \\
 c_{i4} &= \frac{\lambda_{\max}(P_i)}{\sqrt{\lambda_{\min}(P_i)}} \\
 b_{i1} &= \|E_i\| \\
 b_{i2} &= \|C_{zi}\| \\
 l_{12} &= \|A_{12}\| \\
 l_{21} &= \|A_{21}\|
\end{align*}
\]

Farina, Ferrari Trecate ()
Examples

Three-tank system

With $Q_{(1,3)} = -(F_{(1,3)} + F_{(1,3)}^T) = \text{diag}(2,6)$, $P_{(1,3)} = I_2$, $Q_2 = 2(k_2 + 1) = 6$, and $P_2 = 1$

\begin{align*}
c_{i1} &= \sqrt{\lambda_{\text{min}}(P_i)} \\
c_{i2} &= \sqrt{\lambda_{\text{MAX}}(P_i)} \\
c_{i3} &= \frac{\lambda_{\text{min}}(Q_i)}{2\sqrt{\lambda_{\text{MAX}}(P_i)}} \\
c_{i4} &= \frac{\lambda_{\text{MAX}}(P_i)}{\sqrt{\lambda_{\text{min}}(P_i)}} \\
b_{i1} &= \|E_i\| \\
b_{i2} &= \|C_{zi}\| \\
l_{12} &= \|A_{12}\| \\
l_{21} &= \|A_{21}\|
\end{align*}

\[c_{i1} \quad c_{i2} \quad c_{i3} \quad c_{i4} \]

\[b_{i1} \quad b_{i2} \quad l_{12} \quad l_{21} \]

\[M = \begin{bmatrix}
- \frac{c_{23}}{c_{22}} c_{i1} & c_{i2} & c_{i3} & c_{i4}
\end{bmatrix}
\begin{bmatrix}
- \frac{c_{23}}{c_{22}} c_{i1} & c_{i2} & c_{i3} & c_{i4}
\end{bmatrix}

= \begin{bmatrix}
-3 & 0 \\
1 & -1
\end{bmatrix}
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Conclusions

Take-home messages:

- verifying in a centralized fashion the stability properties of a large-scale system:
  - is demanding from the information storage and from the computational load perspective;
  - is not robust with respect to the uncertainties on the interconnections between subsystems;

- such analysis can not be carried out solely by analyzing the stability properties of the independent subsystems;
Conclusions
Take-home messages:

- the composite system method allows to perform stability analysis of large-scale systems
  - it provides sufficient conditions;
  - it reduces the scale of the problem to the number of subsystems (thanks to the use of aggregate models);
  - it is robust with respect to uncertainties in the interconnections (connective stability): plug-and-play features.
- the verification can be performed in a distributed fashion;
- the vector Lyapunov function method is a possible and effective method for such analysis.
Conclusions

The linear case is discussed in [1];

The general non-linear system case is discussed in [2,3].
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Suggested readings

Books