Distributed moving horizon estimation for sensor networks*

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Abstract: This paper focuses on distributed state estimation using a sensor network for monitoring a linear system. In order to account for physical constraints on process states and inputs, we propose a moving horizon approach where each sensor has to solve a quadratic programming problem at each time instant. We discuss conditions guaranteeing convergence of all estimates to a common value by characterizing the dynamics of the unobservable component of the state. Furthermore, we highlight how the performance of the state estimation scheme depends upon various observability properties of the system and discuss how different communication protocols impact on the quality of the estimates.

Keywords: distributed state estimation, constrained linear systems, moving horizon estimation, sensor networks.

1. INTRODUCTION

Nowadays, the application of state estimation and control algorithms to large-scale, multi-agent and distributed systems is an issue of paramount importance. Centralized algorithms provide optimal theoretical performances in ideal conditions. Nevertheless, practical problems in processing and transmitting large amounts of data limit the applicability of centralized methods. Besides, robustness, fault-tolerance and reconfigurability requirements promote the study of distributed algorithms. This is also spurred by the availability of low cost sensing devices, endowed with computational capabilities, which can coordinate their activity through wireless communication networks (i.e., sensor networks).

Sensor networks can be employed in a wide range of applications, such as monitoring, exploration, surveillance or to track targets over specific regions. In this regard, the development of suitable distributed state estimation algorithms is a fundamental theoretical challenge for the full exploitation of their potential.

The estimation problem can be described as follows. Assume that any sensor of the network measures some variables of a linear constrained process, computes a local estimate of the overall state of the system under monitoring, and transmits to its neighbors both the measured values and the computed state estimates. Then, the main challenge is to provide a methodology which guarantees that all the sensor asymptotically reach a common reliable estimate of the state variables, i.e. the local estimates reach a consensus. This goal must be achieved even if the measurements performed by any sensor are not sufficient to guarantee observability of the process state (namely, local observability), provided that all the sensors, if put together, guarantee such property (namely, collective observability).

Consensus algorithms for distributed state estimation based on Kalman filters have recently been proposed. In Olfati-Saber and Shamma (2005); Olfati-Saber (2005); Spanos et al. (2005), consensus on the measurements is used to reduce their uncertainty and Kalman filters are applied by each agent. In Olfati-Saber (2007), three algorithms for distributed filtering are proposed, where consensus on the estimates is performed, so as to force the sensors to converge to a common reliable yet sub-optimal state estimate. However, stability and optimality of the resulting estimation algorithms have not been addressed. Recently, convergence in mean of the local state estimates obtained with the algorithm presented in Olfati-Saber (2005) has been proved in Kamgarpour and Tomlin (2008), provided that the observed process is stable.

In Afriksson and Rantzer (2006) consensus on the estimates is used together with Kalman filters. A two-step procedure is also used in Carli et al. (2008), where the considered observed signal is a random walk. In the proposed algorithm filtering and consensus are performed subsequently, and the estimation error is minimized with respect to both the observer gain and the consensus weights. Recently, an interesting solution to the problem of distributed estimation of a parameter vector, with noisy linear measurements, has been proposed in Calafiore and Abrate (2009), accounting for dynamically changing interconnections, unreliable communication links, and faults.

In this paper we extend some results presented in Farina et al. (2009), where a distributed algorithm based on the concept of Moving Horizon Estimation (MHE) has been proposed, (Rao et al., 2001). This approach has many advantages; first of all, the observer is optimal in a sense, since a suitable minimization problem must be solved on-line at each time instant. Furthermore, constraints on the noise and on the state can be taken into account, (Goodwin et al., 2005). Finally, in Farina et al. (2009) we have proved that, under weak observability conditions, convergence of the state estimate is guaranteed in a deterministic framework. In this work we highlight how the performance of the state estimation scheme depends upon various observability...
properties of the system, we extend the main results of Farina et al. (2009) in case different communication protocols are employed, and discuss how these protocols impact on the quality of the estimates.

The paper is structured as follows. In Section 2 we introduce the observed dynamical system and the structure of the sensor network. We also define a number of observability properties and describe the distributed state estimation algorithm. In Section 3 we investigate the convergence properties of the algorithm. In Section 4 we discuss how the parameters of the communication protocols can be properly tuned, so as to enhance the performance of the estimation scheme. Finally, Section 5 reports some concluding remarks. For space limitations, proofs of the main results are omitted.

2. PROBLEM FORMULATION AND SOLUTION

The observed process obeys to the linear dynamics

\[ x_{t+1} = Ax_t + w_t, \]  

where \( x_t \in \mathbb{R}^n \) is the state vector and the term \( w_t \in \mathbb{W} \subset \mathbb{R}^n \) represents a white noise with covariance equal to \( \mathbb{G} \in \mathbb{R}^{n \times n} \). We assume that the sets \( \mathbb{X} \) and \( \mathbb{W} \) are convex and contain the origin. The initial condition \( x_0 \in \mathbb{R}^n \) is a random variable with mean \( \mu \) and covariance \( \Pi_0 \). The pair \((\mathbb{A}, \sqrt{\mathbb{G}})\) is assumed to be stabilizable. Measurements on the state vector are performed by \( M \) sensors, according to the sensing model (in general different from sensor to sensor)

\[ y_i = C_i x_i + v_i, \quad i = 1, \ldots, M \]  

where the term \( v_i \in \mathbb{R}^{p_i} \), represents white noise with covariance equal to \( R_i \in \mathbb{R}^{p_i \times p_i} \) and the matrix \( C_i \) is non null.

The communication network among sensors is described by the directed graph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \), where the nodes in \( \mathcal{V} = \{1, 2, \ldots, M\} \) represent the sensors and the edge \((j, i)\) in the set \( \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} \) models that sensor \( j \) can transmit information to sensor \( i \). We assume \((i, i) \in \mathcal{E}, \forall i \in \mathcal{V} \). Moreover, we denote with \( \mathcal{V}_i^k \) the set of \( k \)-th order neighbors to node \( i \), i.e., \( \mathcal{V}_i^k = \{ j \in \mathcal{V} : \) there is a path of length at most \( k \) from \( j \) to \( i \) in \( \mathcal{G} \} \).

We associate to the graph the stochastic matrix \( \mathcal{X} \in \mathbb{R}^{M \times M} \), with entries

\[ k_{ij} = 0 \quad \text{if} \quad (j, i) \in \mathcal{E} \]  

\[ k_{ij} = 0 \quad \text{otherwise} \]  

\[ \sum_{j=1}^{M} k_{ij} = 1, \quad \forall i = 1, \ldots, M \]  

We say that a matrix \( K \) is compatible with \( \mathcal{G} \) if the entries \( k_{ij} \) verify (3). Moreover, a stochastic matrix \( K \) with strictly positive diagonal entries induces a graph \( \mathcal{G} \), according to (3a), (3b). There are many degrees of freedom for the choice of the coefficients \( k_{ij} \), which can be exploited to guarantee the convergence of the state estimator described in the following and/or reduce uncertainty of the estimates.

2.1 Communication protocols and models

The first assumption on the communication network is that measurements taken by a sensor at time \( t \) are instantaneously transmitted to its first-order neighboring agents. Secondly, we let \( \mathcal{N}_T \geq 1 \) be the number of transmissions between two sensors within a sampling interval. Two types of data communication protocols can be assumed:

\[ P_1 \) For \( \mathcal{N}_T \geq 1 \), at time \( t \) sensor \( i \) collects the sets of measurement \( \mathcal{Y}_{t/k}^{i} = \{ y_{k}^{i}; \ j \in \mathcal{V}_t^k \} \), for all \( k \in [t - \mathcal{N}_T, t] \).

\[ P_2 \) For \( \mathcal{N}_T = 1 \) and given \( \mathcal{N}_T \geq 1 \), at time \( t \) sensor \( i \) collects

the sets of measurement \( \mathcal{Y}_{t/k}^{i} = \{ y_{k}^{i}; \ j \in \mathcal{V}_t^{k+1} \} \), for all \( k \in [t - \mathcal{N}_T, t] \).

Note that, even if \( \mathcal{Y} \) contains loops, measurements in the sets \( \mathcal{Y}_{t/k}^{i} \) are considered just once. In the case of protocol \( P_2 \), the elements of the sets \( \mathcal{Y}_{t/k}^{i} \) are illustrated in Figure 1. Note also that the protocols can be combined to obtain a more complex information transmission scheme. However, for simplicity, in the following the two cases \( P_1 \) and \( P_2 \) will be addressed independently.

![Fig. 1. Illustration of the communication protocol P_2 for N = 2.](image)

The information available to node 1 at time \( t \) consists of \( \mathcal{Y}_{t-1/k}^{1} \), \( k = t - 1, t - 1, \ldots, t \). The set \( \mathcal{Y}_{t-1/k}^{1} \) (panel A) collects the data measured by nodes 1 and 2 at time \( t \). The set \( \mathcal{Y}_{t-2/k}^{2} \) (panel B) contains data measured by nodes 1 and 2 at time \( t - 1 \) and data collected by node 2 at time \( t - 1 \), that is \( \mathcal{Y}_{t-2/k-1}^{2} = \{ y_{t-1}^{2}, y_{t-1}^{3} \} \). Analogously, \( \mathcal{Y}_{t-2/k-2}^{2} \) contains (panel C) the measurements \( y_{t-2}^{1}, j = 1, \ldots, 4 \).

We introduce now suitable notations for describing measurements available at node \( i \) at time \( t \) with both protocols. Let \( y_{k}^{i} \) be the vector of measurements in \( \mathcal{Y}_{t/k}^{i} \).\(^1\) We denote with \( \bar{y}_{k}^{i} \) the dimension of \( y_{k}^{i} \). Apparently, from matrices \( C_i \) one can build matrices \( \bar{C}_{t-k}^{i} \in \mathbb{R}^{\bar{y}_{k}^{i} \times n} \) such that

\[ y_{k}^{i} = \bar{C}_{t-k}^{i} x_k + \bar{v}_{k}^{i}; \quad t - \mathcal{N}_T \leq k \leq t \]  

where \( \bar{v}_{k}^{i} \) collects noise samples affecting the measurements \( y_{k}^{i} \). Note also that in case of protocol \( P_1 \), matrices \( \bar{C}_{t-k}^{i} \) are all identical.

\(^1\) Note that the order in which elements of \( \mathcal{Y}_{t/k}^{i} \) are listed in \( y_{k}^{i} \) does not play any particular role.
We can now distinguish three types of quantities: local, regional, and collective. Specifically, a quantity is referred to as local (with respect to sensor \( i \)) when it is related to the node \( i \) solely. A quantity is called regional (with respect to sensor \( i \)) if it is related to the sensor \( i \) and the nodes in \( \mathcal{V}_i^{N_t} \) and \( \mathcal{V}_i^{t-k+1} \) for protocols \( P_1 \) and \( P_2 \), respectively. Finally, we say that a quantity is collective, if it is related to the whole network. For the sake of clarity, and consistently with (2) and (4), we use different notations for local, regional and collective variables. Namely, given a variable \( z \), \( z' \) represents its local version, \( z'' \) is its regional counterpart, and \( z \) the collective version. For instance, we refer to \( y_i' \) in (2) as local measurement. Accordingly, \( \bar{y}_i \) in (4) will be referred to as regional measurements. Furthermore, we will make use of Moving Horizon Estimation (MHE), Alessandri et al. (1999); Rao and Rawlings (2000); Rao et al. (2001, 2003); Goodwin et al. (2005), in view of its capability to handle constraints. More specifically, we propose a MHE problem, as defined in (6a)–(6d) under the constraints

\[
\begin{align*}
\tilde{x}_{i+1}^k &= A \hat{x}_i^k + \hat{w}_i^k \quad (6a) \\
\hat{y}_i^k &= \hat{C}_i^k \hat{x}_i^k + \hat{v}_i^k \quad (6b) \\
\hat{w}_i^k &\in \mathcal{W} \quad (6c) \\
\hat{z}_i^k &\in \mathcal{X} \quad (6d)
\end{align*}
\]

where \( k = t, N, \ldots, t \) and the local cost function \( J' \) is given by

\[
J'(t - N, t, \hat{x}_{i-N}^t, \hat{w}_i, \hat{v}_i, \Gamma_{i-N}) = \frac{1}{2} \sum_{k=1}^{t} \| \hat{w}_i^k \|^2_{\hat{R}_{i-k}^{-1} + \hat{v}_i^k} + \frac{1}{2} \sum_{k=1}^{t} \| \hat{v}_i^k \|^2_{\hat{R}_{i-k}^{-1} + \Gamma_{i-N}(\hat{x}_{i-N}; \hat{y}_i^{t - N/t - 1})} \quad (7)
\]

We denote with \( \hat{x}(t - N/t) \) and with \( \{ \hat{w}(k/t) \}_{k=1}^{t} \) the optimizers to (5) and with \( \hat{y}(k/t) \) the local state sequence stemming from \( \hat{x}(t - N/t) \) and \( \{ \hat{w}(k/t) \}_{k=1}^{t} \). Furthermore, \( \hat{x}(t - N/t - 1) \) denotes the weighted average state estimate

\[
\hat{x}(t - N/t - 1) = \frac{1}{M} \sum_{j=1}^{M} k_{ij} \hat{x}_i^j(t - N/t - 1) \quad (8)
\]

where \( k_{ij} \) are the entries of the stochastic matrix \( K^* \) compatible with the graph \( \mathcal{G}^* \) induced by \( K^{N_t} \). Of course the choice \( K^* = K^{N_t} \) is always possible. However, note that agent \( i \) can set the nonzero entries \( k_{ij} \) autonomously, as far as \( \sum_{j=1}^{M} k_{ij} = 1 \), for instance, a possible choice is \( k_{ij} = 1/|\mathcal{Y}_i^{N_t}| \), and this highlights that the choice of coefficients \( k_{ij} \) can be done in a distributed fashion. In (7), the function \( \Gamma_{i-N}(\hat{x}_{i-N}; \hat{y}_i^{t - N/t - 1}) \) is the so-called penalty, defined as follows

\[
\Gamma_{i-N}(\hat{x}_{i-N}; \hat{y}_i^{t - N/t - 1}) = \frac{1}{2} \| \hat{y}_i^{t - N/t - 1} - \hat{x}(t - N/t - 1) \|^2_{\hat{R}_{i-N}^{-1} + \Theta_{i-1}^{2}} \quad (9)
\]

where \( \Theta_{i-1}^{2} \) is the optimal cost defined in (5).

In (7) and hereafter, the notation \( \| z \|^2_{S} \) stands for \( z^T Sz \), where \( S \) is a positive-semidefinite matrix.

Note that \( \hat{x}_i(t - N/t - 1) \) is the estimate of \( x_{i-N} \) computed by sensor \( j \) at time \( t - 1 \) and therefore, in view of the definition of \( k_{ij} \), \( \Gamma(\cdot) \) depends only upon regional quantities. Since also the cost (7) and the constraints (6) depend only upon regional variables, the overall estimation scheme is distributed.

Finally, note that \( \Gamma(\cdot) \) embodies a consensus term, in the sense that it penalizes deviations of \( \hat{x}(t - N/t - 1) \) from \( \hat{x}(t - N/t - 1) \). Consensus, besides increasing accuracy of the local estimates, is fundamental to guarantee convergence of the state estimates to the state of the observed system even if regional observability does not hold. In other words, it allows sensor \( i \) to reconstruct components of the state that cannot be estimated by the \( i \)-th regional model.

The positive-definite symmetric matrix \( \Pi_{i-N}/t-1 \) appearing in (9) plays the role of a covariance matrix and is a design parameter whose choice will be discussed later.

2.3 Regional observability

The regional models (1) and (4) are time-varying models, since the output equation (4) depends upon \( k \), and the definition of regional observability will refer to these kinds of models. In this context, the most suitable observability definition is that of
uniform observability (Rao et al., 2003), which easily applies to time-varying systems, as well as to non-linear systems. For this reason, the following definitions depend upon the size of the estimation horizon and, in particular, upon the number of output-dependent terms \( \| \hat{\theta}_k^2 \| (\hat{R}_{k+i})^{-1} \) appearing in (7) (i.e., \( N + 1 \)).

Given a single sensor model (1) and (2) and the considered communication protocol, the \( s \) step regional observability matrix \( \bar{\theta}_s^i \) for sensor \( i \) is

\[
\bar{\theta}_s^i = \begin{bmatrix} C_{t-1}x_t \bar{A} \cdots \bar{C}_A \bar{A}_n \end{bmatrix}
\]

(10)

Definition 2. The system is regionally observable by sensor \( i \) (or, equivalently, the sensor \( i \) is regionally observable) on horizon \( N \), if \( \ker(\bar{\theta}_s^i) = 0 \).

We define the regionally unobservable subspace as \( \ker(\bar{\theta}_s^i) \).

Let \( \hat{P}_{NO} \) be the orthogonal projection matrix on \( \ker(\bar{\theta}_s^i) \). Similarly, let \( \hat{P}_O \) be the orthogonal projection on the regional observability subspace \( \ker(\bar{\theta}_s^i) \). Next, recall how \( \hat{P}_O \) and \( \hat{P}_{NO} \) can be computed. Let \( r_i = \ker(\bar{\theta}_s^i) \) and denote with \( \bar{z}_t, \ldots, \bar{z}_t \) an orthonormal basis of \( \ker(\bar{\theta}_s^i) \). Let also \( z_t, \ldots, z_t \) be an orthonormal basis of \( \ker(\bar{\theta}_s^i) \) and define the non-singular matrix \( \bar{S} = [\bar{z}_1, \ldots, \bar{z}_n] \). Defining the matrices \( \hat{S}_O \) and \( \hat{S}_{NO} \) as

\[
\hat{S}_O = \begin{bmatrix} I_{t-i} \end{bmatrix}, \quad \hat{S}_{NO} = \begin{bmatrix} 0_{(n-i)\times r} \end{bmatrix}
\]

we have \( \hat{P}_O = \bar{S}_O \hat{S}_O^T = \bar{S}_{NO} \hat{S}_{NO}^T \). Furthermore, defining \( \bar{T} = \text{diag}(\bar{T}_1, \ldots, \bar{T}_M) \), \( \hat{S}_O = \text{diag}(\bar{S}_1, \ldots, \bar{S}_M) \), and \( \hat{S}_{NO} = \text{diag}(\bar{S}_NO_1, \ldots, \bar{S}_NO_M) \), the collective projection matrices are \( \hat{P}_O = \hat{S}_O \hat{S}_O^T \) and \( \hat{P}_{NO} = \hat{S}_{NO} \hat{S}_{NO}^T \). Note that \( \hat{S}_{NO} \) is empty when the system is regionally observable by sensor \( i \). In this case we assume that \( \hat{P}_{NO} = 0_{n\times n} \).

2.4 Computation and boundedness of the matrices \( \Pi_{i\rightarrow N/i-1} \)

In this section we illustrate a distributed procedure for updating the matrices \( \Pi_{i\rightarrow N/i-1} \) used in (9). According to the distributed paradigm, each node must compute matrix \( \Pi_{i\rightarrow N/i-1} \) on the basis, at most, of the information provided by its \( N_t \)-th order neighbors \( j \in \mathcal{N}_i \). Specifically, we propose the following updating rule, see also Farina et al. (2009).

\[
\Pi_{i\rightarrow N/i-1} = 2 \sum_{j=1}^{M} (k_{ij})^2 \Pi_{j\rightarrow N/i-1} \quad (11)
\]

where the matrix \( \Pi_{i\rightarrow N/i-1} \), \( i \in \mathcal{Y} \), is given by one iteration of the difference Riccati equation associated to a Kalman filter for the system

\[
\begin{align*}
x_t & = A_{t-1}x_{t-1} + \omega_{t-1} \\
\hat{z}_t & = \hat{C}_A x_{t-1} + \hat{V}_{i-N}
\end{align*}
\]

where matrix \( \hat{\theta}_s^i \) is defined in (10). If we define

\[
\bar{C}_{N} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\
\end{bmatrix} \in \mathbb{R}^{n_{\Pi_{i\rightarrow N/i-1}}} \quad (12)
\]

\[
\bar{R}_i = \text{diag}(\bar{R}_{i-N}, \ldots, \bar{R}_{0}) \in \mathbb{R}^{n_{\Pi_{i\rightarrow N/i-1}}} \quad (13)
\]

\[
Q_{N-1} = \text{diag}(O, \ldots, Q) \in \mathbb{R}^{n_{\Pi_{i\rightarrow N/i-1}}} \quad (14)
\]

\[
\text{Cov}[w_t] = O \quad (15)
\]

\[
\text{Cov}[\hat{z}_t] = \bar{R}_i + \bar{C}_A \Omega_{N-1} (\bar{C}_A)^T \quad (16)
\]

and set the covariance of the estimate \( \hat{z}_t^i \) as

\[
\Pi_{i\rightarrow N/i-1} = ((\Pi_{i\rightarrow N/i-1})^{-1} + (\bar{C}_A)^T (\bar{R}_i)^{-1} \bar{C}_A)\quad (17)
\]

we obtain the Riccati equation

\[
\Pi_{i\rightarrow N/i-1} = \mathcal{R}(\Pi_{i\rightarrow N/i-1}; Q, \bar{R}_i) \quad (18)
\]

\[
= \mathcal{R}_{i\rightarrow N/i-1} \bar{z}_t + Q - \Pi_{i\rightarrow N/i-1}(\hat{z}_t)^T \times
\]

\[
(\hat{z}_t \Pi_{i\rightarrow N/i-1}(\hat{z}_t)^T + \bar{R}_i)^{-1} \Pi_{i\rightarrow N/i-1}(\hat{z}_t)^T + \bar{R}_i \quad (19)
\]

In the following we sketch the steps that have to be carried out, in practice, in order to apply the proposed algorithm

- Initialization: at \( t = 0 \) all nodes store the matrix \( \Pi_0 \) and the estimate \( \hat{z}_t = 0 \) of \( x_t \), \( i \in \mathcal{Y} \). Recall that \( \Pi_0 \) is the covariance matrix related to the initial condition \( x_0 \).
- if \( 1 \leq t \leq N \), the estimation horizon \( N \) is reduced to \( N = t \) and node \( i \in \mathcal{Y} \) performs the following steps
  - compute \( \Pi_{i\rightarrow N/i-1} = \Pi_{i\rightarrow N/i-1} \) from \( \Pi_0 \) according to (11), for all \( i \in \mathcal{Y} \);
  - solve the problem MHE-i, with initial penalty

\[
\Pi_{i\rightarrow N/i-1} = \frac{1}{2} \| \hat{z}_t - 0 \|_i \quad (20)
\]

- if \( t > N \), at each time instant, every node \( i \in \mathcal{Y} \),
  - computes \( \Pi_{i\rightarrow N/i-1} \) from \( \Pi_{i\rightarrow N/i-1} \) according to (11), (17), and (18);
  - solves the problem MHE-i, with initial penalty

\[
\Pi_{i\rightarrow N/i-1} = \frac{1}{2} \| \hat{z}_t - N = N - N/2 \|_i \quad (21)
\]

A key condition for guaranteeing convergence of DMHE is that the sequence \( \{\Pi_{i\rightarrow N/i-1}\}_{1}^{\infty} \) is bounded, for all \( i \) (at least in case the state variable \( x_t \) is subject to constraints), but unfortunately (11) does not guarantee this property. However, boundedness of \( \{\Pi_{i\rightarrow N/i-1}\}_{1}^{\infty} \) can be enforced by selecting the matrix \( K \) and \( N_t \) in a suitable way. Next, we provide a sufficient condition for boundedness of the sequence \( \{\Pi_{i\rightarrow N/i-1}\}_{1}^{\infty} \).

Before introducing the main result of this section, we need to define the matrices \( C_i \), \( i \in \mathcal{Y} \). If the communication protocol is \( P_i \) [resp. protocol \( P_o \)], the matrix \( C_i \) is composed by the rows of matrices \( C_i \), for all \( j \in \mathcal{Y}_{\Pi_{i\rightarrow N/i-1}} \) [resp. for all \( j \in \mathcal{Y}_{\Pi_{i\rightarrow N/i-1}} \)]. We partition the set \( \mathcal{Y} \) into the subsets \( \mathcal{Y}_o = \{ j \in \mathcal{Y} : (A, \hat{C}) \) is an observable pair \( \} \), \( \mathcal{Y}_{NO} = \{ j \in \mathcal{Y} : (A, \hat{C}) \) is not an observable pair \( \} \).

Theorem 1. If \( \mathcal{Y}_o \) is non-empty and, for all \( i \in \mathcal{Y}_{NO} \), there exists \( k > 0 \) such that \( \mathcal{Y}_o \cap \mathcal{Y}_{\Pi_{i\rightarrow N/i-1}} \neq \emptyset \), then there exists \( K^* \), compatible with the graph induced by \( K^* \), such that the matrices \( \Pi_{i\rightarrow N/i-1} \) resulting from (18) and (11) are bounded for all \( i \in \mathcal{Y} \).
The only assumption of Theorem 1 is that, for each node in \( \mathcal{V}_{NO} \), there exists an incoming directed path stemming from a node in \( \mathcal{V}_O \). Therefore, if one sensor is regionally observable, the assumption of Theorem 1 is verified. The proof of Theorem 1 is constructive, since it is based on the definition of a suitable matrix \( K^* \) compatible with the graph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \), induced by \( \mathcal{K}^{NO} \). A possible and simple way to choose \( K^* \) is summarized in the following algorithm.

**Algorithm 1.**

1. For each \( i \in \mathcal{V}_O \), set \( k_{ii}^* = 1 \);
2. For each \( i \in \mathcal{V}_{NO} \), select \( k_{ii}^* < \frac{1}{\sqrt{2\sigma_i(A)}} \), where \( \sigma_i(A) = \max \{ \lambda_{ij}(A) : \lambda_{ij}(A) \text{ is an unobservable eigenvalue for the pair } (A, \tilde{C}) \} \);
3. For each \( i \in \mathcal{V}_{NO} \) select a node \( j \in \mathcal{V}_O \) and a path from \( j \) to \( i \) in \( \mathcal{G} \), in such a way that each node in the path has at most one neighbor. We denote with \( \mathcal{E}_j \) the set of edges selected in this way;
4. For all edges \( (i, j) \in \mathcal{E}_j \), choose \( k_{ij}^* = 1 - k_{ji}^* \), while for all edges \( (i, j) \in \mathcal{E}_f \setminus \mathcal{E}_j \), set \( k_{ij}^* = 0 \).

Given the availability of methods for computing paths with a computational complexity that scales polynomially with \( |\mathcal{V}| \), see Bertsekas (1991), the overall algorithm is polynomial.

3. CONVERGENCE PROPERTIES OF DMHE

When the network is composed by a single sensor one has \( K = 1 \) and DMHE reduces to the MHE scheme, for which convergence and stability have been established in Rao et al. (2001). The main purpose of this Section is to extend the convergence results of Rao et al. (2001) to the proposed DMHE scheme.

**Definition 3.** Let \( \Sigma \) be system (1) with \( w = 0 \) and denote by \( x_2(t, x_0) \) the state reached by \( \Sigma \) at time \( t \) starting from initial condition \( x_0 \). Assume that the trajectory \( x_2(t, x_0) \) is feasible, i.e., \( x_2(t, x_0) \in \mathbb{X} \) for all \( t \). DMHE is convergent if \( \| x(t) - x_2(t, x_0) \| \to 0 \) as \( t \to \infty \) for all \( i \in \mathcal{V} \). □

Note that, as in Rao et al. (2001), convergence is defined assuming that the model generating the data is noiseless, but the possible presence of noise is taken into account in the state estimation algorithm.

Now, if we define the collective vectors

\[
\hat{x}(t_1/t_2) = \begin{bmatrix} x^1(t_1/t_2) \\ \vdots \\ x^M(t_1/t_2) \end{bmatrix} \in \mathbb{R}^{nM}, \quad x_2(t, x_0) = I_M \otimes x_2(t, x_0)
\]

(19)

where \( \otimes \) denotes the Kronecker product, the following intermediate result can be stated.

**Lemma 1.** If (i) matrices \( \Pi_{1-Ni}^{t} \) are computed as in Section 2.4, (ii) \( \Pi_{1-Ni}^{t} \) are bounded for all \( t \), and for all \( i \in \mathcal{V} \), then the dynamics of the state estimation error provided by the DMHE scheme is given by

\[
\hat{x}(t - N/t) = x_2(t - N, x_0) - \Phi(\hat{x}(t - N - 1/t - 1) - x_2(t - N - 1, x_0)) + \alpha_i
\]

(20)

where

\[
\Phi = P_{NO} K^* A \quad \text{and} \quad K^* = K^* \otimes I_n, \quad I_n \text{ is the } n\text{-th dimensional identity matrix,}
\]

\[
A = \text{diag}(A, ..., A) \in \mathbb{R}^{nM \times nM}, \quad \text{and } \alpha_i \text{ is an asymptotically vanishing term, i.e., } \| \alpha_i \| \to 0.
\]

The next result provides conditions for convergence of DMHE.

**Theorem 2.** Under the assumptions of Lemma 1, DMHE is convergent if the matrix \( \Phi \) is Schur.

We highlight that under the assumptions of Theorem 1, it is always possible to choose a matrix \( K \) compatible with \( \mathcal{G} \) such that condition (ii) of Lemma 1 is satisfied (see Algorithm 1).

The fundamental assumption that matrix \( \Phi \) is Schur is motivated by Lemma 1 and does not require that system (1) is asymptotically stable.

4. COLLECTIVE OBSERVABILITY

In this section we analyze some key implications of collective observability (see Definition 1). First, in Section 4.1 we investigate spectral properties of \( \Phi \). Then, in Section 4.2 we show how the assumptions guaranteeing the convergence of the proposed DMHE estimation scheme can be fulfilled, provided that a collective observability condition is satisfied and by properly tuning \( N \) and \( N_T \).

4.1 Spectral properties of \( \Phi \)

Note that Theorem 2 does not hinge explicitly on observability properties. For instance, it embraces the case where a component of the state of system (1) is not observed by any sensor but the estimation error decays to zero because the state component has the same property. Nevertheless, understanding how observability conditions affect the error dynamics is a topic of great interest. As a motivation, consider the problem of designing a linear state estimator when the network is composed by a single sensor. It is well known that if system (1) is detectable the error dynamics will inherit the unobservable eigenvalues of \( A \). However, if system (1) is observable one can design estimators such that the eigenvalues of the error dynamics do not depend upon the eigenvalues of \( A \).

In our framework, this rises the problem of studying conditions for guaranteeing that the matrix \( \Phi \) does not inherit any (non zero) eigenvalue of \( A \). More formally, let \( \lambda_i^j \) and \( v_i^j \) be the eigenvalues and the eigenvectors of \( A \), respectively, with \( i = 1, ..., n \). Note that, in view of its definition, the eigenvalues of \( A \) are \( \lambda_i^j \), one each with multiplicity \( M \). Moreover, denote by \( e_j \), \( (j = 1, ..., M) \) the canonical basis vectors of \( \mathbb{R}^M \), so that the eigenspace related to \( \lambda_i^j \) is span\{\( e_1 \otimes v_i^j, ..., e_M \otimes v_i^j \)\}. We consider the following property.

**Property 1.** If \( \lambda_i^j \) is a nonzero eigenvalue of \( A \), for all \( x \in \text{span}(e_1 \otimes v_i^j, ..., e_M \otimes v_i^j) \), \( \lambda_i^j \) and \( x \) are not an eigenvalue/eigenvector pair for \( \Phi \). □

**Theorem 3.** Consider a partition of \( \mathcal{G} \) into the irreducible subgraphs \( \mathcal{G}_i = (\mathcal{V}_i, \mathcal{E}_i) \), \( j = 1, ..., l \), assume that \( N \geq n - 1 \), and \( N \geq 1 \). If all the isolated strongly connected subgraphs \( \mathcal{G}_i \) are collectively observable, then Property 1 is verified.

In the case of strongly connected graphs the following result trivially follows.
Corollary 1. Assume that $N \geq n - 1$, and $N \geq 1$. If $\mathcal{G}$ is strongly connected and the system is collectively observable, then Property 1 is verified.

4.2 Collective observability and convergence of DMHE.

In this Section we investigate how the assumptions of Theorems 1 and 2 can be fulfilled, by properly tuning $N$ and $N_T$, and provided that the assumptions of Theorem 3 are satisfied. First we consider the assumptions of Theorem 1. Then, we prove that $N$ and $N_T$ can be chosen in such a way that $\Phi$ is Schur.

I) First of all, for sufficiently large values of $N$ and $N_T$, the assumptions of Theorem 3 imply the assumptions of Theorem 1. In fact, there exists a threshold value $\bar{N}$ such that, at least one node in each isolated irreducible subgraph $\mathcal{G}_j$, $(j = 1, \ldots, l)$ is observable for $N_T \geq \bar{N}$.

II) We now study the matrix $\Phi$. As a limit case, assume that all sensors enjoy regional observability. This yields $P_{NO} = \Phi = 0_{m \times nM}$ and convergence of DMHE follows from Theorem 2. Consider now the two data transmission protocols mentioned in Section 2.1.

In the case $P_T$, regional observability can be enhanced by increasing the number $N_T$ of data transmissions between agents within a sampling interval. The increase of $N_T$ has two accompanying effects.

1) If all the isolated strongly connected subgraphs are collectively observable, there exists a threshold value for $N_T$ (say $\bar{N}_T$) such that regional observability is satisfied by all the sensors for $N_T \geq \bar{N}_T$. So, for $N_T \geq \bar{N}_T$ one has $P_{NO} = 0_{m \times nM}$.

2) If $K^* = K^{\bar{N}_T}$, the modulus of the eigenvalues of matrix $\Phi$ decrease as $N_T$ increases. In fact, the eigenvalues of $K^*$ are equal to the eigenvalues of $K^{\bar{N}_T}$. Since $K$ is stochastic, it has $I$ (being $I$ the number of non empty irreducible subgraphs in which one can partition $\mathcal{G}$) eigenvalues equal to 1, and $M-I$ eigenvalues with modulus strictly less than 1. We denote with $\lambda_j$ the $j$-th eigenvalues of $K$. The corresponding eigenvalues of $K^*$ verify $|\lambda_j^{\bar{N}_T}| \leq |\lambda_j|$, resulting in a decrease of the eigenvalues of $\Phi$.

On the other hand, if the communication protocol $P_2$ is employed, we can enhance regional observability and the Schureness of $\Phi$ by increasing the estimation horizon $N$. As a limit case, if all the isolated strongly connected subgraphs are collectively observable, there exists a value $\bar{N}$ such that, $N \geq \bar{N}$ implies that $P_{NO} = 0_{m \times nM}$, which guarantees the Schureness of $\Phi$.

5. CONCLUSIONS

In this paper, we focused on distributed state estimation using a sensor network for linear systems. In order to account for physical constraints on process states and inputs, we have proposed a moving horizon approach where each sensor has to solve a quadratic programming problem at each time instant. We discussed conditions guaranteeing convergence of all estimates to a common value by characterizing the dynamics of the unobservable component of the state. As a main result, we showed that the only condition required for the application of the proposed estimation scheme is collective observability. This highlights the versatility and the wide applicability of the proposed algorithm.

REFERENCES


