Moving horizon partition-based state estimation of large-scale systems

Marcello Farina, Giancarlo Ferrari-Trecate, Riccardo Scattolini

Abstract
This paper presents three novel Moving Horizon Estimation (MHE) methods for discrete-time partitioned linear systems, i.e. systems decomposed into coupled subsystems with non-overlapping states. The MHE approach is used due to its capability of exploiting physical constraints on states and noise in the estimation process. In the proposed algorithms, each subsystem solves reduced-order MHE problems to estimate its own state and different estimators have different computational complexity, accuracy and transmission requirements among subsystems. In all cases, proper tuning of the design parameters, i.e. the penalties on the states at the beginning of the estimation horizon, guarantees convergence of the estimation error to zero. Numerical simulations demonstrate the viability of the approach.

Key words: Large scale systems, moving horizon estimator, system partitioning.

1 Introduction
Decentralized state-estimation algorithms for large-scale systems decomposed into physically coupled subsystems is of paramount importance in many engineering control problems, such as power networks [24], transport networks [20] and process control [23]. For this reason, many studies focused on the design of decentralized Kalman filters and the different solutions proposed can be classified according to the model used by each subsystem for state-estimation purposes and to the topology of the communication network among subsystems. Early works, e.g. [10,16] aimed at reducing the computational complexity of centralized Kalman filtering by parallelizing computations. However, they require all-to-all communication and assume each subsystem has full knowledge of the whole dynamics. In [14] the focus is on the use of reduced-order and decoupled models for each subsystem. The proposed solutions, beside neglecting coupling, exploit communication networks that are almost fully connected. Subsystems with overlapping states have been considered in [13,21–23]. While the estimation schemes in [23] require all-to-all communication, in [13,21,22] the topology of the network is defined by dependencies among the states of subsystems resulting in a fully decentralized scheme. The extreme case results in distributed state-estimation where the whole process is observed by a sensor network and each sensor measures just some of the system outputs. In this case, as shown in [4–7,12,15] convergence of the estimates to a common value can be achieved through consensus algorithms under weak assumptions on the topology of the communication network.

One drawback of (centralized or decentralized) Kalman filtering is that known physical constraints on noise and state variables are not exploited in the estimation process. This can lead to suboptimal estimates or instability of the error dynamics [19].

In order to overcome these issues, Moving Horizon Estimation (MHE) has been proposed for discrete-time linear [1,17], nonlinear [2,3,18] and hybrid systems [9]. MHE amounts to solve at each time instant an optimization problem whose complexity scales with the number of states, inputs and the estimation horizon. While the algorithms proposed in [1,2] are developed for unconstrained systems affected by bounded disturbances, the methods proposed in [17,18] are capable to cope with state constraints, at the price of greater computational complexity.

In this paper we propose three Partition-based MHE algorithms (PMHE), namely PMHE1, PMHE2 and PMHE3, for linear constrained systems that are decomposed into interconnected subsystems without overlapping states. In all cases each subsystem solves a reduced-order MHE problem...
in order to estimate its own states. This is in sharp contrast with the distributed MHE algorithm described in [6,7], where fully overlapping states are considered, hence calling for consensus algorithms for guaranteeing convergence of the estimates provided by individual sensors. The proposed PMHE solutions have different features in terms of communication requirements among subsystems, accuracy and computational complexity. While PMHE1 and PMHE2 provide a decentralization of the MHE scheme proposed in [17], PMHE3 is inspired to the MHE strategy for unconstrained systems described in [1]. Decentralization is achieved through suitable approximations of covariance matrices and results in suboptimal estimation algorithms, compared to centralized MHE. Nevertheless we provide conditions guaranteeing convergence of the PMHE schemes.

The paper is structured as follows. Section 2 introduces partitioned systems. Section 3 describes the proposed MHE procedures and convergence results are provided in Section 4. As an illustrative example, a compartmental system is considered in Section 5 and the main properties of the PMHE methods are discussed in Section 6. For the sake of readability, the proofs of the main results are collected in the Appendix. The proofs of some intermediate results are omitted here for space constraints, and are given in [8].

**Notation.** I_n and 0 denote the n × n identity matrix and the matrix of zero elements whose dimensions will be clear from the context, respectively. The notation ∥x∥₂ stands for z^T S z, where S is a symmetric positive-semidefinite matrix. Finally, we use the short-hand v = (v_1,...,v_s) to denote a column vector with s (not necessarily scalar) components.

### 2 Partitioned systems

Consider the discrete-time linear system

\[ x_{i+1} = A x_i + w_i, \quad (1) \]

where \( x_i \in \mathbb{R}^n \) is the state vector, while \( w_i \) represents a disturbance with variance \( Q > 0 \). The initial condition \( x_0 \) is a random variable with mean \( m_0 \) and covariance matrix \( \Pi_0 > 0 \). Measurements on the state vector are performed according to the sensing model

\[ y_i = C x_i + v_i, \quad (2) \]

where \( v_i \in \mathbb{R}^p \) is a white noise with variance \( R > 0 \).

Let system (1) be partitioned in \( M \) low order interconnected submodels with non-overlapping states i.e., where a generic submodel has \( x_i^l \in \mathbb{R}^{n_l} \) as state vector, and \( x_i = (x_i^1,...,x_i^M) \). Accordingly, the state transition matrices \( A^l \in \mathbb{R}^{n_l \times n_l} \), \( A^M \in \mathbb{R}^{p \times n_l} \) of the \( M \) subsystems are diagonal blocks of \( A \), whereas the non-diagonal blocks of \( A \) define the coupling terms between subsystems. It results that the \( i \)-th subsystem obeys to the linear dynamics

\[ x_{i+1}^l = A_i^l x_i^l + u_i^l x + w_i^l, \quad (3) \]

where \( x_i^l \) is the state vector, \( u_i^l \) collects the effect of state variables of other subsystems (and will be specified later on), and the term \( w_i^l \) is a disturbance with variance \( Q^l \). In the decomposition we also assume that state and disturbance of the \( i \)-th subsystem verify the bounds \( x_i^l \in \mathbb{R}_{x}^l, w_i^l \in \mathbb{R}_{w}^l \), where \( \mathbb{R}_{x}^l, \mathbb{R}_{w}^l \) are convex sets and 0 \( \in \mathbb{R}_{w}^l \). Correspondingly, constraints for model (1) are \( x_i \in \mathbb{R}_{x}^M, w_i \in \mathbb{R}_{w}^M \). When \( X = \mathbb{R}^n \) and \( W = \mathbb{R}^n \) we say that the system is unconstrained. The initial condition \( x_0^l \) is a random variable with mean \( m_0^l \) and covariance matrix \( \Pi_0^l \).

Note that \( Q^l > 0, R^l > 0 \) and \( \Pi_0^l > 0 \) can be obtained from \( Q, R \) and \( \Pi_0 \). For example, similarly to [22], we can assign \( Q^l, R^l \) and \( \Pi_0^l \) as diagonal blocks of \( Q, R \) and \( \Pi_0 \) of appropriate size.

According to (2) and to the state partition, the outputs of the subsystems are given by

\[ y_i^l = C_i^l x_i^l + u_i^l x + v_i^l, \quad (4) \]

where \( u_i^l \) collects the effect of the state variables of other subsystems (it will be specified later on), and the term \( v_i^l \in \mathbb{R}^p \) represents white noise with variance equal to \( R^l \). We define \( y_i = (y_i^1,...,y_i^M) \) and \( C = [C_1^T \cdots C_M^T]^T \).

**Remark 1** In general, some outputs of system (1) can be considered as outputs of more than one subsystem, i.e., \( p = \sum_{j=1}^{M} p_j \geq p \). Accordingly, there exists a matrix \( H \in \mathbb{R}^{p \times p} \) with rank \( p \), such that \( y_i = H y_i \) and \( C = HC \). Notice, however, that in decentralized control, each local subsystem commonly uses local information, which reduces the amount of transmitted information between subsystems. In this case we have \( H = I_p \).

From now on, we assume that the system partitioning has been carried out in such a way that the following assumption holds.

**Assumption 1** The pairs \((A_i^l, C_i^l)\) are observable, for \( i = 1,...,M \).

Notice that, neither Assumption 1 implies that the pair \((A, C)\) is observable, or observability of (1)-(2) implies Assumption 1.

We define \( n_i^l \) as the observability index of the pair \((A_i^l, C_i^l)\). We introduce the matrices \( A^l = \text{diag}(A_i^1,\ldots,A_i^M), \quad A = \text{diag}(A_1,\ldots,A_M), \quad \text{and} \quad A^l \in \mathbb{R}^{n \times n}. \)

Furthermore \( C = \text{diag}(C_i^1,\ldots,C_i^M), \quad C = \text{diag}(C_1,\ldots,C_M), \quad \text{and} \quad C^l \in \mathbb{R}^{n \times n}. \)

Correspondingly, the inputs \( u_i^1,l \) and \( u_i^M,l \) in (3) and (4) are

\[ \begin{bmatrix} C_i^1 & \cdots & C_i^M \end{bmatrix}^T \] has full column rank \( n_i^l \).
the terms $\hat{A}^{\delta}$, $\hat{B}^{\delta}$, $\hat{C}^{\delta}$, and $\hat{D}^{\delta}$, defined in (5) and (6), encompass also the uncertainty characterizing the terms $A^{\delta}$, $B^{\delta}$, $C^{\delta}$, and $D^{\delta}$.

We define $\hat{A}^{\delta}$ as the estimate of $A^{\delta}$ performed at time $t_2$ by subsystem $i$. Its error covariance matrix is denoted with $\Pi^{\delta}_{t_2|t_2}$. We approximate $\text{Var}(x_k - \hat{x}_k|t_2)$ as $\Pi^{\delta}_{t_2|t_2} = \text{diag}(\Pi^{\delta}_{t_2|t_2}, \ldots, \Pi^{\delta}_{t_2|t_2})$ when $i \neq j$. This approximation assumes that the error terms of different subsystems are uncorrelated. This approximation will allow decentralization of the centralized MHE problem. At time $t$ the estimation model is, for $k = t - N, \ldots, t$:

$$\hat{x}_k^{\delta} = A^{\delta}\hat{x}_{k-1}^{\delta} + \hat{A}^{\delta}\hat{x}_{k-1}^{\delta} + \hat{w}_k^{\delta}$$

(5a) and defines constraints of the centralized MHE problem specified in the next section. In (5), $\hat{x}_{k-1}^{\delta} \in \mathbb{R}^n$ denotes estimates of the subsystem states available at time $t$, and can differ from $\hat{x}_{k-1}^{\delta}$. Next we introduce two models for $\hat{x}_{k-1}^{\delta}$, $w_k^{\delta}$, and $\hat{w}_k^{\delta}$, that are related to different communication protocols: the first one will be used in PMHE1, while the second one will be used in PMHE2 and PMHE3.

Model 1: the system partition induces an interconnected network of subsystems, which can be described by a directed graph $G = (\mathcal{V}, \mathcal{E})$, where the nodes in $\mathcal{V}$ are the subsystems and the edge $(j, j)$ in the set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. The noise appearing in the equations (3), (4) and the estimation error of the variables $u_k^{\delta}$ and $u_k^{\gamma}$ are the terms $\hat{A}^{\delta}$, $\hat{B}^{\delta}$, $\hat{C}^{\delta}$, and $\hat{D}^{\delta}$. Therefore their variance is given by

$$\text{Var}(u_k^{\delta}) = Q^{\delta} + (\hat{A}^{\delta})^T \Pi^{\delta}_{k-1|k-1} (\hat{A}^{\delta})^T$$

(6a) and

$$\text{Var}(u_k^{\gamma}) = R^{\delta} + (\hat{C}^{\delta})^T \Pi^{\delta}_{k-1|k-1} (\hat{C}^{\delta})^T$$

(6b) Moreover, we set $\hat{x}_{k-1}^{\delta} = \hat{x}_{k-1}^{\delta}$. Note that, in (5) and (6), the terms $\hat{A}^{\delta}$, $\hat{B}^{\delta}$, $\hat{C}^{\delta}$, and $\hat{D}^{\delta}$ depend only upon the quantities transmitted by the neighboring subsystems $j \in \mathcal{V}^{\delta} = \{ j : (j, j) \in \mathcal{E} \}$.

Model 2: we assume an all-to-all communication, so that all the subsystems at time $t$ know the vector $\hat{x}_{k-1}^{\delta}$ and, for PMHE2, the matrix $\Pi^{\delta}_{k-1|k-1}$. Accordingly, at time $t$, the $i$-th subsystem estimation model, for $k = t - N, \ldots, t$, is (5), where $\hat{x}_{k-1}^{\delta} = A^{\delta}(t)\hat{x}_{k-1}^{\delta} - N, \ldots, t$.

The noise terms $w_k^{\delta}$ and $\hat{w}_k^{\delta}$ encompass also the uncertainty characterizing the terms $A^{\delta}$, $B^{\delta}$, $C^{\delta}$, and $D^{\delta}$, respectively, and hence their variance is given by

$$\text{Var}(w_k^{\delta}) = Q^{\delta} + A^{\delta} \Pi^{\delta}_{k-1|k-1} (A^{\delta})^T$$

(7a) and

$$\text{Var}(\hat{w}_k^{\delta}) = R^{\delta} + (\hat{C}^{\delta})^T \Pi^{\delta}_{k-1|k-1} (\hat{C}^{\delta})^T$$

(7b) where $\Pi^{\delta}_{k-1|k-1} = \Pi^{\delta}_{k-1|k-1}$ and $\Pi^{\delta}_{k-1|k-1} = A^{\delta}(t-N)\Pi^{\delta}_{k-1|k-1}(A^{\delta}(t-N))^T + \sum_{k=1}^{t} \Pi^{\delta}_{k-1|k-1}$

(7c)

3.2 The PMHE estimation problems

Given an estimation horizon $N \geq 1$, in order to perform the PMHE algorithm (with $r = 1, 2, 3$), each node $i \in \mathcal{V}$ at time $t$ solves the constrained minimization problem MHE-i defined as

$$\Theta_i^{\delta} = \min_{\hat{x}_{k-1}^{\delta}, \hat{w}_k^{\delta}} \left\{ \sum_{k=t-N}^{t} J_i^{\delta}(t-N, t; \hat{x}_{k-1}^{\delta}, \hat{w}_k^{\delta}, \hat{w}_k^{\delta}, \Gamma_r^{\delta}) \right\}$$

(8) where $\hat{w}_k^{\delta}$ and $\hat{w}_k^{\delta}$ stand for $\{\hat{w}_k^{\delta}\}_{k=t-N+1}$ and $\{\hat{w}_k^{\delta}\}_{k=t-N+1}$, respectively, under the constraints

$$\left\{ \begin{array}{l}
\text{System (5) with transmission Model 1 if } r = 1 \\
\text{System (5) with transmission Model 2 if } r = 2, 3
\end{array} \right.$$
\[ L_3^r = \frac{1}{2} \| \hat{x}_i^r \|^2 (\hat{x}_i^r) + \frac{1}{2} \| \hat{x}_i^r \|^2 (\hat{x}_i^r)^T \]  
\[ \Gamma_{3,t-N}^r = \frac{1}{2} \| \hat{x}_i^r \|^2 (\hat{x}_i^r) + \Theta_{3,t-1}^r \]  
with \( r = 1, 2 \). On the other hand, for PMHE3, they are defined as

\[ L_3^r = \frac{1}{2} \| \hat{x}_i^r \|^2 \]  
\[ \Gamma_{3,t-N}^r = \frac{1}{2} \| \hat{x}_i^r \|^2 (\hat{x}_i^r) + \Theta_{3,t-1}^r \]

In (12) and (14), \( \Theta_{3,t-1}^r \) is defined in (8) and, since it is known at time \( t \), it could be neglected when solving the optimization problem (8)-(9). However, it is here maintained for clarity since it plays a major role in establishing the main convergence properties of the algorithm. The positive definite symmetric matrices \( \Pi_{i,N-1}^r \) and \( R_{h,t-1}^r \) are design parameters whose choice is discussed next. In the sequel, \( \bar{x}_{i-N}^r / \bar{x}_{k/N}^r \) are the optimizers to (8) and \( \bar{x}_{j/k}^r, k = t-N, \ldots, t \) is the local state sequence stemming from \( \bar{x}_{i-N}^r / \bar{x}_{k/N}^r \) and \( \bar{x}_{i/N}^r / \bar{x}_{k/N}^r \)

### 3.3 Computation of \( \Pi_{i,N-1}^r \) for PMHE1 and PMHE2

In this section we provide an algorithm for computing \( \Pi_{i,N-1}^r \) given matrices \( Q_{j/t-1}^r, j = t-N, \ldots, t-2 \) and \( R_{h,t-1}^r \), \( h = t-N, \ldots, t-1 \). As shown in Section 4, this choice of \( \Pi_{i,N-1}^r \) is required for guaranteeing convergence of PMHE1 and PMHE2. We compute \( \Pi_{i,N-1}^r \) \( i \in \mathcal{Y} \), as the result of one iteration of the difference Riccati equation associated to a Kalman filter for the system

\[
\begin{aligned}
\dot{x}_{i/N-1}^r &= A^r \dot{x}_{i/N-1}^r + w_{i/N-1}^r \\
\dot{x}_{i-N}^r &= \bar{x}_{i-N}^r / \bar{x}_{k/N}^r + \bar{x}_{i/N}^r / \bar{x}_{k/N}^r + X_{i,N-1/1}^r + Y_{i/N}^r
\end{aligned}
\]

where \( X_{i,N-1/1}^r = (\hat{x}_{i-N-1/2}^r, \ldots, \hat{x}_{i/1}^r) \) are inputs and

\[
\begin{aligned}
\eta_{i/N+1}^r &= \left[ (C^r)^T \cdots (C^r (A^r)^N)^T \right]^T \\
\eta_N^r &= \begin{bmatrix}
C^r & \bar{C}^r & \cdots & 0 \\
C^r A^r & \bar{C}^r & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
C^r (A^r)^N A^r & C^r (A^r)^N & \cdots & \bar{C}^r
\end{bmatrix}
\end{aligned}
\]
Alternative A: For a threshold $\Lambda > 0$ set

$$
\begin{align*}
Q_{k+1}^i &= \text{Var}(\hat{x}_{k+1}^i) \\
R_{k+1}^i &= \text{Var}(\hat{\theta}_{k+1}^i) \\
Q_{k+1}^{i+1} &= Q_{k+1}^i \\
R_{k+1}^{i+1} &= R_{k+1}^i
\end{align*}
$$

if $\|\Pi_{t-N/N-1}^i\|_2 < \Lambda$

Alternative B: set $Q_{k+1}^i = Q_{k+1}^{i+1} = Q_{k+1}^h$, $h = t-N, \ldots, t-1$, $\forall t$.

Alternative C: keep $Q_{k+1}^i$ and $R_{k+1}^i$ constant replacing $\Pi_{k+1}^i$ in (6) and $\Pi_{k+1}^i$ in (7) with the matrix $\Delta = \text{diag}(\Delta^1, \ldots, \Delta^M)$, where $\Delta^i$ verifies the algebraic Riccati equation associated to (16b):

$$
\Delta^i = A^i \Delta^i (A^i)^T + Q^i - A^i \Delta^i \Theta^i (\Theta^i)^T \times
\left( \Theta^i \Delta^i (\Theta^i)^T + R^i \right)^{-1} \Theta^i \Delta^i (\Theta^i)^T
$$

(17)

where $\Delta^i = \text{diag}(R^1, \ldots, R^M) \in \mathbb{R}^{N_{P_i} \times N_{P_i}}$. Note that, under Assumption 1, $(A^i, \Theta^i)$ is an observable pair and, since $Q^i > 0$, (17) has always a unique positive definite solution.

Alternative A, in principle, provides the best performances in terms of estimation accuracy. However, since the matrices $Q_{k+1}^i$ and $R_{k+1}^i$ are updated until $\|\Pi_{t-N/N-1}^i\|_2$ grows too much, Alternative A requires a bigger transmission load than the other choices. In fact, alternative B and C do not require transmission of covariance matrices. Alternative B is simpler than alternative C, but neglects the uncertainty associated with the transmitted information. On the other hand, alternative C accounts, in an approximate way, for the uncertainty on $\hat{x}_{t-N/N-1}^i$.

4 Convergence properties of the proposed estimators

In this section the convergence results reported in [1,17] for centralized estimators (corresponding to the trivial partition) are extended to the proposed PMHE methods in presence of constraints. Similarly to [17], these properties are analyzed in a deterministic setting.

Definition 1 Let $\Sigma$ be system (1) with $w_t = 0$ and denote by $x_t, x_0$ the state reached by $\Sigma$ at time $t$ starting from initial condition $x_0$. Assume that the trajectory $x_{t}(t, x_0)$ is feasible, i.e., $x_{t}(t, x_0) \in \mathcal{X}$ for all $t$. PMHE is convergent if $\|\hat{x}_{t} - x_t(t, x_0)\|_{\infty} \rightarrow 0$.

Note that, as in [17], convergence is defined assuming that the model generating the data is noiseless, but the possible presence of noise is taken into account in the state estimation algorithm. The estimation error is defined as $\hat{x}_{t} = x_t(x_1, x_0) - \hat{x}_t$.

Let $\hat{\Theta}_{N+1} = \text{diag}(\Theta^1_{N+1}, \ldots, \Theta^M_{N+1})$ be the extended observability matrix of the pair $(A^i, C^i)$, where matrices $\Theta^i$ are defined in (15a). Furthermore, let $\hat{\Theta}_{N+1} = (\hat{C}^T \cdots (\hat{C}^M)^T)^T$ be the extended observability matrix of the pair $(A, C)$. Denote by $f_{\min} = \sigma_{\min}(\hat{\Theta}_{N+1})$ and $f_{\max} = \sigma_{\max}(\hat{\Theta}_{N+1})$, the minimum and the maximum singular value of $\hat{\Theta}_{N+1}$, respectively. By Assumption 1, if $N \geq \tilde{n}^0 - 1$, then $\text{rank}(\hat{\Theta}_{N+1}) = n_i$ for all $i \in \mathcal{I}$. From this it follows that $\text{rank}(\Theta^i_{N+1}) = n_i$, and therefore $f_{\min} > 0$.

Furthermore, define $\Delta_f = \|\Theta^i_{N+1} - \Theta^i_{N-1}\|_2$, $\kappa = \|\A\|_2$, and $\kappa^* = \|\A^i\|_2$.

Lemma 1 If matrices $\Pi_{t-N/N-1}^i$ are computed as in Section 3.3 and $N \geq \max\{\tilde{n}^0 - 1, 1\}$, then there exist asymptotically vanishing sequences $\alpha_j^i$ (i.e., $\|\alpha_j^i\|_{\infty} \rightarrow 0$, $j = 1, 2$) such that the dynamics of the state estimation error generated by PMHE1 is given by

$$
\begin{align*}
\hat{e}_{t-N+N-1} &= -\Theta_{N+1}^T \hat{e}_{t-N-1} + \alpha_1^i \\
E_{t-N-1} &= M_{t-N-1} \hat{e}_{t-N+1} + \alpha_2^i
\end{align*}
$$

where $E_{t-N-1} = (e_{t-N/k_1}, \ldots, e_{t-N/k_2})$ and

$$
\begin{align*}
\alpha_1^i &= \begin{bmatrix} \hat{C} & 0 & \cdots & 0 \\ C^i \hat{A} & \hat{C} & \cdots & 0 \\ : & : & \cdots & : \\ C^i (A^i)^{N-1} \hat{A} & C^i (A^i)^{N-2} \hat{A} & \cdots & \hat{C} \end{bmatrix} \\
M_1 &= \begin{bmatrix} A^i & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ (A^i)^N & \cdots & 0 \\ (A^i)^{N+1} & \cdots & \cdots & \cdots \end{bmatrix}, M_2 &= \begin{bmatrix} \hat{A} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ (A^i)^{N-1} \hat{A} & (A^i)^{N-2} \hat{A} & \cdots & 0 \\ (A^i)^{N} \hat{A} & (A^i)^{N-1} \hat{A} & \cdots & \cdots \end{bmatrix}
\end{align*}
$$

(19a)

(19b)

Lemma 2 If matrices $\Pi_{t-N/N-1}^i$ are computed as in Section 3.3 and $N \geq \max\{\tilde{n}^0 - 1, 1\}$, then there exists an asymptotically vanishing sequence $\alpha_j^i$ (i.e., $\|\alpha_j^i\|_{\infty} \rightarrow 0$) such that the dynamics of the state estimation error generated by PMHE2 is given by

$$
\begin{align*}
\hat{e}_{t+N-1} &= (\hat{\Theta}_{N+1} - \Theta_{N+1}) \hat{e}_{t-N/N-1} + \alpha_1^i \\
\hat{e}_{t-N/N-1} &= (\hat{\Theta}_{N+1} - \Theta_{N+1}) \hat{e}_{t-N/N-1} + \alpha_1^i
\end{align*}
$$

(20a)

(20b)

Lemma 3 Assume that $A^i$ is non singular, and that one of the following conditions holds: (a) $\kappa^* \leq 1$, (b) $\kappa^* > 1$ and
III) Under the assumptions of Lemma 3, if the matrix
\[ \Phi_1 = M_2 - M_1 \left( (\mathcal{E}_N^o)^T \mathcal{E}_N^o \right)^{-1} (\mathcal{E}_N^o)^T \mathcal{E}_N^o \]
is Schur, then PMHE1 is convergent.

II) Under the assumptions of Lemma 2, if the matrix
\[ \Phi_2 = \left( (\mathcal{E}_N^o)^T \mathcal{E}_N^o \right)^{-1} (\mathcal{E}_N^o)^T (\mathcal{E}_N^o - \mathcal{E}_N^o) A \]
is Schur, then PMHE2 is convergent.

III) Under the assumptions of Lemma 3, if the matrix \( \Phi_2 \) is Schur then PMHE3 is convergent.

4.1 Simpler conditions for PMHE2 and PMHE3 and a measure of the quality of the partition

We start providing sufficient conditions for the convergence to zero of the error dynamics (20). To this aim, we define the scalars \( v \) and \( a_0 \) as
\[ v = \frac{f_{\text{max}} \Delta f}{f_{\text{min}}}, \quad a_0 = v \kappa \]
(22)

Theorem 2 Under the assumptions of Lemma 2 [resp. Lemma 3], if \( a_0 < 1 \), then PMHE2 [resp. PMHE3] is convergent.

In view of this result, convergence of PMHE2 and PMHE3 depends on the norm \( \kappa \) of the transition matrix \( A \) (which is a function of the problem data), and the constant \( v \), given in (22), which depends on the adopted partition. Therefore, \( v \) can be considered as a measure of the quality of the partition, for state estimation purposes. Specifically, the smaller \( v \), the better the partition. On the other hand, \( \Delta f \) is expected to increase with the number of arcs in the graph and this results in an increase of the index \( v \).

Concerning only PMHE3, Lemma 3, part III of Theorem 1 and Theorem 2 generalize the results of [1] to constrained estimation. In the unconstrained case, with similar arguments to [1] we obtain the following result.

Lemma 4 For all \( i \in \mathcal{V} \), if \( N \geq \max \{ n^i, 1 \} \), the dynamics of the state estimation error provided by PMHE3 in the unconstrained case is given by
\[ e_{r-N} = (\mu I_n + (\mathcal{E}_N^o)^T \mathcal{E}_N^o)^{-1} \times \left[ \mu I_n + (\mathcal{E}_N^o)^T (\mathcal{E}_N^o - \mathcal{E}_N^o) \right] A e_{r-N} \]
(23a)

A sufficient condition for the convergence of unconstrained PMHE3 encompassing the case \( a_0 \geq 1 \) excluded by Theorem 2, is given by the following corollary.

Corollary 1 If \( N \geq \max \{ n^i - 1, 1 \} \), then unconstrained PMHE3 is convergent if
\[ a(\mu) = \frac{\mu + f_{\text{max}} \Delta f}{\mu + f_{\text{min}} \Delta f} \kappa = \frac{\mu + v}{\mu + v} \kappa < 1 \]
(24)

Note that \( a(0) = a_0 \), showing that the condition for convergence of constrained PMHE2 and PMHE3 (i.e., that \( a_0 < 1 \)) is only sufficient for the existence of a value of \( \mu \) guaranteeing the convergence of unconstrained PMHE3. In Figure 1 we plot \( a(\mu) \), and discuss convergence of PMHE2 and PMHE3 with respect to all possible combinations of \( a_0, \kappa \) and \( v \).

Fig. 1. Plots of \( a(\mu) \) (y-axis) with respect to \( \mu \) (x-axis). (A-B): neither PMHE2 nor PMHE3 are guaranteed to converge (Theorem 2 and Corollary 1). D-E: both PMHE2 and PMHE3 are guaranteed to converge. (C): PMHE2 is guaranteed to converge, while convergence of unconstrained PMHE3 is guaranteed by Corollary 1 only if \( \mu < \mu_{\text{max}} \). (F): PMHE2 and PMHE3 are not guaranteed to converge, while convergence of unconstrained PMHE3 is guaranteed only if \( \mu > \mu_{\text{min}} \).

5 Example: a compartmental system

Consider the interconnected system reported in Figure 2-A. The subsystems 1, \ldots, 4 are third order compartmental subsystems, whose structure is depicted in Figure 2-B. If subsystem \( i \) has \( m \) inputs and \( p \) outputs, its discrete-time dynamical model is defined by
Fig. 2. Scheme of the compartmental system in the example. A: connections between subsystems 1, …, 4. B: general structure of the subsystems 1, …, 4.

\[
A^4 = \begin{bmatrix}
1-k_{12} & k_{21} & k_{31} \\
-k_{12} & 1-(k_{21}+k_{23}+p_k_{23}) & 0 \\
0 & k_{23} & 1-k_{13}
\end{bmatrix}
\]

\[
B^4 = \begin{bmatrix}
1 & \ldots & 1 \\
0 & \ldots & 0 \\
0 & \ldots & 0
\end{bmatrix}, \quad C_l^4 = \begin{bmatrix}
0 & k_{21} & 0 \\
0 & k_{23} & 0 \\
0 & k_{23} & 0
\end{bmatrix}
\]

We chose \(k_{ij} = 0.1\) for all \(i\) and \(j\), \(k_{23} = 0.1\), and we introduce the vectors \(b = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T\) and \(c = \begin{bmatrix} 0 & k_{21} & 0 \end{bmatrix}\). If we connect the \(4\) subsystems according to the scheme in Figure 2-A we obtain a 12-states system with the structure (1) and (2), given by

\[
A = \begin{bmatrix}
A^4 & bc & O_{3 \times 3} & bc \\
O_{3 \times 3} & A^2 & bc & O_{3 \times 3} \\
bc & O_{3 \times 3} & A^3 & O_{3 \times 3} \\
O_{3 \times 3} & O_{3 \times 3} & bc & A^4
\end{bmatrix}
\]

and \(C = \text{diag}(c, c, c, c)\). Note that the spectral radius of \(A\) is 1. We assume that states of each subsystem are affected by leakages represented by additive negative noise terms \(w_k\). We take \(w_k = \max(-1, -|e_k|)\), for all \(k\), where \(e_k\) is a white noise signal with zero mean and \(Q = \text{var}(e_k) = \text{diag}(1, \epsilon, 1, \epsilon, 1, \epsilon, 1, \epsilon, 1, \epsilon, 1, \epsilon)\), where \(\epsilon = 10^{-8}\). Therefore, the first state of each subsystem is affected by leakage more severely than the other states. We assume white measurement noise with covariance \(\mathbf{R} = 0.01I_{12}\). Since the states represent masses in the compartments, they are constrained to be non negative. Furthermore, we take \(\Pi_0 = 340I_{12}\).

Next we compare the PMHE1, PMHE2, PMHE3 strategies with a centralized MHE estimator. For the design of PMHE3, we compute \(\kappa^* = 0.9913 < 1\) and, by Lemma 3, all \(\mu > 0\) guarantee convergence of the estimates. We choose \(\mu = 0.001\).

For the considered system, \(\bar{\mu} = 3\), \(\kappa = 1.03\) and \(V = 370\), so that \(a_{ij} \gg 1\). Therefore, the sufficient conditions given in Theorem 2 cannot be used. Nevertheless, the convergence properties of PMHE estimators can be proved using Theorem 1. In order to guarantee the applicability of the four estimators, the estimation horizon is set as \(N = 3\) in all the PMHE schemes (to satisfy the assumptions of Lemmas 1, 2 and 3) as well as in the centralized MHE. PMHE1 and PMHE2 have been run by updating covariances according to Alternative B. In Fig. 3 we compare the estimated and real state trajectories.

We have also explored the effect of the variation of the estimation horizon \(N\) on the estimation performances and on the computational burden through simulations. The root mean square error for \(t \in [15, 45]\) (i.e., neglecting the initial transients) and the time required to run the estimation algorithms, for \(N = 3, 7, 10\), are reported in Table 1.

Interestingly, the time required for each node to perform PMHE1, PMHE2 and PMHE3, is reduced with respect to the time required to perform centralized MHE, at the price of obtaining suboptimal estimations in terms of noise rejection. Although as \(N\) increases a larger set of data is used in the optimization problem, this does not lead to a significant improvement of the accuracy of the results. On the other hand, an increase in \(N\) leads to a significant grow in computational (and transmission) burden (i.e. \(T_r = 3.8\)s if \(N = 3\), \(T_r = 111\)s if \(N = 7\) and \(T_r = 21.9\)s if \(N = 10\)).

6 Discussion and conclusions

In this paper we have proposed three distributed state-estimation algorithms, namely PMHE1, PMHE2 and PMHE3, for partitioned large-scale systems and we have provided sufficient conditions for convergence. The three solutions have different features in terms of communication requirements among subsystems, accuracy and computational complexity. More specifically, PMHE1 relies on a partially connected communication graph in the sense that subsystems exploit a communication network where links are present only if subsystem dynamics are coupled. Algorithms PMHE2 and PMHE3 assume an all-to-all communi-
culation but a reduced amount of information is transmitted over each communication channel. The main difference between PMHE1 and PMHE2 consists in the type of communication required among subsystems, and on how the estimates of $\hat{x}^1_k$ and $\hat{x}^2_k$ are used. While in PMHE1 and PMHE2 the transmitted information amounts to state estimates and estimation error covariances, in PMHE3 no information on the noise variances is required and the weights on the different components of the cost functions are constant. This allows for a significant reduction in terms of transmission and computational load, at the price of a loss in noise filtering performance.

Future research will focus on methods for weakening the requirement of an all-to-all communication for PMHE2 and PMHE3. Moreover we will study how to generalize the proposed methods to the case of subsystems with overlapping states while guaranteeing that estimates of states that are shared by some subsystems converge to a common value. Further studies are also needed for designing PMHE schemes capable to cope with non linear systems and non-idealities in the communication network, such as quantiza-

tion effects and transmission delays.

Finally note that PMHE schemes can be applied even if one chooses matrices $A'$ and $C'$ arbitrarily and assign $\hat{A} = A - A'$ and $\hat{C} = C - C'$, provided that Assumption 1 is verified. This decomposition gives more flexibility in selecting the system partition, but also bias models used in the estimation process and calls for the development of robust MHE schemes for bounding the effect of bias terms.

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References


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<th>PMHE3</th>
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<td>0.34Tc</td>
<td>0.47Tc</td>
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</table>

Table 1

Comparison of the root mean square error (RMSE) $\sqrt{\frac{1}{T} \sum_{k=1}^{45} \| \hat{x}_k - x_k \|^2}$ and of the time required (for each subsystem) to perform the proposed estimates. $I_c$ and $T_c$ denote the RMSE of the centralized MHE estimation error and the time required to perform the centralized MHE, respectively.


A Proofs

Denote by \( J_r \) the sum (for all the \( M \) subsystems) of the local cost functions \( J_r^{i} \), given by (10)

\[
J_r = \sum_{i=1}^{M} J_r^{i} (t - N; t; \hat{x}_r^{i}[t-N], \hat{y}_r^{i}, \Gamma_r^{i}[t-N])
\]  

(A.1)

Define the collective vectors \( \hat{\mathbf{x}}_t = (\hat{x}_r^{i}[t-N], \ldots, \hat{x}_r^{M}[t-N]) \in \mathbb{R}^n \), \( \hat{\mathbf{y}}_t = (\hat{y}_r^{i}[t-N], \ldots, \hat{y}_r^{M}[t-N]) \in \mathbb{R}^p \), \( \hat{\mathbf{w}}_t = (\hat{w}_r^{i}[t-N], \ldots, \hat{w}_r^{M}[t-N]) \in \mathbb{R}^n \) and rewrite \( J_r \) as

\[
J_r = \sum_{k=t-N}^{t} L_r(\hat{w}_r^{i}, \hat{y}_r^{i}) + \Gamma_{r,k-N}(\hat{\mathbf{x}}_t; \hat{\mathbf{x}}_t^{N-k-N})
\]  

(A.2)

For \( r = 1, 2; \)

\[
L_2 = \frac{1}{2} \| \hat{x}_t \|^2 |_{R_{k-1}}^{-1} + \frac{1}{2} \| \hat{w}_k \|^2 |_{Q_{k-1}}^{-1}
\]  

(A.3)

\[
\Gamma_{r,k-N} = \frac{1}{2} \| \hat{\mathbf{x}}_t^{N-k-N} - \hat{\mathbf{x}}_t^{N-k-N} \|^2 |_{\mathbf{P}_{k-1}}^{-1} + \Theta_{r,k-1}^{-1}
\]  

(A.4)

where \( R_{k-1} = \text{diag}(R_{k-1}^{1}, \ldots, R_{k-1}^{M}) \), \( Q_{k-1} = \text{diag}(Q_{k-1}^{1}, \ldots, Q_{k-1}^{M}) \) and \( \Theta_{r,k-1} = \sum_{i=1}^{M} \Theta_{r,k-1}^{i} \). On the other hand, for PMHE3:

\[
L_3 = \frac{1}{2} \| \hat{x}_t \|^2
\]  

(A.5)

\[
\Gamma_{3,k-N} = \frac{1}{2} \| \hat{\mathbf{x}}_t^{N-k-N} - \hat{\mathbf{x}}_t^{N-k-N} \|^2 + \Theta_{3,k-1}^{-1}
\]  

(A.6)

Constraints (9) can be written in the following collective form

\[
\begin{align}
\hat{x}_{k+1} &= A^x \hat{x}_k + \hat{A} \hat{x}_{k/[i-1]} + \hat{w}_k \\
\hat{y}_k &= C^x \hat{x}_k + \hat{C} \hat{x}_{k/[i-1]} + \hat{v}_k \\
\hat{\mathbf{x}}_k &\in \mathbb{X}
\end{align}
\]  

(A.7a)

(A.7b)

(A.7c)

(A.7d)

with \( k = t - N, \ldots, t \). Notice that \( \hat{x}_{k/[i-1]} \) are computed differently, depending on the adopted scheme. The solution to

\[
\sum_{k=t-N}^{t} J_r(t - N; t; \hat{x}_{k,N}, \hat{y}_k, \hat{\mathbf{x}}_k, \Gamma_{r,k-N})
\]  

(A.8)

where \( \hat{\mathbf{w}} \) and \( \hat{\mathbf{y}}^i \) are shorthand notation for \{ \( \hat{\mathbf{w}}_k \) \} \( k=t-N \) and \{ \( \hat{\mathbf{y}}_k^i \) \} \( k=t-N+1 \) respectively, is equivalent to the solution of the MHE-i problems (8), in the sense that \( \hat{\mathbf{w}}^i_k \) \( k=t-N \) is a solution to (8) if and only if \( \hat{\mathbf{x}}_{k,N}^i \) \( k=t-N \) is a solution to (A.8), where \( \hat{\mathbf{x}}_{k,N}^i = (\hat{w}_r^{i}[k-N], \ldots, \hat{w}_r^{M}[k-N]) \). In fact, at time \( t \), variables \( \hat{x}_{k,N}^i \) are computed as

\[
\begin{align}
\hat{\mathbf{x}}_{k,N}^i &= \mathbf{R}_{k-N}^{-1} (\hat{\mathbf{x}}_{k,N}^i \hat{\mathbf{y}}_k^i) \\
\hat{\mathbf{x}}_{k,N}^i &= \mathbf{R}_{k-N}^{-1} (\hat{\mathbf{x}}_{k,N}^i \hat{\mathbf{y}}_k^i)
\end{align}
\]  

(A.9)

The key condition to prove Lemma 1 and Lemma 2 (see also [17,18]) is that, for all \( \hat{x} \in \mathbb{X} \)

\[
\Gamma_{r,k-N}(\hat{x}; \hat{\mathbf{x}}_t^{N-k-N}) \leq \mathbf{R}_{t-N} \| \hat{x} \|_{\mathbb{X}} \quad \forall t \geq N
\]  

(A.10)

Under (A.10) the first step towards the convergence of the PMHE estimators is the following lemma.

Lemma 5 If (A.10) holds then, for all the PMHE schemes

\[
\sum_{k=t-N}^{t} L_r(\hat{w}_r^{i,k}, \hat{y}_r^{i,k}) \rightarrow 0
\]  

(A.11)

The following results are specific for each estimation scheme. We first provide conditions guaranteeing (A.10) to hold, then we investigate the implications of (A.11), in terms of estimation error convergence.

Lemma 6 For PMHE1 and PMHE2, if (16) are satisfied then, for all \( \hat{x} \in \mathbb{X} \), (A.10) holds.

Lemma 7 For PMHE3, if \( \lambda^* \) is non singular and (a) \( \lambda^* \leq 1 \) or \( \lambda^* > 1 \) and \( \mu < \mu_{max} \), where \( \mu_{max} = \frac{f_{cap}}{(\lambda^*)^2-1} \) then (A.10) holds.
**Lemma 8** Assume that $N \geq \max\{\hat{n}, 1, 1\}$ and equation (A.11) holds. Then

a) if $Q_{j-1}$ and $R_{j-1}$ are bounded for all $t$ and for all $k = t-N, \ldots, t$, then the dynamics of the state estimation error provided by PMHE1 is given by (18);

b) if $Q_{j-1}$ and $R_{j-1}$ are bounded for all $t$ and for all $k = t-N, \ldots, t$, then the dynamics of the state estimation error provided by PMHE2 is given by (20);

c) the dynamics of the state estimation error provided by PMHE3 is given by (20).

**Proof of Lemmas 1, 2 and 3**

For PMHE1 and PMHE2 [resp, for PMHE3], in view of Lemma 6 [resp, Lemma 7], (A.10) holds. Therefore, from Lemma 5, (A.11) is verified. The proofs are concluded by resorting to Lemma 8.

The proofs of Lemma 5, Lemma 6, Lemma 7 and Lemma 8 are omitted for lack of space and are given in [8].

**Proof of Theorem 1**

First, note that $\theta_{N+1}$ has full column rank, in view of Assumption 1. To prove (I), pre-multiply both sides of (18a) by the pseudo-inverse of $\theta_{N+1}$

$$e_{-N/\ell} = -((\theta_{N+1})^T \theta_{N+1})^{-1} (\theta_{N+1})^T e_{t/\ell} + ((\theta_{N+1})^T \theta_{N+1})^{-1} (\theta_{N+1})^T a_t$$

From (18b), we obtain that $e_{t+1/\ell} = \Phi_1 e_{t/\ell} + a^2$. Therefore, convergence of the error to zero is guaranteed if $\Phi_1$ is Schur.

To prove (II) and (III), pre-multiply both sides of (20) by the pseudo-inverse of $\theta_{N+1}$

$$e_{-N/\ell} = ((\theta_{N+1})^T \theta_{N+1})^{-1} (\theta_{N+1})^T (\theta_{N+1} - e_{N+1}) \times$$

$$\times A e_{-N/\ell}^- + ((\theta_{N+1})^T \theta_{N+1})^{-1} (\theta_{N+1})^T a_t$$

(A.12)

Recalling (21), convergence of the error to zero is guaranteed if $\Phi_2$ is Schur.

**Proof of Theorem 2**

Under the assumptions of Lemma 2 [resp, Lemma 3] the dynamics of the estimation error of PMHE2 [resp, PMHE3] is given by (A.12). Let $\varepsilon_t = \|e_{-N/\ell}\|_2$, and

$$s_t = \|((\theta_{N+1})^T \theta_{N+1})^{-1} (\theta_{N+1})^T a_t\|_2,$$

which satisfies $s_t \to 0$ as $t \to \infty$. From (A.12) one has

$$\varepsilon_t \leq \|((\theta_{N+1})^T \theta_{N+1})^{-1}\|_2 \|\theta_{N+1}\|_2 \times$$

$$\times \|\theta_{N+1} - e_{N+1}\|_2 \|a_t\|^2 + s_t$$

from (23a) we obtain

$$\varepsilon_t \leq \|\mu I_n + (\theta_{N+1})^T \theta_{N+1}\|_2 \times$$

Recall that $\|\mu I_n + (\theta_{N+1})^T \theta_{N+1}\|^2 = (\mu + \mu_{\min})^{-1}$, and that $\|\mu I_n + (\theta_{N+1})^T (\theta_{N+1} - e_{N+1})\|^2 \leq \mu^2 + \|\theta_{N+1}\|^2 \|e_{N+1}\|^2$ since $\mu \geq 0$. Being $f(\mu)$ defined as in (24), it follows that $\varepsilon_t \to 0$ as $t \to \infty$ if $f(\mu) < 1$. 

**Proof of Lemma 4**

Let $\tilde{y}_t = \bar{C} x_t(k, x_0)$. For $r = 3$, the cost function (A.1) is

$$J_3(\cdot) = \frac{1}{2} \sum_{k=t-N}^t \|CA^{k-t+N} x_{\tilde{y}_t}(t-N, x_0) +$$

$$- C^*(A^*)^{k-t+N} \tilde{y}_t - M_{k-t+N} \tilde{y}_t - t/\ell + 1\|_2^2 +$$

$$+ \frac{\mu}{2} \|\tilde{y}_t - \tilde{y}_t(t-N)\|_2^2$$

where

$$M_{k-t+N} = C^* \sum_{j=1}^k A^*(A^*)^j$$

Therefore

$$J_3(\cdot) = \frac{1}{2} \|\theta_{N+1} - e_{N+1}\|_2^2 \times$$

(A.13)

From (A.13), the unconstrained ($X = \mathbb{R}^n$) minimization problem (A.8) is solved by

$$\hat{x}_{t-N/\ell} = (\mu I_n + (\theta_{N+1})^T \theta_{N+1})^{-1} \left[(\mu I_n + (\theta_{N+1})^T \times$$

$$(\theta_{N+1} - e_{N+1}) \right] \hat{x}_{t-N/\ell} + \left((\theta_{N+1}^T \theta_{N+1})^{-1} \right) \hat{y}_{t-N/\ell}$$

From this, it follows that the dynamics of the estimation error is given by (23).

**Proof of Corollary 1**

From (23a) we obtain

Recall that $\|\mu I_n + (\theta_{N+1})^T \theta_{N+1}\|^2 = (\mu + f_{\min})^{-1}$, and that $\|\mu I_n + (\theta_{N+1})^T (\theta_{N+1} - e_{N+1})\|^2 \leq \mu^2 + \|\theta_{N+1}\|^2 \|e_{N+1}\|^2$ since $\mu \geq 0$. Being $f(\mu)$ defined as in (24), it follows that $\varepsilon_t \to 0$ as $t \to \infty$ if $f(\mu) < 1$. 

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