Control schemes based on the wave equation for consensus in multi-agent systems with double-integrator dynamics

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Abstract—In this paper, we consider the problem of driving a group of agents communicating through an undirected and weighted network towards a consensus point. We assume that agents obey to double-integrator dynamics and study decentralized control schemes for consensus of the position variables. In particular we revisit control policies proposed in [1] and [2] in the unified framework of Partial difference Equations over graphs and highlight the link between the closed-loop system dynamics and the damped wave equation describing the motion of a free elastic beam.

I. INTRODUCTION

The problem of studying how the individual actions of intercommunicating agents can give rise to a coordinated behavior is an issue of notable importance in many fields of scientific research like biology [3], physics [4], [5] and computer graphics [6].

In control engineering, the last decades have manifested an increasing interest in research on distributed control of multi-agent systems, due to the number of relevant applications, e.g. control of unmanned autonomous vehicles, data fusion in sensor networks, and design of cooperating systems specialized for research, surveillance, monitoring and exploration tasks (see [7], [8], [9] and the references therein).

This paper deals with the so-called consensus problem. It consists, given a group of dynamically uncoupled agents connected by a communication network, in the determination of decentralized control strategies guaranteeing the convergence of agents’ state variables to a common value, called consensus point.

Most of the existing results deal with agents’ dynamics described by the single integrator model [10], [7], [11], [12], [13]. Recently, the extension to double integrator models has been considered. The control input is acceleration and the goal is to guarantee consensus on position and velocity. Such a problem is particularly important in vehicle coordination, where the agents obey to point-mass dynamics. Control schemes guaranteeing consensus for this class of systems were developed in [1] and [2], where the proof of consensus is based on spectral properties of Laplacian matrices (see also [14] for a generalization to the case of graphs with time-varying topology). The double-integrator dynamics was also considered in [15].

The aim of the present paper is to revisit the results in [1] and [2] for time-invariant bi-directional networks by means of a unified framework, based on the formalism of Partial difference Equations (PdEs) over graphs [16], [17]. PdEs mimic the classical Partial Differential Equations (PDEs) on spatial domains having a graph structure. Thanks to this analogy, the definition of appropriate control laws is supported by physical intuition. For instance, in [17] it has been shown that well-known consensus results based on linear control policies exploiting graph Laplacians [18] are in complete accordance with the theory of the heat equation. These results were generalized in [19] in order to account for communication delays. Another example is provided by the decentralized control schemes studied in this paper, that make the closed-loop system mimic the damped wave equation. This analogy is also exploited for choosing an energy function to prove consensus.

The paper is organized as follows: Section II describes the model of the communication network as well as some key results about PdEs. Section III contains the agents’ dynamic model, a formal definition of the consensus problem and the analysis of a control policy proposed in in [1]; moreover we show how to compute the consensus point in the case of non-zero initial velocities of the agents, which was not taken into account in [1]. Section IV is devoted to the proof of the main result on consensus. A variation of the control scheme based on the wave equation originally proposed in in [2] is discussed in Section V and a simulation example is presented in Section VI. The paper ends with some concluding remarks.

II. PRELIMINARIES ON PDEs OVER GRAPHS

We consider a group of $n$ agents communicating through a network described by an undirected, weighted, time-invariant graph $G = (\mathcal{N}, \mathcal{E}, \omega)$, where $\mathcal{N} = \{1, \ldots, n\}$ is the set of nodes, $\mathcal{E} \subseteq \{(i, j) : i, j \in \mathcal{N}, i \neq j\}$ is the set of edges and $\omega : \mathcal{N} \times \mathcal{N} \mapsto \mathbb{R}^+$ is a weight function verifying

$$
\begin{cases}
\omega(i, j) > 0 & \text{if } (i, j) \in \mathcal{E} \\
\omega(i, j) = 0 & \text{otherwise}
\end{cases}
$$

(1)
with \( \omega(i, j) = \omega(j, i), \forall i, j \in \mathcal{N} \). Nodes represent agents and two agents \( i, j \) are neighbors if \((i, j) \in \mathcal{E}\) or, with an alternative notation, \(i \sim j\). The neighboring relation means that agents \( i \) and \( j \) share measurements of their state. The function \( \omega(i, j) \) can be used to assign weights to the links between neighbors.

Two nodes \( i \) and \( j \) are connected by a path if there is a finite sequence of nodes \( \{\sigma_0 \sigma_1 \ldots \sigma_m\}, \sigma_0 = i, \sigma_m = j, \) with \( \sigma_r \sim \sigma_{r+1}, r = 0, \ldots, (m - 1) \). The graph \( G \) is connected if each pair of nodes \((i, j) \in \mathcal{N} \times \mathcal{N}\) is connected by a path, and complete if all pairs \((i, j)\) with \( i \neq j \) are in \( \mathcal{E} \).

Next we recall concepts of functional analysis developed by Bensoussan and Menaldi in [16] for functions \( f : \mathcal{N} \mapsto \mathbb{R} \) defined over a graph. As it will be apparent they play a central role in defining and analyzing PdEs. From now on, partial derivatives of \( f \) are defined as

\[
\partial_j f(i) = \frac{\partial}{\partial j} f(i)
\]

and enjoy the following elementary properties:

\[
\begin{align*}
\partial_j f(i) &= -\partial_i f(j) \\
\partial_i f(i) &= 0 \\
\partial^2 f(i) &= \partial_j f(j) - \partial_j f(i) = -\partial_j f(i) 
\end{align*}
\]

The weighted Laplacian of \( f \) is given by

\[
\Delta f(i) = -\sum_{j \sim i} \omega(j, i) \partial_j^2 f(i) = \sum_{j \sim i} \omega(j, i) \partial_j f(i)
\]

and the integral and the average of \( f \) are defined, respectively, as

\[
\int_G f = \sum_{i \in \mathcal{N}} f(i), \quad \langle f \rangle = \frac{1}{n} \int_G f
\]

Let \( L^2(G, \mathbb{R}^q) \) (or simply \( L^2 \) for short) be the Hilbert space composed by all functions \( f : \mathcal{N} \mapsto \mathbb{R}^q \) equipped with the scalar product and the norm

\[
(f, g)_{L^2} = \int_G f^T(\cdot)g(\cdot), \quad \|f\|_{L^2}^2 = \int_G \|f(\cdot)\|^2
\]

where \( \|\cdot\| \) is the Euclidean norm on \( \mathbb{R}^q \). We now introduce the space \( H^1(G, \mathbb{R}^q) \) composed by all functions in \( L^2(G, \mathbb{R}^q) \) with zero average. Correspondingly, the symbol \( H_0^1 \) will be used to denote the space of constant functions over \( \mathcal{N} \). This notation points out that the spaces \( H^1 \) and \( H_0^1 \) are \( L^2 \)-orthogonal, as proven in [16].

As shown in [16], if \( G \) is connected, \( H^1 \) is an Hilbert space endowed with the scalar product

\[
(f, g)_{H^1} = \sum_{i \in \mathcal{N}} \sum_{j \sim i} \omega(i, j)(\partial_j f(i))^T \partial_j g(i)
\]

Consider a subspace \( \mathcal{V} \subset L^2 \) and let \( \mathcal{V}_\perp \) be its \( L^2 \)-orthogonal complement. Let \( P_\mathcal{V} : L^2 \mapsto \mathcal{V} \) be the \( L^2 \)-orthogonal projection operator on \( \mathcal{V} \) and define \( f_\mathcal{V} = P_\mathcal{V} f \). One can easily verify that

1. \( f \in L^2 \) and \( \int_G f^T c = 0, \forall c \in \mathcal{V}_\perp \Rightarrow f \in \mathcal{V} \);
2. \( f, g \in L^2 \) and \( \int_G f^T c = \int_G g^T c, \forall c \in \mathcal{V} \Rightarrow f_\mathcal{V} = g_\mathcal{V} \).

The next Theorem summarizes fundamental properties of the Laplacian operator.

**Theorem 1** Let \( G \) be a connected graph. Then,

1. the operator \( \Delta : H^1 \mapsto H^1 \) is symmetric (i.e. \( (f, \Delta g)_{L^2} = (\Delta f, g)_{L^2} \)) has \((N - 1)q \) strictly negative eigenvalues and the corresponding eigenfunctions form a basis for \( H^1 \);
2. for \( f \in L^2 \), \( \Delta f = 0 \) if and only if \( f \in H_0^1 \).

**Remark 1** Note that the space \( L^2 \) is finite dimensional and isomorphic to \( \mathbb{R}^{n^2} \). This means that functions over graphs and linear operators on \( L^2 \) can be represented in terms of vectors and matrices, respectively. In particular, if \( L(G) \) is the Laplacian matrix of the graph \( G \) (see [20] for a definition) then, as shown in [17] and [19], the operator \( \Delta \) provides an alternative representation of the matrix \(-L(G) \otimes I_q\), where \( \otimes \) is the Kronecker product and \( I_q \) is the identity matrix of order \( q \).

Based on the previous definitions, we can introduce Partial Difference Equations (PdEs) over graphs. Given a function of two variables \( z(i, t) : \mathcal{N} \times \mathbb{R}^+ \mapsto \mathbb{R}^q \), consider the initial value problem

\[
\begin{align*}
\dot{z}(\cdot, t) &= F(z(\cdot, t)) \\
z(\cdot, 0) &= z_0(\cdot)
\end{align*}
\]

where \( F : L^2(G, \mathbb{R}^q) \mapsto L^2(G, \mathbb{R}^q) \) is a continuous operator and \( z_0(\cdot) \in L^2 \). In the sequel, we will assume that \( F(0) = 0 \). In (8a) the symbol \( \dot{z}(i, t) \) denotes the partial derivative with respect to time, i.e. \( \dot{z}(i, t) = \frac{\partial z(i, t)}{\partial t} \).

We call the equality (8a) a continuous-time PdE with initial conditions (8b) and refer to \( z(i, t) \) as the state of the PdE. For this system, which can be recast into an equivalent system of ODEs [17], it is possible to formulate a definition of stability of the equilibria. In particular, for analyzing consensus properties we are interested in the effect of perturbations on the projection of \( z(i, t) \) on suitable subspaces.

**Definition 1** The origin of (8a) is Globally Asymptotically Stable (GAS) on \( \mathcal{V} \) if for all \( \tilde{z}(\cdot) \in L^2 \) the following conditions simultaneously hold:

- **Stability on \( \mathcal{V} \)**: \( \forall \epsilon > 0, \exists \delta > 0 : \|\tilde{z}(\cdot)\|_{L^2} \leq \delta \Rightarrow \|\tilde{z}(\cdot, t)\|_{L^2} \leq \epsilon, \forall t \geq 0 \).
- **Global attractivity on \( \mathcal{V} \)**: \( \forall \epsilon > 0, \forall \tau > 0, \exists T > 0 \) such that \( \|\tilde{z}(\cdot)\|_{L^2} \leq \tau \Rightarrow \|\tilde{z}(\cdot, t)\|_{L^2} \leq \epsilon, \forall t \geq T \).

We conclude this Section by stating a LaSalle-Krasowski theorem for checking when the origin of (8a) is GAS. Recall that a functional \( W : \mathcal{V} \mapsto \mathbb{R} \) is

1. positive definite if \( W(0) = 0 \) and \((v \in \mathcal{V}, v \neq 0) \Rightarrow W(v) > 0 \);
2. radially unbounded if \( W(v) \to +\infty \) as \( \|v\|_{L^2} \to +\infty \).

We also say that a set \( \Omega \subset \mathcal{V} \) is positively \( \mathcal{V} \)-invariant with respect to (8a) if \( \tilde{z}(\cdot) \in \Omega \Rightarrow z(\cdot, t) \in \Omega, \forall t \geq 0 \).

\(^1\)These eigenvalues will be called "the eigenvalues of \( \Delta \) on \( H^1 \)."
Theorem 2 (LaSalle-Krasowski) Assume that there exists a unique solution to (8) and that $P_{F} = FP_{F}$. Let $W : \mathcal{V} \rightarrow \mathbb{R}$ be a continuously differentiable, radialy unbounded, positive definite functional such that $W(z_{\mathcal{V}}(\cdot, t)) \leq 0, \forall z_{\mathcal{V}} \in \mathcal{V}$. Let $E = \{v \in \mathcal{V} : \dot{W}(v) = 0\}$ and assume that the largest positive invariant set in $E$ is $M = \{0\}$. Then the origin of (8a) is GAS on $\mathcal{V}$.

Proof: We first summarize some key properties of the level sets $\Omega_{\beta}$, $\{v \in \mathcal{V} : \|v\|_{L^{2}} \leq \varepsilon\}$. It is possible to show that, under the assumptions of the Theorem, $\exists \beta > 0$ such that $\Omega_{\beta}$ is in the interior of $B_{r}$. The positive definiteness of $W$ also guarantees that there exists $\delta > 0$ such that $B_{\delta} \subset \Omega_{\beta} \subset B_{r}$. Then one has, $\forall \varepsilon > 0$,

$$z_{\mathcal{V}}(\cdot, t) \in B_{\delta} \Rightarrow \tilde{z}_{\mathcal{V}}(\cdot, t) \in \Omega_{\beta} \Rightarrow z_{\mathcal{V}}(\cdot, t) \in \Omega_{\beta} \Rightarrow z_{\mathcal{V}}(\cdot, t) \in B_{r}$$

where the second implication follows from the invariance of $\Omega_{\beta}$. Formula (10) shows that the origin is stable on $\mathcal{V}$.

The next step is to prove global attractivity on $\mathcal{V}$ of the origin. Note that each $v \in \mathcal{V}$ is contained in the compact, positive $\mathcal{V}$-invariant set $\Omega$, with $r = W(v)$. Thanks to the assumptions that (8a) has a unique solution and that $P_{F} = FP_{F}$, it is possible to apply the LaSalle principle stated by Theorem 3 in [17]. More precisely, using the notation of [17], we set $\Omega = \Omega_{\varepsilon}$, and since the largest positive $\mathcal{V}$-invariant set in $E$ is $M = \{0\}$, one has:

$$\lim_{t \rightarrow +\infty} \|z_{\mathcal{V}}(\cdot, t)\|_{L^{2}} = 0$$

Formula (11) proves the global attractivity on $\mathcal{V}$ of the origin.

III. PROBLEM FORMULATION

Let $x(i, t) \in \mathbb{R}^{q}$, $v(i, t) \in \mathbb{R}^{q}$ and $u(i, t) \in \mathbb{R}^{q}$ denote the position, velocity and control input of agent $i \in \mathcal{N}$ at time $t \in \mathbb{R}$. Agents with double-integrator dynamics yield the collective model

$$\dot{x}(\cdot, t) = v(\cdot, t)$$
$$\dot{v}(\cdot, t) = u(\cdot, t)$$

with initial conditions $x(\cdot, 0) = \bar{x}(\cdot) \in L^{2}$, $v(\cdot, 0) = \bar{v}(\cdot) \in L^{2}$. We consider the effect of the following feedback control law, originally proposed in [1]:

$$u(i, t) = \Delta x(i, t) - \gamma v(i, t), \gamma > 0$$

that is decentralized, in the sense that $u(i, t)$ only depends on the state of agent $i$ and the position of agents $j \sim i$ at time $t$.

Using (13), the closed-loop collective model is given by the PdE

$$\dot{x}(\cdot, t) = v(\cdot, t)$$
$$\dot{v}(\cdot, t) = \Delta x(\cdot, t) - \gamma v(\cdot, t)$$

with state $z(\cdot, t) = [x(\cdot, t)^{T}v(\cdot, t)]^{T}$ and initial conditions $x(\cdot, 0) = \bar{x}(\cdot) \in L^{2}$, $v(\cdot, 0) = \bar{v}(\cdot) \in L^{2}$.

Our main goal will be to prove that (13) guarantees the achievement of consensus, in the sense of the following definition.

Definition 2 The multi-agent system (14) achieves GAS consensus if the origin $z = 0$ is GAS on $H^{1}(G|\mathbb{R}^{2q})$.

The interpretation of such a definition is as follows. Consider the decomposition of state variables

$$x = x_{1} + \bar{x}$$
$$v = v_{1} + \bar{v}$$

where $x_{1}, v_{1} \in H^{1}(G|\mathbb{R}^{q})$ and $\bar{x}, \bar{v} \in H^{1}(G|\mathbb{R}^{q})$. Obviously, $\bar{x}(i, t) = \langle x(i, t) \rangle$, and $\bar{v}(i, t) = \langle v(i, t) \rangle$ with $\forall i \in \mathcal{N}, \forall t \geq 0$. GAS consensus implies that both $x_{1}$ and $v_{1}$ vanish as $t \rightarrow +\infty$ and therefore, asymptotically, all agents will have the total value of $c$.

Notice that system (14) can also be rewritten in the form:

$$\ddot{x}(i, t) = \Delta x(i, t) - \gamma \dot{x}(i, t), \gamma > 0$$

This equation mirrors the classical damped wave equation.

In order to study consensus properties of (14) we first characterize the evolution of the $H^{1}$ and $H^{1}_{\perp}$ components of the state.

Lemma 1 The functions $x, v$ are solutions to the PdE (14) if and only if $x_{1}, v_{1}$ and $\bar{x}, \bar{v}$ are solutions to the PdEs

$$\Sigma^{1} : \left\{ \begin{array}{l}
\dot{x}_{1} = v_{1} \\
\dot{v}_{1} = \Delta x_{1} - \gamma v_{1}
\end{array} \right\} \Sigma : \left\{ \begin{array}{l}
\dddot{x} = \bar{v} \\
\dddot{\bar{v}} = -\gamma \dot{\bar{v}}
\end{array} \right\}
$$

with initial conditions $x_{1}(\cdot, 0) = P_{H^{1}}\bar{x}(\cdot), \bar{x}(\cdot, 0) = \langle \bar{x}(\cdot) \rangle$, $v_{1}(\cdot, 0) = P_{H^{1}}\bar{v}(\cdot)$ and $\bar{v}(\cdot, 0) = \langle \bar{v}(\cdot) \rangle$.

Proof: By means of the decomposition (15) we obtain

$$\int_{G} (\dot{x}(\cdot, t))^{T} c = \int_{G} (\dot{v}(\cdot, t))^{T} c + \int_{G} (v_{1}(\cdot, t))^{T} c$$
where the second term on the right-hand side is zero because of the orthogonality between \(v_1(i, t) \in H^1\) and \(c \in H^1\). It is thus possible to conclude that
\[
\dot{x} = \bar{v}.
\]  
(17)

Applying the same procedure to equation (14b) we obtain:
\[
\int_G (\dot{v}(i, t))^T c = \int_G ((\Delta \bar{x}(i, t))^T + \int_G (\Delta x_1(i, t))^T c - \gamma \int_G v_1^T(i, t)c - \gamma \int_G \bar{v}^T(i, t)c.
\]  
(18)

where the only non-null term in the right-hand side is \(-\gamma \int_G \bar{v}^T(i, t)c\). One thus obtains:
\[
\dot{v} = -\gamma \bar{v}.
\]  
(19)

Substituting (17) in (14a) one has: \(\dot{x}_1 = v_1\). From (14b) one also gets \(\dot{v}_1 = \Delta x_1 - \gamma v_1\), that completes the proof.

Notice that from (19) it easily follows that \(\lim_{t \to +\infty} \bar{v}(i, t) = 0\) and then
\[
\lim_{t \to +\infty} \dot{x}(i, t) = 0.
\]  
(20)

This means that there exists a constant vector \(x^* \in \mathbb{R}^q\) such that \(\lim_{t \to +\infty} \bar{x}(i, t) = x^*, \forall i \in N\).

IV. MAIN RESULT

We are now in a position to state the main result showing that the control law (13) ensures that all the agents’ positions converge to a common value and their velocities tend to zero. For the proof of consensus, we must show that
\[
\lim_{t \to +\infty} x_1(i, t) = \lim_{t \to +\infty} v_1(i, t) = 0
\] as it is rigorously stated in the following Theorem.

**Theorem 3** If \(G\) is connected, the multi-agent system (14) achieves GAS consensus. Moreover \(\lim_{t \to +\infty} v(i, t) = 0\).

**Proof:** The largest part of the proof will be devoted to checking the assumptions of Theorem 2. First note that (14) is a linear PDE and then it has a unique solution for any given initial condition. Consider the subspace \(V = H^1(G; \mathbb{R}^q)\). Writing (14) in the form (8) one has
\[
F(z) = F(x, v) = \begin{bmatrix} v \\ \Delta x - \gamma v \end{bmatrix}
\]
and since \(\Delta x = \Delta x_1 + \Delta \bar{x} = \Delta x_1\) the condition \(P_\gamma F = F P_\gamma\) holds.

We consider the following energy function on \(V\):
\[
W(x_1, v_1) = \frac{1}{2} \|v_1(i, t)\|^2 \|\Delta x_1(i, t)\|^2 - \frac{1}{2} \int_G (^\Delta x_1(i, t))^T x_1(i, t)
\]  
(21)

Now we show that there exists a constant \(k > 0\) such that
\[
W(x_1, v_1) \geq k \|x_1(i, t)\|^2 \|v_1(i, t)\|^2 \|\Delta x_1\|^2
\]  
(22)

From Theorem 1, the Laplacian operator on \(H^1\) has all negative eigenvalues. Let \(\lambda_{max}\) be the maximal eigenvalue of \(\Delta\) on \(H^1(G; \mathbb{R}^q)\) and define \(k = \min \left(\frac{1}{2}, \frac{-\lambda_{max}}{\gamma} \right)\). Note that \(\lambda_{max} < 0\). Then, one has
\[
W(x_1, v_1) \geq \frac{1}{2} \|v_1(i, t)\|^2 - \frac{1}{2} \lambda_{max} \|x_1(i, t)\|^2 \geq
\]
\[
\geq k \left(\|x_1(i, t)\|^2 + \|v_1(i, t)\|^2 \|\Delta x_1\|^2\right) = k \left(\|x_1(i, t)\|^2 \|v_1(i, t)\|^2 \|\Delta x_1\|^2\right)
\]  
(23)

Inequality (22), together with the fact that \(W(0, 0) = 0\), implies that \(W\) is positive definite and radially unbounded. Observing that \(\frac{\partial (\Delta x_1(i, t))}{\partial \bar{t}} = \Delta x_1(i, t)\), and using the definition of vector product in (6), one has
\[
\dot{W}(x_1, v_1) = \int_G \dot{v}_1(i, t) v_1(i, t) - \frac{1}{2} \int_G (v_1(i, t))^T \Delta x_1(i, t)
\]
\[
- \frac{1}{2} \int_G x_1(i, t)^T \Delta x_1(i, t)
\]  
(24)

From the symmetry of the Laplacian operator, it holds \((x_1(i, t), \Delta v_1(i, t)) = (v_1(i, t), \Delta x_1(i, t))\) and hence
\[
\dot{W}(x_1, v_1) = \int_G \dot{v}_1(i, t)^T v_1(i, t) - \int_G (v_1(i, t))^T \Delta x_1(i, t)
\]  
(25)

Hence \(W(x_1, v_1) \leq 0\) on \(H^1\), and the set \(E\) where \(\dot{W}(x_1, v_1) = 0\), defined in Theorem 2, is given by \(E = \{(x_1, v_1) : v_1 = 0\}\). From \(\Sigma^1\) one has that, if \((x_1, v_1) \in E, \forall t \geq 0\), then \(\dot{x}_1 = 0\) and \(v_1 = \Delta x_1\). Moreover, the condition \(v_1 = 0, \forall t \geq 0\), implies that \(\Delta x_1 = 0\) and hence \(x_1 = 0\), because \(x_1 \in H^1\). It follows that the only state trajectory that can stay in \(E\) is \((x_1, v_1) = (0, 0)\). The final result is a direct application of Theorem 2.

**Remark 2** Note that \(\bar{x}\) in Lemma 1 is a second-order system. A straightforward calculation reveals that the consensus is related to the initial states by the formula
\[
x^* = \lim_{t \to +\infty} \bar{x}(t) = \frac{1}{\gamma} (\bar{v}(\cdot)) + \bar{x}(\cdot)
\]  
(26)

Hence, when \(\langle \bar{v}(\cdot) \rangle = 0\) we have average-consensus. This observation is in agreement with [1], where average-consensus has been proven when \(\bar{v}(\cdot) = 0\). Secondly, if \(q = 1\), given the initial conditions \(\bar{v}(\cdot)\) and \(\bar{x}(\cdot)\) one can enforce convergence to a desired consensus point \(x^*\) by selecting \(\gamma\) as:
\[
\gamma = \frac{\langle \bar{v}(\cdot) \rangle}{x^* - \bar{x}(\cdot)}
\]

**Remark 3** Proof of Theorem 3 critically depends on the choice of the energy function (21), that has been determined.
in analogy with the energy of a vibrating string. It therefore consists in the sum of a kinetic term, \( \frac{1}{2} \| v_1(\cdot, t) \|_{L^2}^2 \), and an elastic potential term, \(-\frac{1}{2} \int_G (\Delta x_1(i, t))^T x_1(i, t)\).

V. Generalization

In this section, we propose a slight modification of the control law (13) that guarantees consensus with a non-zero asymptotic velocity. To this aim consider the control law

\[
  u(i, t) = \Delta x(i, t) + \gamma \Delta v(i, t), \quad \gamma > 0
\]

where the damping is proportional to the Laplacian of the velocity. This control policy has been first considered in [2]. In the PDE context this type of equation is often called strongly damped wave equation, see [21]. In this case the closed-loop system dynamics is:

\[
  \dot{x}(\cdot, t) = v(\cdot, t) \tag{28a}
\]

\[
  \dot{v}(\cdot, t) = \Delta x(\cdot, t) + \gamma \Delta v(\cdot, t) \tag{28b}
\]

with state \( z(\cdot, t) = [x(\cdot, t)^T v(\cdot, t)]^T \) and initial conditions \( x(\cdot, 0) = \tilde{x}(\cdot) \in L^2, v(\cdot, 0) = \tilde{v}(\cdot) \in L^2 \). By using the same rationale adopted for proving Lemma 1, we can formulate the following result:

**Lemma 2** The functions \( x, v \) are solutions to the PDE system (28) if and only if \( x_1, v_1 \) and \( \tilde{x}, \tilde{v} \) are solutions to the PDEs

\[
  \Sigma_1 : \begin{cases} 
    \dot{x}_1 = v_1 \\
    \dot{v}_1 = \Delta x_1 + \gamma \Delta v_1
  \end{cases} \quad \Sigma_1 : \begin{cases} 
    \dot{\tilde{x}} = \tilde{v} \\
    \dot{\tilde{v}} = 0
  \end{cases}
\]

**Proof:** As in the proof of Lemma 1, testing (28a) against \( c \in H_1^{\perp} \) yields to equation (17). By applying the same procedure to equation (28b), we obtain

\[
  \int_G (\dot{v}(\cdot, t))^T c = \int_G (\Delta \tilde{v}(\cdot, t))^T c + \int_G (\Delta x_1(i, t))^T c 
  + \gamma \int_G (\Delta \tilde{v}(i, t))^T c + \gamma \int_G (\Delta v_1(i, t))^T c = 0
\]

(29)

It therefore results that \( \dot{\tilde{v}}(i, t) = 0 \). Substituting (17) in (28a) yields \( \dot{x}_1 = v_1 \) and substituting \( \dot{\tilde{v}} = 0 \) in (28b) yields \( \dot{\tilde{v}}_1 = \Delta x_1 + \gamma \Delta v_1 \).

The next Theorem shows that also the control law (27) guarantees consensus. However, from \( \Sigma_1 \) one obtains \( \tilde{v}(\cdot) = \langle \tilde{v}(\cdot) \rangle, \forall t \geq 0 \) and \( \tilde{x}(\cdot) = \langle \tilde{x}(\cdot) \rangle + (\tilde{x}(\cdot)) \), and then the agents asymptotically achieve consensus while jointly drifting with constant velocity.

**Theorem 4** If \( G \) is connected, the multi-agent system (28) achieves GAS consensus. Moreover, \( \lim_{t \to +\infty} v(\cdot, t) = \langle \tilde{v}(\cdot) \rangle \).

**Proof:** The PDE (28a) is linear and then has a unique solution for given initial conditions. Let \( V = H^1(G[\mathbb{R}^2]^q) \). Writing (28a), (28b) in form (8) one has

\[
  F(z) = F(x, v) = \begin{bmatrix} v \\
  \Delta x + \gamma \Delta v \end{bmatrix}
\]

and since \( \Delta x = \Delta x_1 + \Delta \tilde{x} = \Delta x_1, \Delta v = \Delta v_1 + \Delta \tilde{v} = \Delta v_1 \) the condition \( P_V F = FP_V \) is fulfilled.

We consider the energy \( W(x_1, v_1) \) defined in (21). As shown in the proof of Theorem 3, \( W \) is positive definite and radially unbounded on \( V \). Moreover, from (24) and the symmetry of the Laplacian operator one has

\[
  \dot{W}(x_1, v_1) = \int_G (\Delta x_1(\cdot, t) + \gamma \Delta v_1(\cdot, t))^T v_1(\cdot, t) 
  - \int_G v_1(\cdot, t)^T \Delta x_1(\cdot, t) 
  = \gamma \int_G v_1(\cdot, t)^T \Delta v_1(\cdot, t) \leq \gamma \lambda_{\text{max}} \| v_1(\cdot, t) \|_{L^2}^2
\]

(30)

where \( \lambda_{\text{max}} \) is the maximum eigenvalue of the Laplacian operator on \( H^1(G[\mathbb{R}^q]) \) and verifies \( \lambda_{\text{max}} < 0 \). The remainder of the proof is analogous to the part of the proof of Theorem 3 starting after formula (25).

VI. A Simulation Example

In this Section we illustrate the results provided by Theorem 3 through a simulation example. We consider a network of five agents moving in a two-dimensional space \((i.e., q = 2)\) and governed by the decentralized control law (13). The communication network is represented by the undirected graph given in Fig. 1, which is unweighted \((i.e., if (i, j) \in \mathcal{E}, then \omega(i, j) = 1)\).

![Fig. 1. The communication graph of the multi-agent system considered in Section VI.](image)

The initial conditions of the agents are selected as follows:

<table>
<thead>
<tr>
<th>i</th>
<th>( \tilde{x}(i) )</th>
<th>( \tilde{v}(i) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>[10 10]</td>
<td>[5 5]</td>
</tr>
<tr>
<td>2</td>
<td>[-10 10]</td>
<td>[-5 15]</td>
</tr>
<tr>
<td>3</td>
<td>[-10 -10]</td>
<td>[5 -10]</td>
</tr>
<tr>
<td>4</td>
<td>[10 -10]</td>
<td>[-5 5]</td>
</tr>
<tr>
<td>5</td>
<td>[-5 0]</td>
<td>[-10 0]</td>
</tr>
</tbody>
</table>

By letting \( \gamma = 1 \), from (26) we expect all the agents to approach the position \( x^* = [-3, 3]^T \) with asymptotically zero velocity. This is confirmed by the plots given in Fig. 2 and Fig. 3.

VII. Conclusions

In this paper we proposed two decentralized control schemes inspired to the damped wave equation for guaranteeing consensus in multi-agent systems where agents obey to a point-mass dynamics. We assumed that the communication network is described by a weighted, undirected and time-invariant graph, and investigated global stability of consensus by exploiting the framework of PDEs. Future research will
focus on the application of similar control laws to multi-agent systems with nonholonomic constraints, as in the case of wheeled vehicles.

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