Analysis of Coordination in Multi-Agent Systems through Partial Difference Equations

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Abstract—In this paper we introduce the framework of Partial difference Equations (PdEs) over graphs for analyzing the behavior of multi-agent systems equipped with decentralized control schemes. Both leaderless and leader-follower models are considered. PdEs mimic Partial Differential Equations (PDEs) on graphs and can be studied by introducing concepts of functional analysis strongly inspired to the corresponding ones arising in PDEs theory. We generalize different models proposed in the literature by introducing errors in the agent dynamics and analyze agent coordination through the joint use of PdEs and automatic control tools. Moreover, for the simplest control schemes, we show that the resulting PdEs enjoy properties that are similar to those of well-known PDEs like the heat equation, thus allowing to exploit physical-based reasoning for conjecturing formation properties.

I. INTRODUCTION

In the last few years, the problem of understanding when the individual actions of interacting agents give rise to a coordinated behavior has received a considerable attention in many fields. For instance, this issue appears in biology, statistical physics and computer graphics. For a thorough review of the literature in various fields, we defer the interested reader to [1] and [2].

In the control community, the interest in coordination phenomena has been recently promoted by the need of controlling groups of unmanned autonomous vehicles, like airplanes or robots [3]. A fairly simplified setup is to consider a group of $N$ mobile agents, each one described by a dynamical system capturing the evolution of its heading angle [1] or its position and velocity [4]. Different agents share information through a communication network: agents connected by a communication link are neighbors and position and velocity of each one is instantaneously available to the others for regulating their own trajectory. When agents communicate with a limited number of neighbors, one faces the problem of designing a decentralized control scheme (where each agent uses only the neighbors information) in order to orchestrate the collective behavior. Decentralization implies that the control action can be computed in a distributed fashion.

The main purpose of this paper is to propose a new modeling framework for the analysis of multi-agent systems. Our approach exploits the formalism of Partial difference Equations (PdEs) over graphs proposed by [5] and summarized in Section II. Conceptually, PdEs mimic PDEs (Partial Differential Equations) in spatial domains having a graph structure and, in [5], the basic mathematical framework for static problems of elliptic type is provided. In order to account for the temporal dynamics of the agents, we generalize the models described in [5] to continuous-time PdEs. One major difference between PDEs and PdEs is that the latter can be recast into systems of Ordinary Differential Equations (ODEs). However, we argue that PdE models can be more expressive than their equivalent ODE form, for many reasons. First, many mathematical tools for analyzing PdEs are completely analogous to the ones developed for PDEs. Then, the PdEs formalism establishes a direct link between classic functional analysis and control theory that can be fruitfully exploited for studying systems linked by a communication network. Second, PdEs provide a mathematical description of the collective dynamics where spatial phenomena (due to the graph structure) and temporal evolution of the agent states are kept separated and described through operators acting either on space or time. Third, the PdEs framework leads to equations that may be reminiscent of PDEs arising in physics and this can be of great help for conjecturing sensible properties of the collective dynamics.

Along the paper, we consider communication networks with a time-invariant topology and agent obeying to a point-mass dynamics affected by deterministic, exponentially decreasing errors. As recalled in Section V, this error model arises when agents are equipped with internal feedback schemes for counteracting the effects of constant perturbations. We focus on the use of two decentralized control laws both for leaderless and leader-follower schemes. The first one is the “Laplacian control”, a linear control strategy inspired to the one proposed in [6] and analyzed in [7] and [8]. The second one is the “elastic control”, a potential field based control law similar to the one proposed in [4], [2], the only difference being that nonzero safety distances between pairs of communicating agents are allowed.

The goal is to assess whether these control schemes are capable to guarantee alignment, i.e. (i) for leaderless models, all agents asymptotically move with the same velocity, (ii) for leader-follower models all agents asymptotically move with the leader's velocity. For elastic control, we also discuss the properties of collision avoidance (i.e. at each time instant, the distance between two communicating agents does not fall below a safety threshold) and cohesion (i.e. the distance between pairs of communicating agents asymptotically converges to a given setpoint).

We stress that the properties of Laplacian control for errorless models have been already studied in [1], [7] and [8]. Similarly, alignment, cohesion and collision avoidance for the elastic control in errorless models have been studied in [4], [9], [2] both for time-invariant and time-varying communication links.

However, we highlight that the main focus of the present paper is not on the differences between the models we consider and similar ones that have appeared in the literature, but rather on showing how PdEs provide a framework in which different agent models and control laws can be analysed with similar techniques. The PdE models for leaderless multi-agent systems are presented in Section III. Section IV is devoted to the analysis of leaderless models. Leader-follower models are introduced and analyzed in Section VI.

II. PARTIAL DIFFERENCE EQUATIONS AND STABILITY THEOREMS

We introduce basic notions of graph theory, deferring the reader to [10] for details. Let $G$ be an undirected graph defined by a nonempty set $\mathcal{N} = \{1, 2, \ldots, N\}$ of nodes and a set $\mathcal{E} \subset \mathcal{N} \times \mathcal{N}$ of edges. In our case, each node will represent an agent. Two nodes $x$ and $y$ are neighbors if $(x, y) \in \mathcal{E}$. This means that the agent $x$ and $y$ share the information about their state. We use the notation $x \sim y$ for neighboring nodes and assume that $x \sim x$ always holds.

Two nodes $x$ and $y$ are connected by a path if there is a finite sequence $x_0 = x, x_1, \ldots, x_n = y$ such that $x_{i-1} \sim x_i$. The graph $G$ is connected when each pair of nodes $(x, y) \in \mathcal{N} \times \mathcal{N}$ is connected by a path.
Let $S$ be a nonempty, connected subgraph of the connected graph $G$ and let $\mathcal{S}$ be the set of nodes of $S$. The boundary of $S$ is defined as $\partial S = \{ y \in G \setminus S : \exists x \in S : x \sim y \}$. Hereafter, we suppose for simplicity that $\mathcal{N} = \mathcal{S} \cup \partial S$.

Next, we summarize the main concepts of functional analysis for functions $f : \mathcal{N} \to \mathbb{R}^q$ defined over a graph $G$ by following closely the exposition of [5], where scalar functions are considered. The partial derivative of $f$ is defined as

$$\partial_y f(x) = f(y) - f(x)$$  (1)

and enjoys the following elementary properties: $\partial_y f(x) = -\partial_x f(y)$, $\partial_x f(x) = 0$ and $\partial^2_x f(x) = -\partial_y f(x)$. The Laplacian of $f$ is given by

$$\Delta f(x) = -\sum_{y \sim x} \partial_y^2 f(x) + \sum_{y \sim x} \partial_x f(x).$$  (2)

The integral and the average of $f$ are defined, respectively, as

$$\int_G f \, d\mathcal{N} = \sum_{x \in \mathcal{N}} \int_{G(x)} f \, d\mathcal{N}, \quad \langle f \rangle = \frac{1}{N} \int_G f \, d\mathcal{N}.$$  (3)

Let $L^2(G) \subseteq \mathbb{R}^q$ be the Hilbert space composed by all functions $f : \mathcal{N} \to \mathbb{R}^q$ equipped with the scalar product and the norm

$$\langle f, g \rangle_{L^2} = \int_G f \, g \, d\mathcal{N}, \quad \| f \|_{L^2} = \int_G f^2 \, d\mathcal{N}.$$  (4)

where $\| \cdot \|$ is the euclidean norm on $\mathbb{R}^q$. Note that $L^2(G) \subseteq \mathbb{R}^q$ is isomorphic to $\mathbb{R}^{Nq}$. We introduce now two subspaces of $L^2(G) \subseteq \mathbb{R}^q$ which will be relevant in our analysis. They are defined as

$$H^1(G) = \{ f \in L^2(G) \subseteq \mathbb{R}^q : f(0) = 0 \}$$  (5)

and

$$H^1_0(G) = \{ f \in L^2(G) \subseteq \mathbb{R}^q : f(x) = 0 \, \forall x \in \partial S \}. \quad (6)$$

We will use the shorthand notation $L^2$, $H^1$, and $H^1_0$ when no ambiguity is possible. If $G$ is connected, both $H^1$ and $H^1_0$ are Hilbert spaces endowed with the norm [5]:

$$\| f \|_H^2 = \sum_{x \in \mathcal{N}} \sum_{y \sim x} \| \partial_y f(x) \|^2.$$  (7)

In other words, if $G$ is connected, for any $f \in L^2$ one has that $\| f \|_H = 0$ if and only if $f$ is constant. Let $H^1_2(G) \subseteq H^1_0(G)$ denote the space orthogonal to $H^1_0(G)$. Apparently, $H^1_2$ is the space of constant functions on $G$ and $\dim H^1_2 = q$. Moreover, the decomposition $L^2 = H^1 \oplus H^1_2$ is $L^2-$orthogonal, i.e., $\int_G f^2 \, d\mathcal{N} = 0, \forall f \in H^1$. Hence, the next theorem clarifies the eigenstructure of the Laplacian operator [10], [5].

**Theorem 1** Let $G$ be a connected graph. Then,

1) the operators $\Delta : H^1 \to H^1$ and $\Delta : H^1_0 \to H^1$ have both strictly negative eigenvalues and the corresponding eigenfunctions form a basis for $H^1$ and $H^1_0$, respectively.

2) for $f \in L^2$, $\Delta f = 0$ if and only if $f \in H^1_2$.

Note that when $\Delta$ is defined on $L^2$, it has $Nq$ eigenvalues. In particular, in view of the decomposition $L^2 = H^1 \oplus H^1_2$, $(N - 1)q$ eigenvalues are those considered in point (1) of Theorem 1 and the remaining $q$ eigenvalues are zeros (this follows directly from point (2) of Theorem 1). Since $\Delta$ is a linear and finite-dimensional operator, its eigenvalues can be easily computed. As shown in [11], they are directly related to those of the Laplacian matrix of the graph $G$. In [11] it is also shown that Theorem 1 is in complete accordance with the well-known spectral theory for Laplace operator on Sobolev spaces.

We are now in a position to introduce continuous-time Partial difference Equations (PdEs) on graphs. Let $z(x, t) : \mathcal{N} \times \mathbb{R}^q \to \mathbb{R}^q$ be a function of two variables and consider the initial value problem

$$\dot{z}(x, t) = F(z(x, t)) \quad (8a)$$

$$z(x, 0) = \bar{z}(x) \quad (8b)$$

where $F : L^2 \to L^2$ is a continuous, locally Lipschitz operator. Note that, for example, one can have $F = \Delta$ thus motivating the term “PdE” used for (8). Since the space $L^2$ is finite-dimensional, it is also easy to show that (8) is equivalent to a (possibly nonlinear) system of order $Nq$ (see [11] for details). In the sequel we assume that there exists a unique function $z$ verifying (8) for $t \in [0, +\infty[$.

Next, we provide tools for analyzing the effect of perturbations on the projection of $z$ on suitable subspaces. Assume that $\bar{F}(0) = 0$ and consider a subspace $V \subseteq L^2(G) \subseteq \mathbb{R}^q$. We denote by $\bar{f}_V = P_V f$ the projection of $f \in L^2$ on $V$.

**Definition 1** The origin of (8) is stable on $V$ if for all $t \geq 0$

$$\forall \varepsilon > 0, \exists \delta > 0 : \| z_V \|_{L^2} < \delta \implies \| z_V(t, \cdot) \|_{L^2} \leq \varepsilon.$$

If, in addition, there exists $k > 0, \eta > 0$ such that $\forall \varepsilon \in L^2, \forall t \geq 0$ it holds

$$\| z_V(t, \cdot) \|_{L^2} \leq k e^{-\eta t} \| z_V \|_{L^2}$$

then, the origin is globally exponentially stable on $V$.

Note that if $V = L^2$, stability on $V$ coincides with the standard notion of stability of the origin [12]. The next Theorem, that is a straightforward generalization of the second method of Lyapunov (see [12]), can be used for checking exponential stability of the origin on $V$.

**Theorem 2** Assume that there exists a unique solution $z(x, t)$ to (8), $\forall \varepsilon \in L^2, \forall t \geq 0$. If there exist a continuously differentiable functional $W : V \to \mathbb{R}$ and constants $k_1, k_2, k_3, \alpha > 0$ such that

$$k_1 \| \varepsilon \|_{L^2}^2 \leq W(\varepsilon) \leq k_2 \| \varepsilon \|_{L^2}^2, \forall \varepsilon \in V$$

$$W(\varepsilon) \leq -k_3 \| \varepsilon \|_{L^2}^2$$

then, the origin of (8) is globally exponentially stable on $V$.

Next, we introduce the Lasalle invariance principle on subspaces.

**Definition 2** A set $\Omega \subseteq V$ is positively $V$-invariant with respect to (8a) if $\exists \varepsilon \in \Omega \implies \varepsilon(\cdot, t) \in \Omega, \forall t \geq 0$

**Theorem 3** Assume that there exists a unique solution to (8), $\forall t \geq 0$ and that $P_V F = P_V \bar{F}$. Let $\Omega \subseteq V$ be a positively $V$-invariant compact set in $L^2$ and let $W : V \to \mathbb{R}$ be a continuously differentiable functional verifying $W(\varepsilon) \leq 0$. Consider the set $E = \{ v \in V : W(v) = 0 \}$. Then, for every initial condition verifying $\exists \varepsilon \in \Omega$, the projected solution $\bar{z}_V$ approaches $E$, i.e.

$$\lim_{t \to \infty} \inf_{v \in E} \| \bar{z}_V(\cdot, t) - P_V \bar{z}(\cdot) \|_{L^2} = 0.$$
III. The Collective Dynamics

The communication network between agents is modeled in form of a connected graph \( G \). Let \( r(x,t) \) be the position of the agent \( x \) at time \( t \), where\(^1 \) \( r \in \mathbb{R}^2 \) and \( q \) is the dimension of the physical space. Similarly, the agent velocity, input and errors are denoted with \( v(x,t) \), \( u(x,t) \) and \( e(x,t) \). By assuming that each agent obeys to a point-mass dynamics, we consider the collective model

\[
\begin{align*}
\dot{r} &= v \\
\dot{v} &= u + \beta e, \\
\dot{e} &= -\alpha e
\end{align*}
\]

(12a)

(12b)

(12c)

where \( \beta \neq 0 \) and \( \alpha > 0 \) are constants. Then, the state of system (12) is \( z = [r^T, v^T, e^T]^T \). As detailed in Section V, the error model (12c) is not artificial but rather due to the compensation of persistent perturbations affecting the velocity dynamics (12b). Since \( e \) is independent of \( r \) and \( v \), the errorless dynamics corresponds to setting \( \dot{e} = 0 \).

Definition 3 We say that the system (12) achieves (i) alignment if there exists a time-invariant velocity \( u^* \in \mathbb{R}^q \) such that \( v(x,t) \rightarrow u^* \) as \( t \rightarrow +\infty \) for all agents \( x \in \mathcal{N} \); (ii) collision avoidance with safety distances \( \bar{r}_{xy} > 0 \) when the set \( \mathcal{R} = \{ r \in \mathbb{R}^2 : \| \partial_x r(x) \| > \bar{r}_{xy}, \forall (x,y) \in \mathcal{E} \} \) is positively invariant for the PdE (12).

In order to obtain these properties, we consider two different control laws:

\[
\begin{align*}
u(x,t) &= u_L(x,t) = \Delta v(x,t) \quad (13a) \\
\dot{u}(x,t) &= u_E(x,t) = \Delta v(x,t) - U(r(:,t)) \quad (13b)
\end{align*}
\]

where \( U(\cdot) \) is an elastic force defined as

\[
\begin{align*}
U(r(:,t))(x) &= \nabla_{r(x)} \bar{V}(x, \partial_x r(x)_{y=x}). \\
\bar{V}(x, \partial_x r(x))_{y \in \mathcal{G}} &= \sum_{y \sim x} V(x,y, \| \partial_y r(x,t) \|^2) 
\end{align*}
\]

(14)

(15)

and \( V(x,y,\cdot) \geq 0 \) is an elastic potential verifying

1) \( V(x,y,\zeta) = \bar{V}(x,y,\zeta) \);

2) \( V(x,y,\zeta) \rightarrow \infty \) as \( \zeta \rightarrow \tilde{r}_{xy}^2 \) where \( \tilde{r}_{xy} \geq 0 \) represent given safety distances;

3) for all \( x,y \in \mathcal{N}, V(x,y,\zeta) \) attains its unique minimum when \( \zeta = \tilde{r}_{xy}^2 \), where \( \tilde{r}_{xy} > \bar{r}_{xy} \) are given desired distances;

4) \( V(x,y,\zeta) \rightarrow \infty \) as \( \zeta \rightarrow +\infty \).

We refer to (13a) as Laplacian control (considered in [7] and [8] and similar to the one proposed by Vicsek [6]) and to (13b) as “elastic control” (a slight generalization of the control law proposed in [4], [9] and [2] because we allow for strictly safety distances). The role of \( v_{xy} \) is to reciprocally attract agents \( x \) and \( y \) when \( \| r_x - r_y \| > \bar{r}_{xy} \) and repulse them when \( \| r_x - r_y \| > \bar{r}_{xy} \). The growth at infinity for \( \| r_x - r_y \| = \bar{r}_{xy} \) intuitively guarantees that if \( \| r_x(0) - r_y(0) \| > \bar{r}_{xy} \), then the distance between agents \( x \) and \( y \) will never fall below the safety threshold \( \tilde{r}_{xy} \). Note that the Laplacian control is independent of the agent positions. It is then apparent that it cannot guarantee collision avoidance.

IV. Analysis of Leaderless Models

In the sequel, we investigate the collective behavior of the leaderless model (12) when Laplacian and elastic control laws are used.

\(^1\)For sake of conciseness, for a function \( f(x,t) : \mathcal{N} \times \mathbb{R}^+ \rightarrow \mathbb{R} \) we will often write \( f \in \mathbb{R}^2 \) instead of \( f(:,t) \in \mathbb{R}^2 \).

The next Lemma highlights the key properties of \( u_L \) and \( u_E \), used repeatedly for analyzing (12).

Lemma 1 For both choices (13), \( u(\cdot, t) \in H^1 \) for all \( t > 0 \).

\([\text{Proof}]\) Let \( f \in \mathbb{R}^2 \). Then \( f = f_1 + \langle f \rangle \), with \( f_1 \in H^1 \). Thus, in view of Theorem 1, one has \( \Delta f_1 = \Delta f_1 \in H^1 \). Direct calculations show that \( U(f(x)) \in H^1 \) for all \( f \in \mathbb{R}^2 \) (see [11] for details).

In order to highlight how the mathematical tools so far introduced can be used for proving alignment, we start considering the simpler case where \( u = u_L \) and errors are absent. Then, the PdE (12c) becomes

\[
\dot{v} = \Delta v, \quad v(\cdot, 0) = \bar{v} \in \mathbb{R}^2
\]

(16)

that formally coincides with the heat equation, by interpreting \( x \) as a point in an open, bounded and regular set \( \mathcal{G} \subset \mathbb{R}^N \), \( v(x,t) \in \mathbb{R} \) as the temperature and \( \bar{v} \) as an initial temperature distribution. Due to diffusion effect of the Laplace operator, it is not surprising that the temperature becomes asymptotically constant on \( \mathcal{G} \). In view of the similarities between classic Laplacian and the Laplacian on graphs, highlighted in [11], asymptotic convergence of the velocity \( v \) to a function in \( H^1_\mathcal{G} \) is expected. This means that, asymptotically, all agents will move with the same velocity, i.e. alignment will be achieved.

This physical argument can be proved by means of a simple variational technique. First we introduce the decomposition \( v = v_1 + \bar{v}, \quad v_1 \in H^1_\mathcal{G}, \quad \bar{v} \leftrightarrow \langle v \rangle \in H^1_\mathcal{G} \) and look for the dynamics of \( v_1 \) and \( \bar{v} \). For this purpose, we test each side of \( \dot{v} = \Delta v \) against all \( c \in H^1_\mathcal{G} \) thus obtaining

\[
\int_G \bar{v}^2 \kappa = \int_G (\Delta v_1)^2 \kappa + \int_G (\Delta \bar{v})^2 \kappa
\]

(17)

Note that we have \( \int_G (\Delta v_1)^2 \kappa = 0 \) because, from Lemma 1, it holds \( \Delta v_1 \in H^1 \). Then, (17) reduces to \( \int_G \bar{v}^2 \kappa = 0 \) that implies \( \bar{v} = 0 \). From (16) we obtain \( v_1 + \bar{v} = \Delta v_1 \) and then \( \dot{v}_1 = \Delta v_1 \). In summary, the velocity dynamics is captured by the PDEs

\[
\begin{align*}
\dot{v}_1 &= \Delta v_1, \quad v_1(\cdot, 0) = P_g \bar{v} \\
\dot{\bar{v}} &= 0, \quad \bar{v}(0) = P_{H^1_\mathcal{G}} \bar{v}
\end{align*}
\]

(18a)

(18b)

Equations (18) highlight that the dynamics of \( v_1 \) and \( \bar{v} \) are decoupled and proving alignment amounts to show that \( v_1 \rightarrow 0 \) as \( t \rightarrow +\infty \). In this case, we have that \( v \rightarrow \bar{v} \) as \( t \rightarrow +\infty \). Moreover, from (18b) we also deduce that \( \bar{v} \) is time-invariant and equal to \( \langle \bar{v} \rangle \). Similarly, exponentially stable alignment amounts to check the exponential stability on \( H^1_\mathcal{G} \) of the origin of (18a). This property (that is proved in Theorem 4 for the more general case of agents affected by errors), can be readily shown by using the fact that all the eigenvalues \( \Delta \) on \( H^1_\mathcal{G} \) are strictly negative (see Theorem 1).

In what follows, the above argument is generalized to prove alignment for model (12) when \( u_L \) or \( u_E \) are used. In Section VI, we also show that a very similar rationale is adopted for proving alignment in leader-follower models.

We consider now the dynamics (12) and introduce the splittings:

\[
\begin{align*}
u &= v_1 + \bar{v}, \quad v_1(\cdot, t) \in H^1_\mathcal{G}, \quad \bar{v} \leftrightarrow \langle \bar{v} \rangle \in H^1_\mathcal{G} \\
e &= e_1 + \bar{e}, \quad \bar{e} \leftrightarrow \langle \bar{e} \rangle \in H^1_\mathcal{G}
\end{align*}
\]

(19a)

(19b)

\( ^2 \)More rigorously, it coincides with the heat equation on \( G \) with homogeneous Neumann boundary conditions [13], i.e. with zero heat flow through the boundary of \( G \).
By mimicking the calculations done for the PdE (18), Lemma 1 allows to show that model (12) is equivalent to the PdEs

\[
\begin{align*}
\Sigma_1 : & \quad \begin{cases} 
\partial_t \bar{v} = \partial_x v_1, \quad \forall y \in G \\
 \bar{v}_1 = u + \beta e_1 \\
 \hat{e}_1 = -\alpha e_1
\end{cases} \\
\Sigma_2 : & \quad \begin{cases}
\hat{v} = \beta \bar{e} \\
\hat{e} = -\alpha \bar{e}
\end{cases}
\end{align*}
\]

(20)
equipped with the initial conditions \(\partial_t r = \partial_x \bar{v}, v_1(\cdot, 0) = P_{H^1} \bar{v}, e_1(\cdot, 0) = P_{H^1} \bar{e}, \bar{v}(0) = P_{H^1} \hat{v} \) and \(e(0) = P_{H^1} \hat{e}\). Note that \(u^k\) depends on \(\partial_x r\) rather than \(r\) and this explains why the former quantity has been used in (20).

In view of (20), alignment is achieved if \(v_1 \to 0\) as \(t \to \infty\). In fact, for the PdE \(\Sigma_1\), it is easy to check that \(\bar{v}\) converges to a time-invariant function \(\bar{v}^* \in H^1\). In the errorless case, one has \(\bar{v}^* = (\hat{\bar{v}})\) while, in presence of errors, \(\bar{v}^*\) depends also on the dynamics of \(\hat{\bar{e}}\). We are now in a position to state the main result.

**Theorem 4**

1. Let be the Laplacian control. Then, the origin of (12) is globally asymptotically stable on \(V = \{0\} \times H^1 \times H^1\), i.e. \(u^k\) guarantees exponentially stable alignment.

2. Let be as in (13b), and assume the initial positions satisfy the collision avoidance condition \(r(-, 0) \in R\). Then, alignment and collision avoidance (see Definition 3) are guaranteed.

**Proof:**

1. The choice of \(V\) and the decomposition (20) allow us to consider the stability of the PdEs

\[
\begin{align*}
\dot{v}_1 = u + \beta e_1, \quad \dot{e}_1 = -\alpha e_1
\end{align*}
\]

regardless of the variables \(r, \bar{v}, \bar{e}\). Set \(W(v_1, e_1) = \frac{1}{2} \|v_1\|_{L^2} + \frac{\gamma}{2} \|e_1\|_{L^2}\), where \(\gamma\) is a parameter to be chosen. The functional \(W\) verifies the bounds (11a) for \(k_1 = \min\{\frac{1}{2}, \frac{3}{2}\}\), \(k_2 = \max\{\frac{1}{2}, \frac{3}{2}\}\) and \(a = 2\). By computing \(W\), one obtains

\[
\begin{align*}
\dot{W} = & \int_G v_1^T(\Delta v_1 + \beta e_1) - \alpha \gamma \int_G \|e_1\|^2 \\
\leq & \lambda \|v_1\|^2 + \beta \int_G v_1^T e_1 - \alpha \gamma \int_G \|e_1\|^2 \\
\end{align*}
\]

(21)

where \(\lambda\) is the maximum eigenvalue of the Laplacian operator defined on \(H^1\). In view of Theorem 1, we have \(\lambda < 0\). It can be shown that the bound (11b) is verified with

\[
k_3 = \min\left\{-\lambda, \frac{\beta \gamma}{2\lambda}\right\}.
\]

By choosing \(\gamma\) big enough, one obtains \(k_3 > 0\) and global exponential stability of the origin on \(V\) follows from Theorem 2.

2. Set

\[
V_R = \left\{ \begin{array}{c}
\partial_t v^T \cdots \partial_N v^T v_1^T e_1^T \\
\text{with } v_1 \in H^1, e_1 \in H^1
\end{array} \right\}
\]

(22)
along with the energy \(W_R : V_R \rightarrow \mathbb{R}\),

\[
W_R(\{\partial_t r\}_i) = \int_G \bar{V}(x, \{\partial_x r\}_y) \\
+ \frac{1}{2} \|v_1\|_{L^2}^2 + \frac{\gamma}{2} \|e_1\|_{L^2}^2
\]

(23)

where \(\gamma > 0\) is a parameter. Let \(\delta = W_R \left( \left\{ d_i \right\}_{i=1}^{N}, P_{H^1} \bar{v}, P_{H^1} \hat{e} \right) \) and consider the set

\[
\Omega_3 = \{ \epsilon \in \mathcal{V}_R : W_R(\epsilon) \leq \delta \}.
\]

(24)

Applying exactly the same argument used in the proof of [2, Theorem 1], one can show that \(\Omega_3\) is a compact set. Now, we compute \(W_R\) and prove that \(\dot{W}_R \leq 0\). We exploit the identity:

\[
\frac{1}{2} \frac{d}{dt} \|v\|_{L^2}^2 + \int_G \bar{V}(x, \{\partial_x r\}_y) = \int_G v_1^T U(r(t), t)
\]

(25)

and Lemma 1 to obtain:

\[
\dot{W}_R = \int_G v_1^T (\Delta v_1 + \beta e_1) - \alpha \gamma \int_G \|e_1\|^2
\]

(26)

Simple algebraic manipulations show that \(W_R \leq 0\) if the parameter \(\gamma\) is chosen big enough (see [11] for further details). As a consequence, the set \(\Omega_3\) is positively \(V\)-invariant. The implication

\[
\left[ \begin{array}{c}
\partial_t v^T \cdots \partial_N v^T v_1^T e_1^T \\
\end{array} \right] \in \Omega_3, \quad \forall t \geq 0 \Rightarrow \forall (t, r) \in \mathcal{R}, \quad \forall t \geq 0,
\]

can be easily checked by contradiction.

The \(V_R\)-invariance of \(\Omega_3\) implies also that \(v_1\) is bounded, \(\forall t \geq 0\). Then, the solution to the PdE \(\Sigma_1\) in (20) is uniquely defined, \(\forall t \geq 0\). Indeed, it is apparent that only \(v_1\) could have a finite escape time but this cannot happen because \(v_1\) remains bounded.

Now, we apply Theorem 3 to conclude. To this purpose, we verify that \(P_{V_R} F = FP_{V_R}\). The operator \(F\) corresponding to the PdE (20) is given by

\[
F(\bar{z}) = \left[ \begin{array}{c}
\partial_t v^T \cdots \partial_N v^T (\Delta v - U + \beta e)^T - \alpha e^T \\
\end{array} \right] T
\]

(27)

and the equality \(P_{V_R} F = FP_{V_R}\) is equivalent to the existence of the splitting (20). The set \(E\) considered in Theorem 3 is given by

\[
E = \left\{ \begin{array}{c}
\{ \partial_t v^T \cdots \partial_N v^T \} v_1^T e_1^T \\
\text{such that } v_1 = 0 \text{ and } e_1 = 0
\end{array} \right\}
\]

(28)

and the fact that \(v_1 \to 0\) and \(e_1 \to 0\), as \(t \to \infty\), follows. Note that the elastic control does not guarantee cohesion, i.e. that

\[
\lim_{t \to \infty} \|\partial_x r(x, t)\|^2 = \bar{r}_{xy}, \quad \forall (x, y) \in E.
\]

(29)

In fact, since the desired distances \(\bar{r}_{xy}\) are arbitrary, a necessary condition for cohesion is that there exists a function \(r \in L^2\) fulfilling the conditions

\[
\|\partial_x r(x, t)\|^2 \leq \bar{r}_{xy}, \quad \forall (x, y) \in E.
\]

(30)

In [2] it has been proved that, in the errorless case and when the graph has a tree structure, equation (30) can be always solved and it implies cohesion. The study of the solvability of (30) for general connected graphs, and the conjecture that (30) is sufficient for achieving cohesion, are still open issues.

V. ROBUSTNESS OF THE COLLECTIVE DYNAMICS AND THE ERROR MODEL

A few remarks about robustness of the control laws (13) are due. Since \(u(t) \in H^1\), the control is unable to counteract perturbations having components in \(H^1\). The following elementary example shows that this might be easily the case. Suppose that (12) is not affected by errors but the velocity dynamics is affected by a persistent perturbation \(cv\) where \(c \in \mathbb{R}\) is unknown. Then, equation (12b) becomes

\[
\dot{\hat{v}} = cv + u
\]

(31)

Set, for simplicity, \(u = \Delta v\) as in (13a). The splitting obtained through Lemma 1 can still be applied and gives the following dynamics for the average velocity: \(\hat{v} = \bar{v}\). Thus, if \(c > 0\), the average velocity goes to infinity and, if \(c < 0\), \(v\) converges to 0 for all \(x \in N\). In other words, alignment to a nonzero velocity cannot be achieved. In order to
counteract this undesirable behavior, an internal compensation \( \tilde{u}(x,t) \) has to be designed. The compensated velocity dynamics becomes:

\[
\dot{v} = \epsilon v + u + \tilde{u}
\]

(32)

where, for each \( x \in \mathcal{N} \), \( \tilde{u}(x,t) \) depends only upon \( r(x,t) \) and \( v(x,t) \). A detailed design of the internal compensation is beyond the scope of this paper. In [11, Appendix A] it shown that by using a standard sliding-mode technique, \( \tilde{u} \) can be designed in order to guarantee that, after a finite time, each agent behaves according to model (12) with a prescribed error decrease rate \( \alpha \) and with \( \beta = \alpha^2 \). Moreover, the same result can be obtained for more general perturbation models (see [11] for details).

VI. ANALYSIS OF LEADER-FOLLOWER MODELS

In this Section we use PdEs for analyzing the collective motion of the agents in presence of a leader, i.e. a vehicle that moves according to a prescribed constant velocity, independently of the motion of all other vehicles, however, followers connected to the leader use information on the leader state in order to compute their control inputs. Let \( x_L \in \mathcal{N} \) be the node representing the leader and define the subgraph \( \mathcal{S} \) such that \( \partial \mathcal{S} = \{x_L\} \) and \( \mathcal{N} = \mathcal{S} \cup \partial \mathcal{S} \), where \( \partial \mathcal{S} \) is the set of nodes of \( \mathcal{S} \).

We denote by \( v_L \) the velocity of the leader \( x_L \). Let, by abuse of notation \( v_L(x) \equiv v_L \) for all \( x \in \mathcal{N} \). Note that \( \Delta v_L = 0 \), because \( v_L \in H^1_0 \). It turns out that the velocity \( v \in L^2 \) can be split as

\[
v = v_0 + v_L, \quad v_0 \in H^1_0 \quad \text{and alignment to the leader velocity corresponds to the condition } v_0 \to 0 \text{ as } t \to \infty.
\]

If the followers make use of a control law \( u \), the collective dynamics results in the following PdE with boundary conditions

\[
\begin{align*}
\dot{v} &= v_0 + v_L, \quad x \in G \\
\dot{v}_L &= v_0 = u + \beta e, \quad x \in S \\
\dot{e} &= -\alpha e, \quad x \in S \\
v_0 &= 0, \quad x \in \partial \mathcal{S} \quad \text{(34d)}
\end{align*}
\]

endowed with the initial conditions \( r(\cdot,0) = \tilde{r} \in L^2, v_0(\cdot,0) = v_0 \in H^1_0, e(\cdot,0) = \tilde{e} \in L^2(\mathcal{S};\mathbb{R}^m) \). Note that: (i) each agent, but the leader, behaves exactly according to the model (12); (ii) since \( \Delta v_L = 0 \), the control laws (13) read as \( u^L = \Delta v_0(x,t) \) and \( \tilde{u} = \Delta v_0(x,t) - U(r(x,t)) \); (iii) the boundary condition (34d) forces the velocity dynamics to remain in \( H^1_0 \) at all times.

The following theorem is the counterpart of Theorem 4:

Theorem 5

1) Let \( u \) be the Laplacian control and \( z = [r^T, v_0^T, e^T] \) be the state of (34). Then, the origin of (34) is globally asymptotically stable on \( \mathcal{V} = \{0\} \times H^1_0 \times L^2(\mathcal{S};\mathbb{R}^m) \), i.e. \( u^L \) guarantees exponentially stable alignment to the leader velocity.

2) Let \( u \) be the elastic control and assume that the initial positions verify the collision avoidance condition \( r(\cdot,0) \in \mathcal{R} \). Then, (i) \( r(\cdot,t) \in \mathcal{R}, \forall t \geq 0 \) (collision avoidance at all times); (ii) \( v_0 \to 0 \) as \( t \to \infty \) (alignment to the leader velocity).

Proof: A detailed version of the proof can be found in [11]. Next, we just sketch the main lines of the argument adopted.

1) By taking \( W_L = \frac{1}{2} \|v_0\|^2 + \frac{1}{2} \|e\|_{L^2(\mathcal{S})}^2 \geq 0 \) as a Lyapunov functional, one obtains \( W_L = \int_G v_0^T (\Delta v_0 + \beta e) - \alpha \int_G \|e\|^2 \). In view of Theorem 1, one can prove the result by using the same technique as in point 1 of Theorem 4, after noting that \( \int_G v_0^T e = \int_G v_0^T e \) (since \( v_0 = 0 \) on \( \partial \mathcal{S} \)).

2) We set

\[
W_{LR} \doteq \left\{ \begin{array}{l}
\{ \partial \mathcal{R} \} \times H^1_0 \times L^2(\mathcal{S};\mathbb{R}^m) \\
\end{array} \right\} \in L^2(\mathcal{G};\mathbb{R}\times H^1_0 \times L^2(\mathcal{S};\mathbb{R}^m))
\]

and proceed as in point 2 of Theorem 4.

VII. DISCUSSION AND CONCLUDING REMARKS

In this paper we proposed the framework of continuous-time PdEs for analyzing coordination phenomena in multi-agent systems. We showed that PdEs allow to unify and generalize many results on the analysis of the collective dynamics scattered in the control literature. Although we considered a fairly simplified setup (i.e. agents move according to a point-mass dynamics perturbed by errors and the structure of the communication network is time-invariant) we believe that PdEs provide a useful mathematical framework even when dealing with (i) more complex agent models accounting for the effects of various perturbations (e.g. stochastic effect of wind on the motion of aerial vehicles) (ii) more complex control laws guaranteeing also obstacle avoidance [14] (iii) time-varying communication links [9], [11] [14]. As an example, in [15] PdEs have been fruitfully used for studying alignment of agents affected by various models of time-delays and equipped with linear control schemes.

Moreover, the profound similarity between PdEs and PDEs describing physical phenomena can be inspiring for devising new decentralized control schemes. For instance, linear and nonlinear elasticity models might be inspiring for designing distributed control laws regulating geometric features of the formation.

REFERENCES


