Computing Observability Regions for Discrete-time Hybrid Systems

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Abstract—In this paper we focus on observability for hybrid systems in the mixed-logic dynamical form. We show that the maximal set of observable states, that is usually union of finitely many polytopic regions, can be represented as the union of finitely many polytopic regions. The argument, that is based on multi-parametric programming theory, is constructive and provides an algorithm for the computation of the regions.

I. INTRODUCTION

Many different modeling frameworks are currently available for representing hybrid systems. Among others, one may cite hybrid automata [1], hybrid dynamical systems [8], [18], switched systems [17], linear complementarity systems [14] and Mixed Logic Dynamical (MLD) systems [4]. In all the cases, the aim is to provide an unified description of phenomena where dynamical and logical components interact. Roughly speaking, an hybrid model can be taught as a set of dynamical systems (called modes) together with a rule that orchestrates the switching among them.

The interest in hybrid systems goes beyond their use for simulation. In fact, each class of hybrid models provides a mathematical framework for addressing analysis and synthesis problems. Concerning analysis, a large stream of research focused on the development of methods for verifying structural properties such as stability [7], [16], [17], [11] controllability [10], [3] and observability [10], [2], [19], [15], [3].

The main difficulty that often arises in analysis problems (and independent of the class of hybrid systems adopted) is that the structural properties of the overall system cannot be inferred from the corresponding ones of its modes. In other words, the switching law plays a central role and cannot be neglected by the analysis procedures. Another big challenge is due to the fact that many interesting properties, like stability, are undecidable (see for instance [5]). This led many researchers to focus on either stronger properties or restricted subclasses of hybrid systems.

Consider for instance observability. In [10], sufficient conditions are provided for the limited class of linear, piecewise-constant, time-varying models. In [2], the observability of the overall hybrid system is defined by taking only partially into account the interactions between the automaton governing the switching and the modes. Finally, in [19], the observability of discrete-time linear switched systems is characterized by assuming a minimum dwell-time between two consecutive switches.

A fairly complete study of observability for MLD systems is reported in [3] and it is instrumental to the design of moving-horizon state observers enjoying convergence properties [13]. In particular, by proving the equivalence between MLD and piecewise affine systems, the authors show, through counter examples, three important facts. First, observability over an infinite time horizon is undecidable. Therefore, for a general MLD system, one should resort to a notion of observability in finite time. Second, the observability/unobservability of the overall system cannot be deduced from the observability/unobservability of the affine modes. Third, the set of states that are observable may be non convex and disconnected. Despite these difficulties, numerical tests based on mixed-integer linear programming have been proposed in order to check if a subset of the state space is observable [3].

The main drawback of these algorithms is that the region to be tested must be specified a priori. Therefore, if it contains a large set of observable states and a small set of unobservable states, such tests give a negative answer without revealing the presence of the observable subregion. This problem is critical in view of the fact that the computational cost of trial-and-error procedures for finding observable regions is usually prohibitive.

The present paper focuses on the automatic computation of the maximal observability region (i.e. the maximal set of observable states) for an MLD system. In this sense, it is the natural prosecution of [3]. The rationale underlying our procedure can be summarized in the following steps. First, we show that the set of states indistinguishable from a given state \(x\) can be represented through linear mixed-integer inequalities parametrized by \(x\). Moreover, the computation of the state \(x'\), indistinguishable from \(x\) and maximizing \(\|x - x'\|_1\), can be recast into a multi-parametric Mixed Integer Linear Programming (mp-MILP) problem with parameter \(x\). Basic results of mp-MILP theory show that the map \(\Gamma(x) = x'\) is piecewise affine and mp-MILP solvers allow to computed it in closed-form [9], [6]. Finally, the maximal observability region is found as the set of fixed points of \(\Gamma\). Quite
remarkably, the piecewise affine structure of \( \Gamma \) implies that the maximal observability region is the union of finitely many polytopic sets \( \mathcal{C}_i \). Moreover, the linear inequalities defining each set \( \mathcal{C}_i \) can be explicitly computed.

Section II provides the necessary theoretical background on MLD systems, observability theory and mp-MILP theory. The main algorithm is presented in Section III and illustrated in IV through a simple example.

II. THEORETICAL BACKGROUND

A. MLD systems

We consider discrete-time hybrid systems in the MLD form [4],

\[
\begin{align*}
x(t+1) &= Ax(t) + B_1 u(t) + B_2 \delta(t) + B_3 z(t) \quad (1a) \\
y(t) &= C x(t) + D_1 u(t) + D_2 \delta(t) + D_3 z(t) \quad (1b) \\
g(\delta(t), z(t), u(t), x(t)) &\leq 0 \quad (1c)
\end{align*}
\]

where \( x \in \mathbb{R}^n \times \{0,1\}^{n_1} \) are the continuous and binary states, \( u \in \mathbb{R}^{m_u} \times \{0,1\}^{m_u} \) are the inputs, \( y \in \mathbb{R}^{n_y} \times \{0,1\}^{n_y} \) are the outputs, and \( \delta \in \{0,1\}^{n_\delta} \), \( z \in \mathbb{R}^{n_z} \) represent auxiliary binary and continuous variables, respectively. The auxiliary variables are introduced when translating propositional logic into linear inequalities. For a detailed description of the modeling capabilities of MLD systems, we defer the reader to [3], [4], [12].

For the sake of simplicity, we consider MLD systems without logical states, i.e. \( x \in \mathbb{R}^n \). Moreover, as customary for MLD systems, states and inputs are constrained within polytopic regions \( \mathcal{X} \subset \mathbb{R}^n \) and \( \mathcal{U} \subset \mathbb{R}^{m_u} \times \{0,1\}^{m_u} \), respectively. For an MLD system, we introduce the feasibility set

\[
\mathcal{F} = \{(x,u) \in \mathcal{X} \times \mathcal{U} : \exists \delta, z : g(\delta, z, u, x) \leq 0 \}
\]

and the point-to-set map \( \mathcal{G} : \mathcal{X} \rightarrow 2^\mathcal{U} \)

\[
\mathcal{G}(x) = \{u \in \mathcal{U} : \exists \delta, z : g(\delta, z, u, x) \leq 0 \}
\]

gives the set of feasible inputs corresponding to each state. In order to guarantee that the state update \( x(t+1) \) and the output \( y(t) \) are uniquely defined, we assume that the MLD system (1) is well-posed [4], i.e. \( \forall (x(t), u(t)) \in \mathcal{F}, \) the vectors \( \delta(t) \) and \( z(t) \) verifying the inequalities (1c) are unique.

Note that the well-posedness does not imply that the state \( x(t) \) is defined at all time instants. In particular, the system evolution is blocked at time \( t \) if either \( x(t+1) \notin \mathcal{X} \) (violation of the state bounds) or \( \mathcal{G}(x(t+1)) = \emptyset \) (no admissible input at time \( t+1 \)).

Remark 1: The rationale used in [4] for translating propositional calculus into the mixed-integer linear inequalities, requires the use of “\( c \)-tolerances” in the inequalities (1c). This may produce a small subset of states \( \mathcal{X}_c \subset \mathcal{X} \) such that the inequalities (1c) are infeasible for \( x \in \mathcal{X}_c \) (see [4] for more details). We point out that such tolerances do not affect the computational implementation of an MLD system, if they are in the order of the machine precision. However, from the mathematical point of view, an MLD system is well-defined only for state/inputs pairs belonging to the feasibility set \( \mathcal{F} \), that can be strictly included in \( \mathcal{X} \times \mathcal{U} \).

To a discrete-time vector-valued signal \( f(t) \) and a time horizon \( T > 0 \), we associate the capitalized vector \( F(T) = [f(0)^T \ldots f(T-1)^T]^T \) that collects the samples of \( f \). By using the linearity of the equations and the inequalities in (1), the state trajectory of an MLD system over a time window of length \( T \) can be written in compact form

\[
\begin{align*}
X(T) &= A x_0 + \hat{B}_1 U(T) + \hat{B}_2 \Delta(T) + \hat{B}_3 Z(T) \quad (4a) \\
\hat{E}_2 \Delta(T) + \hat{E}_3 Z(T) &\leq \hat{E}_4 U(T) + \hat{E}_5 x(0) + \hat{E}_5 \quad (4b) \\
U(T) &\in \mathcal{U}^T, \ X(T) \in \mathcal{X}^T \quad (4c)
\end{align*}
\]

where \( \hat{A}, \hat{B}_1, \hat{B}_2, \hat{B}_3 \) and \( \hat{E}_j, j = 1,\ldots, 5 \) are suitable matrices (see [4]). Analogously, the output trajectory can be represented by (4) and the equation

\[
Y(T) = \hat{C} x(0) + \hat{D}_1 U(T) + \hat{D}_2 \Delta(T) + \hat{D}_3 Z(T) \quad (5)
\]

where \( \hat{C}, \hat{D}_1, \hat{D}_2, \hat{D}_3 \) are suitably defined. Occasionally, we will use the notation \( X(T|x(0), U(T)) \) and \( Y(T|x(0), U(T)) \) for highlighting the dependence of \( X(T) \) and \( Y(T) \) on the initial state and the inputs. Note that infeasibility of the inequalities (1c) for \( t \leq T - 1 \) renders the definition of \( X(T|x(0), U(T)) \) meaningless.

Then, we say that \( X(T|x(0), U(T)) \) is well-defined if \( x(0) \) and \( U(T) \) are such that the evolution of the MLD system (1) is not blocked at any time \( t \leq T - 1 \). This leads naturally to the introduction of the following feasibility set:

\[
\mathcal{X}_T^* = \{x \in \mathcal{X} : \exists U(T) \in \mathcal{U}^T \text{ such that } X(T|x(0), U(T)) \text{ is well-defined} \}
\]

B. Observability theory

We first specialize the notion of indistinguishable states to MLD systems.

Definition 1: For an MLD system, two states \( x, \hat{x} \in \mathcal{X} \) are indistinguishable in \( T \) steps if there exists \( U(T) \in \mathcal{U}^T \) such that \( X(T|x, U(T)) \) and \( X(T|\hat{x}, U(T)) \) are well-defined and \( Y(T|x, U(T)) = Y(T|\hat{x}, U(T)) \).

If \( x \) and \( \hat{x} \) are indistinguishable, we use the notation \( x \sim \hat{x} \). Note that “\( \sim \)” defines a relation over \( \mathcal{X} \times \mathcal{X} \), but, differently from the case of linear systems, it may not be an equivalence relation because the transitive property may not hold. More in detail, if there exist \( U_1(T) \in \mathcal{U}^T \) such that \( x_1 \sim x_2 \) and \( U_2(T) \in \mathcal{U}^T \) such that \( x_2 \sim x_3 \), one cannot guarantee that there exists \( U_3(T) \in \mathcal{U}^T \) such that

\[
\begin{align*}
x_1 &\sim x_3 \quad (violation of the state bounds)
\end{align*}
\]
such that \( x_1 \leq x_3 \), because of the input constraints and the intrinsic nonlinear behavior of the MLD systems. However, it is easy to prove that \( \sim \) is an equivalence relation for autonomous MLD systems.

Let \( \mathcal{O}_T(x) \subseteq \mathcal{X} \) be the set of states indistinguishable from \( x \) in \( T \) steps. We stress the fact that if \( x \in \mathcal{X}_T^0 \), then \( \mathcal{O}_T(x) \) is not empty since it holds \( x \sim_T x \). On the other hand, if \( x \not\in \mathcal{X}_T^0 \), we have \( \mathcal{O}_T(x) = \emptyset \). Next, we provide the definition of observable states and observable regions.

**Definition 2:** A state \( x \in \mathcal{X} \) is observable in \( T \) steps if \( \mathcal{O}_T(x) = \{x\} \). A set \( \mathcal{O}_T \subseteq \mathcal{X} \) is an observability region (in \( T \) steps) if all the states \( x \in \mathcal{O}_T \) are observable. The maximal observability region (in \( T \) steps) \( \bar{\mathcal{O}}_T \) is the union of all the observability regions.

**Remark 2:** As in [3], our definition of observability is tailored to the use of observers together with a regulator for MLD systems. In fact, since the control sequence \( U(T) \) is not known at time \( t = 0 \), we require that a state is observable for all admissible control inputs.

**Remark 3:** The finiteness of the parameter \( T \) has a practical meaning. In fact, \( T \) is the horizon over which output data must be collected before being able to reconstruct the initial state. Then, in a realistic scenario it is reasonable to fix a maximal horizon of interest \( T_{\text{max}} \) and classify states that are observable for \( T > T_{\text{max}} \) as practically unobservable [3]. We also outline that Definition 2 is slightly different from the corresponding Definition 1 given in [3] because a minimum level of distinguishability between different states is not required.

**C. Multi-Parametric Mixed Integer Linear Programming**

A key ingredient for computing observability regions is the availability of algorithms for solving mp-MILP problems. Consider the following MILP problem

\[
J^*(\theta) = \min_{\nu} e^T \nu \\
G \nu \leq LB + g
\]

(7)

where \( \nu = [\nu_e' \nu_o']' \in \mathbb{R}^{n_e} \times \{0,1\}^{n_o} \) is the vector collecting the unknowns and \( \theta \in \mathbb{R}^n \) is fixed. Problem (7) is feasible if there exists at least one vector \( \nu \) verifying the constraints. In multi-parametric programming, the vector \( \theta \) is considered as a parameter varying in a nonempty polytope \( \Theta \subseteq \mathbb{R}^n \) and one aims at determining:

1) The set \( \Theta^* \subseteq \Theta \) of parameters \( \theta \) such that the mp-MILP (7) is feasible and \( J^*(\theta) \) is finite.

2) The expression of the optimizer \( \nu^*(\theta) \). In the case of multiple solutions \( \nu^*(\theta) \) can be chosen arbitrarily among all possible optimizers.

The main result on mp-MILP is summarized in the following theorem [6].

**Theorem 1:** Consider the mp-MILP (7) and assume that there exists \( \theta \in \Theta \) such that (7) is feasible. Then, there exists finitely many polytopes \( C_j \in \mathbb{R}^n, j = 1 \ldots s \), such that:

1) \( \Theta^* = \bigcup_{j=1}^s C_j \)

2) The optimizer \( \nu^*(\theta) \), and the value function \( J^*(\theta) \) are affine functions on each polytope \( C_j \). ■

Since the region \( C_j \) is a polytope, it can be represented through a system of linear inequalities \( H_j \theta \leq h_j \). From the computational side, it is important to note that there are algorithms for solving mp-MILP problems, i.e. for finding the number \( s \) of polytopes, the corresponding matrices \( H_j \) and \( h_j \), \( j = 1, \ldots, s \), and the affine expressions of both the value function and the optimizer on each \( C_j \). For the implementation details, we defer the reader to [6], [9]. We highlight that the regions \( C_j \) appearing in Theorem 1 are not unique and both their number and shape depends on the strategy adopted for exploring the parameter set [6]. However, an important point of Theorem 1 is that the set \( \Theta^* \) that is in general non-convex and disconnected can be represented as the union of finitely many polytopic regions.

**Remark 4:** A detailed analysis of the computational time needed for solving mp-MILP can be found in [6]. However, the maximal observability region of an MLD system is typically calculated off-line, and, in this respect, the computational complexity for solving the associated mp-MILP is not a fundamental issue.

**III. COMPUTATION OF THE OBSERVABILITY REGIONS**

As outlined in the introduction, the rationale for computing the maximal observability region of an MLD system is structured in three steps. First, we characterize the sets \( \mathcal{O}(x) \) in terms of mixed-integer linear inequalities parametrized in \( x \). By using (4) and (5), we have that \( \hat{x} \in \mathcal{O}_T(x) \) if and only if there exists \( U(T), \Delta(T), \tilde{\Delta}(T), Z(T) \) and \( \tilde{Z}(T) \) such that

\[
U(T) \in U^T, \Delta(T), \tilde{\Delta}(T) \in \{0,1\}^{T \epsilon}, \quad (8a)
\]

\[
X(T), \hat{X}(T) \in \mathcal{X}^T \quad (8b)
\]

\[
X(T) = \hat{A}x + \hat{B}_1U(T) + \hat{B}_2\Delta(T) + \hat{B}_3Z(T) \quad (8c)
\]

\[
\hat{X}(T) = \tilde{A}\hat{x} + \tilde{B}_1U(T) + \tilde{B}_2\tilde{\Delta}(T) + \tilde{B}_3\tilde{Z}(T) \quad (8d)
\]

\[
\tilde{E}_2\Delta(T) + \tilde{E}_3Z(T) \leq \tilde{E}_1U(T) + \tilde{E}_4\hat{x} + \tilde{E}_5 \quad (8e)
\]

\[
\tilde{E}_2\tilde{\Delta}(T) + \tilde{E}_3\tilde{Z}(T) \leq \tilde{E}_1U(T) + \tilde{E}_4\hat{x} + \tilde{E}_5 \quad (8f)
\]

\[
\tilde{D}_2(\Delta(T) - \Delta(T)) + \tilde{D}_3(\tilde{Z}(T) - Z(T)) = -\tilde{C}(\hat{x} - x) \quad (8g)
\]

In fact, the inequalities (8c), (8e) and (8d), (8f) represent the evolution of an MLD system over the horizon \( T \) when initialized with \( x \) and \( \hat{x} \), respectively. Moreover, the equality (8g) represents the condition \( Y(T|x, U(T)) = Y(T|\hat{x}, U(T)) \). The inequalities (8) can be exploited for deducing the following important property.

**Theorem 2:** For each \( x \in \mathcal{X} \), the set \( \mathcal{O}_T(x) \) is compact.
Proof. Note that \( \Im_T(x) = \emptyset \), for all \( x \in \mathcal{X} \setminus \mathcal{X}_T^* \). Therefore, if \( \mathcal{X}_T^* \) is empty, the thesis trivially follows. Otherwise, fix \( x \in \mathcal{X} \cap \mathcal{X}_T^* \). The inequalities (8) define an MP-MILP with zero cost, parameter vector \( \theta = \hat{x} \in \mathcal{X} \) and optimization vector \( \nu = [U(T)' \ \Delta(T)' \ \Delta(T)' \ Z(T)' \ Z(T)' \ \mathcal{Z}(T)' \ \eta]' \). Since \( x \in \mathcal{X}_T^* \), we have \( x \in \Im(x) \) thus motivating the feasibility of the MP-MILP problem (8) for \( \theta = x \). Then, Theorem 1 can be applied so deducing that the set \( \Theta^*(x) \) of parameter vectors rendering (8) feasible is compact. The thesis follows by noting that \( \Im_T(x) = \Theta^*(x) \).

For a given MLD system, consider the optimization problem

\[
\max_{\hat{x} \in \Im_T(x)} \|x - \hat{x}\|_1 \quad (9)
\]

where \( x \in \mathcal{X} \) is a parameter. We highlight that problem (9) is feasible for all and only the parameters \( x \in \mathcal{X}_T^* \) and, in such cases, Theorem 2 and the continuity of \( \|\cdot\|_1 \) guarantee that the maximum is well-defined.

Consider the function \( \Gamma : \mathcal{X}^* \in \mathcal{X} \) that associate to each parameter \( x \) the optimizer \( \hat{x}^*(x) \) of problem (9) (in case of multiple solutions, \( \hat{x}^*(x) \) can be chosen arbitrarily among all possible optimizers). By using Definition 2, the maximal observability region can be represented as

\[
\hat{O}_T = \{x \in \mathcal{X}_T^* : \Gamma(x) = x\} \quad (10)
\]

In the sequel, we show how to recast (9) into an MP-MILP so obtaining the closed-form expression of \( \Gamma \). The proposed procedure is the same used in [3] for recasting an optimization problem with 1-norm cost into a mixed-integer optimization problem with linear cost. We introduce the slack variables \( \eta \in \mathbb{R}^n \) and \( \mu \in \{0,1\}^n \) defined as

\[
\mu_i = \begin{cases} 
1 & \text{if } (x - \hat{x})_i \leq 0 \\
0 & \text{if } (x - \hat{x})_i > 0
\end{cases} \quad (11)
\]

\[
\eta_i = \begin{cases} 
-(x - \hat{x})_i & \text{if } \mu_i = 1 \\
(x - \hat{x})_i & \text{if } \mu_i = 0
\end{cases} \quad (12)
\]

where \( (x - \hat{x})_i \) is the \( i^{th} \) component of the vector \( x - \hat{x} \). Note that the cost functional in (9) can be rewritten as \( \sum_{i=1}^n \eta_i \). On the other hand, by using the same rationale employed in [4] for translating propositional logic into mixed integer linear inequalities, one can compute matrices \( M, N \) and \( N_c \) such that \( \forall x, \hat{x} \in \mathcal{X} \), the vectors \( \mu \) and \( \eta \) defined as in (11) and (12), respectively, are the unique solution to the system of inequalities:

\[
M \begin{bmatrix} \mu \\ \eta \\ \hat{x} \end{bmatrix} \leq Nx + N_c, \quad \eta \in \{0,1\}^n \quad (13)
\]

By summarizing the previous results, one obtains

\[
\begin{align*}
&\max_{\hat{x}} \|x - \hat{x}\|_1 \text{ s. t. } \hat{x} \in \Im_T(x) \\
&\max_{\hat{x}, \eta, \mu} \sum_{i=1}^n \eta_i \text{ s. t. } \hat{x} \in \Im_T(x) \text{ and (13)} \\
&\max_{\nu} \sum_{i=1}^n \eta_i \text{ s. t. (8) and (13)}
\end{align*} \quad (14)
\]

\[
\begin{align*}
&\max_{\hat{x}} \|x - \hat{x}\|_1 \text{ s. t. } \hat{x} \in \Im_T(x) \\
&\max_{\hat{x}, \eta, \mu} \sum_{i=1}^n \eta_i \text{ s. t. } \hat{x} \in \Im_T(x) \text{ and (13)} \\
&\max_{\nu} \sum_{i=1}^n \eta_i \text{ s. t. (8) and (13)}
\end{align*} \quad (15)
\]

where, in (16),

\[
\nu = \left[ \hat{x}' \ U(T)' \ \Delta(T)' \ \Delta(T)' \ Z(T)' \ \hat{Z}(T)' \ \eta' \ \mu' \right]'.
\]

The mixed-integer linear inequalities (8) and (13) can be written in compact form as

\[
G\nu \leq g + Lx 
\]

(17)

Then, the optimization problem (16) is equivalent to the following one:

\[
\max_{\nu} c'\nu \\
\text{s.t. } G\nu \leq g + Lx
\]

(18)

where \( c'\nu = \sum_{i=1}^n \eta_i \). By direct comparison with (7), problem (18) define an mp-MILP problem with parameter vector \( x \in \mathcal{X} \).

Note that if \( \mathcal{X}_T^* \neq 0 \), Theorem 1 can be applied. In fact, if \( \hat{x} \in \mathcal{X}_T^* \), it holds \( \hat{x} \in \Im(x) \) that motivates the feasibility of (9) and then of the mp-MILP problem (18).

Therefore, by solving the mp-MILP problem (18) we obtain:

- a collection of polytopic regions \( \{C_j\}_{j=1}^s \subseteq \mathcal{X} \) (represented by the linear inequalities \( H_j x \leq h_j \)) such that \( \mathcal{X}_T^* = \bigcup_{j=1}^s C_j \).
- The matrices \( F_j \) and the vectors \( f_j \) such that \( \nu^*(x) = F_j x + f_j, \forall x \in C_j \).

In view of the equivalence between problems (9) and (18), the closed-form expression of \( \Gamma \) is given by

\[
\Gamma(x) = \Gamma_j(x) \quad \text{if } x \in C_j
\]

(19)

\[
\Gamma_j(x) = F_j^* x + f_j^*
\]

(20)

where \( F_j^* = [I_n \mid 0] F_j \) and \( f_j^* = [I_n \mid 0] f_j \).

The last step of the algorithm is the computation of the maximal observability region. From (10), we have that each set

\[
\hat{O}_j = \hat{O}_T \cap C_j = \{x \in C_j : F_j^* x + f_j^* = x\}
\]

(21)

is a (possibly empty) observability region. Moreover, (21) implies that \( \hat{O}_j \) is represented by the inequalities

\[
\begin{bmatrix} H_j \\ F_j^* - I_n \\ -F_j^* + I_n \end{bmatrix} x \leq \begin{bmatrix} h_j \\ -f_j \\ f_j \end{bmatrix}
\]

(22)

and that the maximal observability region can be recovered as by \( \hat{O}_T = \bigcup_{j=1}^s \hat{O}_j \).
Remark 5: One drawback in solving mp-MILP problems is that the number of region $C_j$ generated by the solver can be very large. However, since they are found in a sequential way, it is not necessary to store all of them in the memory. In fact, as soon as a new region $C_j$ and the corresponding function $\Gamma_j$ are found one can immediately compute $O_j$ and store it only if it is not empty.

Quite interesting, even if the overall maximal observability region is disconnected, it can be represented through a single system of mixed-integer linear inequalities $L \left[ x' \delta \right]' \leq l$, where $\delta = [\delta_1 \ldots \delta_n]'$ and $\delta_j \in \{0,1\}$ are additional auxiliary variables. A constructive procedure for obtaining the matrix $L$ and the vector $l$ is given in [4]. This implies that we can force the state of an MLD system to lie in $O_T$ by simply introducing the auxiliary variables $\{\delta_j(t)\}_{j=1}^n$ and adding the inequalities $L \left[ x(t) \delta(t) \right]' \leq l$ to (1c). In particular, when using this “augmented” MLD form in a Model Predictive Control (MPC) scheme (see [4] for an introduction of MPC control for MLD systems), the control law will guarantee that the state never leaves $O_T$. We guess that this observation, although trivial, might play a key role in designing output-feedback MPC schemes. The reason is that if the state $x(t)$, belongs to $O_T$, then $O_T$, the observer proposed in Section 4-D of [3] provides a perfect reconstruction of the state trajectory $x(t)$, $t > T - 1$ by using the last $T$ inputs $u(k)$ and outputs $y(k)$, $k = t, \ldots, t + T - 1$. Therefore, such a deadbeat observer can be applied in a moving-horizon fashion for guaranteeing perfect state reconstruction at each time instant $t \geq T - 1$. The real problem is how to control the system at the time instants $t = 0, 1, \ldots, T - 1$ when the deadbeat observer does not provide a correct state estimate. Moreover, the deadbeat property strongly depends on the assumption that the measurements are noiseless. For a discussion on the performance deterioration of the deadbeat observer in the noisy case we defer the reader to [13].

IV. AN ILLUSTRATIVE EXAMPLE

In this section we show the correctness of our algorithm through a toy example for which the observability region can be computed by hand.

Consider the piecewise linear system described by the state equation

\[
x(t + 1) = \begin{cases} 
  x(t) & \text{if } |x(t)| \leq 2 \\
  \frac{1}{2} x(t) & \text{if } |x(t)| \geq 2 + \epsilon
\end{cases} \tag{22}
\]

\[
y(t) = \begin{cases} 
  x(t) & \text{if } |x(t)| \leq 2 \\
  0 & \text{if } |x(t)| \geq 2 + \epsilon
\end{cases} \tag{23}
\]

where $\epsilon$ is a small tolerance. By imposing the state constraints $x \in [-10, 10]$, the system (22)-(23) can be written in the MLD form [4]. For sake of conciseness we do not give the matrices of the MLD form and simply highlight that the feasibility set $F$ defined in (2) is given by

\[
F = M_1 \cup M_2 \cup (-M_2)
\]

where $M_1 = [-2, 2]$ and $M_2 = [2 + \epsilon, 10]$. By choosing $T = 3$ we also have, from (6),

\[
X_T^* = M_1 \cup M_3 \cup (-M_3)
\]

where $M_3 = M_2 \backslash \{4, 4 + 2\epsilon \cup 8, 8 + 4\epsilon\}$.

In fact, if $x(0) \in \{4, 4 + 2\epsilon \cup 8, 8 + 4\epsilon\}$, it is easy to see that state will evolve, within the next two steps, in the set $[2, 2 + \epsilon]$, where the system dynamics is not defined (an analogous remark applies to $x(0) \in [-4 - 2\epsilon, -4 \cup -8 - 4\epsilon, -8]$). A simple calculation reveals that the maximal observability region is $O_T = [-8, 8] \cap X_T^*$. By running our algorithm, the set $X_T^*$ is partitioned in the intervals $C_1, \ldots, C_7$ given in Table I along with the corresponding optimizers $\hat{x}(x)$. According to (10), the maximal observability region is found as the union of the intervals $C_2, \ldots, C_6$, verifying $\hat{x}(x) = x$. Note that, if $x \in C_1$, the optimizer is constant. In view of (9), $\hat{x}$ corresponds to the farthest state in $F$ that is indistinguishable from all $x \in C_1$. The same remarks apply to the region $C_7$.

From the computational point of view, the result have been obtained in 30 sec. by running Matlab 5.3 on a 1Ghz Pentium III and using a Matlab implementation of the branch and bound algorithm.

V. CONCLUSIONS

In this paper we provided an algorithm for computing the maximal observability region of an hybrid system in the MLD form. In particular, multi-parametric programming is used as a tool for computing the polyhedral sets composing the maximal observability region.

We believe that these results, beside being instrumental to the synthesis of state observers, play also a key role in addressing open problems like model reduction and output-feedback controllers design for MLD systems.

ACKNOWLEDGMENTS

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<table>
<thead>
<tr>
<th>Region</th>
<th>$\hat{x}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1$</td>
<td>$-10, -8 - 4\epsilon$</td>
</tr>
<tr>
<td>$C_2$</td>
<td>$-8, -4 - 2\epsilon$</td>
</tr>
<tr>
<td>$C_3$</td>
<td>$-4, -2 - \epsilon$</td>
</tr>
<tr>
<td>$C_4$</td>
<td>$-2, 2$</td>
</tr>
<tr>
<td>$C_5$</td>
<td>$2, 4 + \epsilon$</td>
</tr>
<tr>
<td>$C_6$</td>
<td>$4, 8 + 2\epsilon$</td>
</tr>
<tr>
<td>$C_7$</td>
<td>$8, 10 + 4\epsilon$</td>
</tr>
</tbody>
</table>

TABLE I

RESULTS OF THE mp-MILP
VI. REFERENCES


