Moving Horizon Estimation for Hybrid Systems

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Abstract

We propose a state smoothing algorithm for hybrid systems based on Moving Horizon Estimation (MHE) by exploiting the equivalence between hybrid systems modeled in the Mixed Logic Dynamical form and piecewise affine systems. We provide sufficient conditions on the time horizon and the penalties on the state at the beginning of the estimation horizon to guarantee asymptotic convergence of the MHE scheme. Moreover, we propose two practical algorithms for the computation of such penalties that allow to implement MHE by solving a Mixed-Integer Quadratic Program.

Keywords

Hybrid systems, state estimation, observability, piecewise affine systems, mixed-integer quadratic programming

1 Introduction

In recent years the attention of both academia and industry has been attracted by hybrid systems, namely systems where dynamical and logical components interact. The motivation is twofold: first, many industrial plants naturally exhibit a hybrid behaviour because they are constituted from dynamical components at the lower level and upper level logical components. Second, the rapid advances in computer and information technology are making it possible to efficiently solve problems that only a few years ago were prohibitive, for example, optimization involving both continuous and logical variables [16] or symbolic model checking [13, 33]. This promoted the development of models for hybrid systems [20] as well as the generalization of classical topics in control and system theory to the hybrid case such as stability [11, 6], regulator design [12, 6, 36, 1], controllability and observability [22, 32, 3].

In [35, 6] it was shown that, by recasting logical propositions into linear inequalities on Boolean variables, it is possible to develop a unified model for a wide range of discrete-time hybrid systems, the *Mixed Logic Dynamical (MLD)* system form [6]. For instance, linear hybrid dynamical systems, hybrid automata, piecewise affine systems and some classes of discrete-event

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systems can be represented in the MLD form. Apart from their modeling features, MLD models are also suitable to solve many analysis and control problems by expressing them as mixed-integer programs, like model predictive control [6], verification problems [7, 9], observability and controllability tests [3].

In this paper we focus on state-estimation for hybrid systems in the MLD form. Very little research has focused on observer theory for hybrid systems, apart from the work of Sontag [32]. It is apparent that the ability to reconstruct the state of the system is fundamental for developing both state feedback control schemes and fault detection algorithms.

For the MLD case, the state-estimation problem has been addressed first in [5], where a *Moving Horizon Estimation (MHE)* strategy was shown to perform effectively for fault detection, although the theoretical properties of such a scheme were not investigated.

The ideas of MHE date back to the early nineties [23]. MHE is appealing because of its capability to incorporate nonlinearities and constraints on states and disturbances. Moreover, from a computational point of view, MHE algorithms are suitable for practical implementation because they amount to optimization problems of finite dimension. In [29], Rao and coworkers applied MHE to constrained linear systems, showing that it can guarantee stability of the estimate. This fact is important because in [27] and [34] examples are reported where the presence of constraints can force instability of the classical Kalman filter. The use of an MHE scheme for nonlinear plants was investigated in [23, 24, 30] and a detailed analysis of its stability properties was given in [28] under the assumption that the state-transition and output maps are Lipschitz continuous.

All the results of this paper hinge on the equivalence between MLD and PWA systems [3] that is recalled in Section 2. PWA systems stemming from the translation of MLD forms have typically discontinuous state-transition and output maps. In this case, even if constraints are not present, local convergence of the Extended Kalman Filter cannot be guaranteed because it is usually proved by exploiting some degree of smoothness of the system [31]. Nevertheless, the ideas of [28] are the basis for studying the convergence of the smoothed estimate obtained via MHE. In particular, as in [28], our analysis does not rely on the knowledge of the noise statistics, even though MHE can be viewed as an approximation of a Bayesian estimation strategy [27].

The MHE scheme we consider is introduced in Section 3, and the convergence analysis is developed in Section 4. Under mild assumptions, we derive sufficient conditions for convergence involving both the horizon length and the penalties on the state at the beginning of the horizon (hereafter termed initial penalties) that have to be used in order to properly take into account the effect of the neglected data. In Section 5 the implementation of MHE as a *Mixed-Integer Quadratic Program* is discussed. The key issue in computation is how to generate quadratic initial penalties in an efficient way. For this purpose, we propose two algorithms: the first one is faster, the second one more accurate. The noise-rejection benefits gained by properly choosing the initial penalties are further discussed in Section 6 through an illustrative example.

2 The Mixed Logic Dynamical Form of Hybrid Systems

In this paper we consider hybrid systems that can be described in the MLD form introduced in [6]. The equations describing an MLD system are

$$x(t+1) = Fx(t) + G_1u(t) + G_2\delta(t) + G_3z(t) \tag{1a}$$
$$y(t) = Hx(t) + D_1u(t) + D_2\delta(t) + D_3z(t) \tag{1b}$$
$$E_2\delta(t) + E_3z(t) \le E_1u(t) + E_4x(t) + E_5 \tag{1c}$$

where $x \in \mathbb{R}^{n_c} \times \{0,1\}^{n_\ell}$ are the continuous and binary states, $u \in \mathbb{R}^{m_c} \times \{0,1\}^{m_\ell}$ are the inputs, $y \in \mathbb{R}^{p_c} \times \{0,1\}^{p_\ell}$ the outputs, and $\delta \in \{0,1\}^{r_\ell}$, $z \in \mathbb{R}^{r_c}$ represent auxiliary binary and continuous variables respectively. The auxiliary variables are introduced when translating propositional logic into linear inequalities. For a detailed description of the modeling capabilities of MLD systems, we defer to [6, 3, 14]. All constraints on state, input, and auxiliary variables are summarized in the inequality (1c). Note that, despite the fact that equations (1a)-(1b) are formally linear, the nonlinearity is hidden in the integrality constraints over the binary variables. We assume that system (1) is *completely well-posed* [6], or, equivalently, that given $x(t)$ and $u(t)$, the values of $\delta(t)$ and $z(t)$ are uniquely defined through the inequalities (1c). This assumption is not restrictive and is satisfied automatically when real-world plants are described in the MLD form [6].

Recently, a formal equivalence between the class of MLD and Piece-Wise Affine (PWA) systems was established in [3]. PWA systems are described by the state-space equations

$$\begin{array}{rcl} x(t+1) & = & A_ix(t) + B_iu(t) + f_i \\ y(t) & = & C_ix(t) + g_i \\ (x,u) \in \mathbb{X} & & \end{array} \quad , \text{ for } \left[\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right] \in \mathcal{X}_i \tag{2}$$

where the state+input set $\mathbb{X} \subset \mathbb{R}^{n_c} \times \{0,1\}^{n_\ell} \times \mathbb{R}^{m_c} \times \{0,1\}^{m_\ell}$ is a bounded polyhedron, $\{\mathcal{X}_i\}_{i=1}^s$ is a polyhedral partition[1] of $\mathbb{X}$ and f_i, g_i are constant vectors of suitable dimension. In [6] it is shown that PWA systems can be represented in the MLD form. The converse is stated in the next proposition.

Proposition 1 ([2]). *Consider generic trajectories $x(t)$, $u(t)$, $y(t)$ of the MLD system (1). Then there exists a finite polyhedral partition $\{\mathcal{X}_i\}_{i=1}^s$ of the state+input set*

$$\{(x,u) \in \mathbb{R}^{n_c} \times \{0,1\}^{n_\ell} \times \mathbb{R}^{m_c} \times \{0,1\}^{m_\ell} \textit{ such that } \exists z \in \mathbb{R}^{r_c}, \delta \in \{0,1\}^{r_\ell} \textit{ satisfying (1c) } \}$$

and 5-tuples $(A_i, B_i, C_i, f_i, g_i)$, $i = 1, \ldots, s$, such that $x(t)$, $u(t)$, $y(t)$ satisfy (2).

Proposition 1 was proved in [2] by using a constructive argument that allows the explicit computations of the sets $\mathcal{X}_i$ and the matrices A_i, B_i, C_i, f_i and g_i defining the PWA system.

[1] Each set $\mathcal{X}_i$ is a (not necessarily closed) convex polyhedron s.t. $\mathcal{X}_i \bigcap \mathcal{X}_j = \emptyset$, $\forall i \neq j$, $\bigcup_{i=1}^s \mathcal{X}_i = \mathbb{X}$.

3 State-Estimation for Hybrid Systems: an MHE Scheme

In view of Proposition 1, the state-estimation problem for MLD systems can be formulated, in an equivalent way, for the class of PWA systems.

As customary in filtering theory, when noises are present on the input and the output we consider *autonomous* PWA systems of the form

$$\hat{\Sigma}: \quad x(t+1) = A_i x(t) + f_i + w(t) \tag{3a}$$

$$y(t) = C_i x(t) + g_i + v(t), \quad \text{for } x(t) \in \mathcal{X}_i \tag{3b}$$

$$x \in \mathbb{X} \tag{3c}$$

$$w \in \mathbb{W} \tag{3d}$$

where $v(t) \in \mathbb{R}^{p_c} \times \{0,1\}^{p_\ell}$ and $\mathbb{W} \subset \mathbb{R}^{n_c} \times \{0,1\}^{n_\ell}$ is a bounded polyhedron containing the origin (i.e. there exist two matrices E_6 and E_7 such that $E_6 w \leq E_7$) . The signals w and v model unmeasured input and output disturbances respectively. Note that we allow binary disturbances on the binary components of the state and the output. Moreover the condition $w \in \mathbb{W}$ takes into account constraints on the input disturbances.

Model (3) embraces also the common case when the input noise affects only the first n_c continuous components of the state. In this case the noise model is $w(t) = \left[\begin{smallmatrix} w_c \\ w_\ell \end{smallmatrix}\right](t) = \left[\begin{smallmatrix} \tilde{B} \\ 0 \end{smallmatrix}\right]\tilde{w}(t)$, $\tilde{w}(t) \in \tilde{\mathbb{W}} \subset \mathbb{R}^q$, where $\tilde{\mathbb{W}}$ is a bounded polyhedron. Indeed, it is apparent that the linear map $\left[\begin{smallmatrix} \tilde{B} \\ 0 \end{smallmatrix}\right]$ defines a new polyhedron $\mathbb{W}$ in the w-space.

We introduce the cost functional

$$J(\tau, t, w, v, x(\tau), \Gamma_\tau) \triangleq \sum_{k=\tau}^{t-1} \|v(k)\|_R^2 + \|w(k)\|_Q^2 + \Gamma_\tau(x(\tau)),$$

where $\tau, t \in \mathbb{N}$, $\tau < t$, Γ_τ is a continuous function and Q and R are positive-definite matrices of suitable dimension.

Consider a generic time instant T and assume that the output samples $y(k)$, $k = 0, \ldots, T-1$ have been measured. The *Full Information (FI)* observer is defined as the estimation of the state trajectory $x(k)$ obtained by solving the constrained minimization problem

$$\min_{x(0),w} J(0, T, w, v, x(0), \Gamma_0) \text{ subj. to (3).} \tag{4}$$

From (4) it is apparent that the matrices Q and R weight the deviation between measured and predicted state/output. In a probabilistic setting, they should be chosen as the inverse of the covariance matrices of the noises w and v. Similarly, Γ_0 should reflect the confidence we have about how much the *true* initial state of (3) differs from an *initial guess* $\bar{x}$ of the state $x(0)$. Then, we introduce the following assumption on the shape of Γ_0.

Assumption 1. *The function Γ_0 is bounded on $\mathbb{X}$ and such that $\Gamma_0(x) \geq 0$, $\forall x \in \mathbb{X}$ and $\arg\min_{x \in \mathbb{X}} \Gamma_0(x) = \bar{x}$.*

The FI observer uses all the collected output samples $y(k)$, $k = 0, 1, \ldots, T$ in order to reconstruct the state trajectory. The use of all the available information is desirable for estimation but is unappealing from a computational point of view. Indeed, the main drawback of the FI scheme is that, as T increases, the size of the optimization problem (4) grows without bounds. To overcome this problem, one can use forward dynamic programming. Consider a time instant $T > M$, where $M \in \mathbb{N}^+$ is a fixed horizon. Then one can rewrite (4) as

$$\min_{x(T-M),w} J(T-M, T, w, v, x(T-M), \bar{\Gamma}_{T-M}) \text{ subj. to (3)} \tag{5}$$

where

$$\bar{\Gamma}_{T-M}(z) \triangleq \min_{\substack{x(0),\, w \\ x(T-M)=z}} J(0, T-M, w, v, x(0), \Gamma_0) \text{ subj. to (3).} \tag{6}$$

and z belongs to the reach set at time $T-M$. The size of the optimization (5) is now bounded and the penalty $\bar{\Gamma}_{T-M}$ (the so-called *arrival cost*) relates the fixed-horizon problem (5) to the FI problem. As discussed in [28], the arrival cost intuitively summarizes the information carried by the past data $y(k)$, $k = 0, \dots, T-M-1$.

If the system (3) is linear and unconstrained, we can compute the arrival cost by using the Kalman Filter covariance update recursion. For the general PWA form, the exact calculation of the arrival cost is a formidable problem. In a MHE scheme we maintain the computational advantages of (5) by solving, at each time instant $T \geq 0$

$$\Theta^*{}_T = \min_{x(T-M),w} J(T-M, T, w, v, x(T-M), \Gamma_{T-M}) \text{ subj. to (3),} \tag{7}$$

where, by convention $T-M$ is set to zero if $T < M$ and Γ_{T-M} is a sequence of penalties at the beginning of the horizon (hereafter called *initial penalties*) that should "approximate" in a suitable sense the arrival cost $\bar{\Gamma}_{T-M}$. In fact, note that (7) and the FI estimator (5) coincide if $\bar{\Gamma}_{T-M} = \Gamma_{T-M}$. We will use the following notations: The smoothed estimates of the state obtained by solving (7) is denoted as $\hat{w}(k|T)$, $\hat{v}(k|T)$, $\hat{x}(k|T)$, for $k = T-M, \dots, T-1$. The predicted state at time T is $\hat{x}(T|T)$.

Remark 1. The estimation scheme described above can be applied to the fault detection problem of hybrid systems described in MLD form. A first application was reported in [5]. We assume that the dynamics of the system is known in the presence of each fault and that the faults are modeled as binary states in MLD form. Since it is often convenient to describe the fault evolution with logical relations and the corresponding mixed-integer inequalities rather than a direct state update, we can associate a binary variable δ with each fault. The scheme is also applicable to the case of off-line process data analysis, commonly referred to as "data mining" [26]. In this case the data records are used to explain possible malfunctions of the process a posteriori .

Remark 2. From a mathematical point of view we cannot ensure that a solution to the optimization problems (4), (5), (6) and (7) exists. This is due to the fact that the regions $\mathcal{X}_i$ are not necessarily closed sets and the state-transition and output maps may be discontinuous on the boundary $\partial\mathcal{X}_i$ of $\mathcal{X}_i$. In fact it can happen that in order to minimize J, an estimate of the state must be pushed from the interior of some open set $\mathcal{X}_i$ arbitrarily close to $\partial\mathcal{X}_i$. In this case the operator "min" should be replaced with "inf" and the minimizer is not well-defined. Note also that this is the only reason for which the minimum can not be attained. To circumvent this technical issue, we implicitly assume that there is a solution to the minimization problems we consider. In practice this assumption is of little relevance because computations are always performed with finite-precision and an approximate solution of a minimization problem is always obtained.

4 Convergence Analysis of MHE

Before analyzing convergence properties of MHE for PWA systems, we recall what *observer convergence* means. Assume that the data $y(t)$ are generated by the noiseless model

$$\Sigma: \quad \begin{aligned} x(t+1) &= A_i x(t) + f_i \\ y(t) &= C_i x(t) + g_i, \quad \text{for } x(t) \in \mathcal{X}_i \\ x(0) &= x_0,\ x \in \mathbb{X}, \end{aligned}$$

where x_0 is the *true* initial condition. We denote with $x_\Sigma(t, x(\tau))$ the state $x(t)$ reached by Σ at time t starting from $x(\tau)$, $\tau < t$. The output map is $y_\Sigma(t, x(\tau))$. Analogously, $x_{\hat{\Sigma}}(t, x(\tau), w)$ and $y_{\hat{\Sigma}}(t, x(\tau), w, v)$ are the state and output trajectories of the PWA model $\hat{\Sigma}$, when the noise sequences $\{w(k)\}_{k=\tau}^{t-1}$ and $\{v(k)\}_{k=\tau}^{t}$ are applied.

Assumption 2. $x_\Sigma(t, x_0) \in \mathbb{X}$, $\forall t \in \mathbb{N}^+$.

Assumption 2 states that the state-trajectory we aim at reconstructing fulfills the constraints at every time instant and corresponds to the assumption that the state evolution is always described by the model Σ. This fact implies the feasibility of the optimization problem (7) at every time instant since the choice $x(T-M) = x_\Sigma(T-M, x_0)$, $w(k) = 0$, $v(k) = 0$, $i = T-M, \dots, T-1$ is feasible.

Definition 1. *For a given $\tau \in \{0, 1, \dots, M\}$, the MHE scheme (7) is τ-convergent if*

$$\lim_{T \to +\infty} \|x_\Sigma(T-\tau, x_0) - \hat{x}(T-\tau|T)\| = 0.$$

In other words, a τ-convergent MHE scheme can track the state trajectory $x(T-\tau)$ of the autonomous system Σ in a correct way, at least asymptotically (for $T \to +\infty$) and independently from the initial guess $\bar{x}$. In particular, for $\tau = 0$ we recover the usual definition of convergence of the predicted state $\hat{x}(T|T)$ to $x(T)$. Definition 1 is introduced because we will focus on the convergence of the *smoothed estimate* $\hat{x}(T-\tau|T)$ rather than convergence of $\hat{x}(T|T)$, as usual in observers theory.

The MHE scheme (7) has mainly two degrees of freedom that can be adjusted in order to achieve convergence: The length M of the horizon and the initial penalty Γ_{T-M}. The former is related to the incremental observability of Σ.

Definition 2. *The PWA system (2) is* incrementally observable in $\bar{T}$ steps on $\mathbb{X}$ *or simply* incrementally observable *if there exist two norms* $\|\cdot\|_S$ *(on* $\mathbb{R}^{n_c+n_\ell}$*) and* $\|\cdot\|_R$ *(on* $\mathbb{R}^{p_c+p_\ell}$*) and a positive scalar* w_{RS} *such that* $\forall x_1, x_2 \in \mathbb{X}$

$$\sum_{t=0}^{k-1} \|y(t,x_1) - y(t,x_2)\|_R \geq w_{RS}\|x_1 - x_2\|_S \tag{9}$$

for $k = \bar{T}$ *and (9) does not hold for* $k < \bar{T}$.

Remark 3. For a linear system Definition 2 reduces to the classical Grammian test. For a nonlinear system it is similar to the O property given in [18]. In the nonlinear case, it is often difficult to check incremental observability by using Definition 2 directly. In particular, as discussed in [3], the observability of the overall PWA system Σ cannot be deduced from the observability of the component subsystems (A_i, f_i, C_i, g_i), $i = 1, \ldots, s$. Moreover the observability index $\bar{T}$ is not related to the dimension of the state-space, as happens for linear systems. By exploiting the equivalence between MLD and PWA systems, an algorithm for checking the incremental observability based on mixed-integer linear programming was given in [3].

As described in [3], incremental observability implies the existence of a deadbeat observer for system Σ when the input/output noises are neglected in the estimation scheme. Indeed, incremental observability is a minimal requirement for the design of a state observer. However, deadbeat observers have poor filtering properties and their performance deteriorates when inputs and outputs are affected by noise.

In our analysis, the incremental observability condition is instrumental for determining the minimum horizon length that must be chosen in order to guarantee convergence of MHE.

Assumption 3. *The system* Σ *is incrementally observable in* $\bar{T}$ *steps and* $M \geq \bar{T}$.

The second critical issue for convergence is the choice of the initial penalty Γ_{T-M}. In fact, an incorrect choice of the initial penalty results in a misleading treatment of the information carried by the past data [24]. In order to guarantee convergence, we aim at approximating the arrival cost with Γ_{T-M}. Since the computation of the arrival cost amounts to an optimization problem of increasing size, we approximate it over a finite horizon in a recursive way. With a slight abuse of terminology, hereafter we shall refer as arrival cost also to the map

$$\Xi_T(z) \triangleq \min_{\substack{x(T-M), w \\ x(T) = z}} J(T-M, T, w, v, x(T-M), \Gamma_{T-M}) \text{ subj. to (3)}, \tag{10}$$

where z belongs to the reach set at time T and $T - M$ is set to zero if $T < M$. The next assumption gives the key condition on the initial penalties Γ_{T-M} in order to ensure convergence of MHE.

Assumption 4. *The initial penalties* Γ_T *satisfy* $\Theta^*{}_T \leq \Gamma_T(x) \leq \Xi_T(x)$, $\forall T \in \mathbb{N}^+$, $\forall x \in \mathbb{X}$.

Remark 4. Note that it holds $\Theta^*{}_{T-M} = \Xi_{T-M}(\hat{x}(T-M|T-M))$ and therefore, from Assumption 4, $\hat{x}(T-M|T-M)$ must be a minimizer of Γ_{T-M}. Then, in practice we choose

Γ_{T-M} penalizing the deviations between the estimated state $\hat{x}(T-M|T-M)$ obtained by solving (7) and $\hat{x}(T-M|T)$.

Note that $\hat{x}(T-M|T-M)$ is not the optimal initial guess for $x(T-M)$ because also $\hat{x}(T-M|T-1)$, (that takes into account the data $y(k)$, $k = T-M-1, \dots, T-1$) is available at time T. Nevertheless, the former choice will allow to design efficient algorithms for computing non-trivial initial penalties that are lower bounds to the arrival cost (see Section 5).

Before stating the main convergence result we introduce a condition concerning the behaviour of the MHE scheme on the *edge set* induced by the partition $\{\mathcal{X}_i\}_{i=1}^s$.

Definition 3. *The state $x \in \mathbb{X}$ is an edge point if there exists at least two indices i,j, $i \neq j$ such that $\forall \epsilon > 0$, $B(x,\epsilon) \cap \mathcal{X}_i \neq \emptyset$ and $B(x,\epsilon) \cap \mathcal{X}_j \neq \emptyset$ ($B(x,\epsilon)$ denotes the open ball of radius ϵ centered in x). We denote with Δ the edge set, i.e. the set of edge points.*

Assumption 5. *There is only a finite number of time instants T such that $\exists t \in \{T-M, \dots, T-1\}$ yielding $x_\Sigma(t, \hat{x}(T-M|T)) \in \Delta$.*

Assumption 5 is mainly technical (see Lemma 5 in Appedix A) and requires that the free evolution of the smoothed estimates $\hat{x}(T-M|T)$ over M steps does not belong (at least asymptotically) to the edge set. We point out that in practical applications this assumption is seldom restrictive because the set Δ has zero measure.

Theorem 1. *If Assumptions 1-5 hold, the MHE scheme is τ-convergent for all $\tau \in \{\bar{T}, \dots, M\}$.*

Proof. The proof is reported in Appendix A. □

Note that Theorem 1 guarantees convergence of the smoothed estimate with a delay of at least $\bar{T}$ steps, where $\bar{T}$ depends on the incremental observability property. For smooth systems, it is possible to prove also the convergence of $\hat{x}(T|T)$ to $x(T)$ by invoking a Lipschitz argument [28] but unfortunately it is not possible to use the same rationale for a general PWA system.

Remark 5. When applied to the fault detection problem, the MHE scheme allows to estimate at time T the faults at time $T-\bar{T}$. The delay for fault estimation is therefore at least $\bar{T}$ time steps. Since longer estimation horizons naturally lead to more accurate estimations, the choice of $M \geq \bar{T}$ determines the trade-off between accuracy and real time requirements. Note, however, that a delay between fault occurrence and its estimation is inherent in many fault detection schemes. In [21], where a qualitative process model is used and fault detection is performed with a stochastic automaton on a three-tank system, the authors report unambiguous fault detection after about 20 time steps. A parameter estimation scheme for the same system, as described in [19] leads to estimation delays that critically depend on the parameters of the estimation scheme. The delays are in the order of 5 time steps.

5 Determining the Initial Penalty Γ_T

In order to completely specify the MHE scheme, one has to choose the initial penalty Γ_T according to Assumption 4. The trivial choice $\Gamma_T = \Theta^*{}_T$ is always possible but this corresponds to ignore completely the effect of past data $y(t)$, $t = 0, \dots, T-1$. This is not a major problem if the output measurements are generated by the noiseless system Σ. However, when disturbances are present, this choice leads to a MHE scheme that is very sensitive to noise, especially if the horizon M is short [34]. In this case, the filtering property depends on our ability to compute $\Gamma_T(x)$ as close to $\Xi_T(x)$ as possible.

On the other hand, when looking for good initial penalties, one has to take into account also the computational implementation of MHE. By using the equivalence between MLD and PWA systems, we can write (7) as

$$\min_{\substack{x(T-M)\\ \{w(t),\delta(t),z(t)\}_{t=T-M}^{T-1}}} \sum_{k=T-M}^{T-1} \|v(k)\|_R^2 + \|w(k)\|_Q^2 + \Gamma_{T-M}(x(T-M)) \tag{11}$$

subject to

$$x(t+1) = Fx(t) + G_2\delta(t) + G_3 z(t) + w(t) \tag{12a}$$

$$y(t) = Hx(t) + D_2\delta(t) + D_3 z(t) + v(t) \tag{12b}$$

$$E_2\delta(t) + E_3 z(t) \le E_4 x(t) + E_5 \tag{12c}$$

$$E_6 w(t) \le E_7 \tag{12d}$$

where the inequalities $E_6 w(t) \le E_7$ represent the condition $w \in \mathbb{W}$. Due to the presence of integer variables, the minimization (11) is, in general a *Mixed-Integer Nonlinear Program* that can be computationally very demanding. However, the cost functional in (11) can be made quadratic if Γ_{T-M} is a quadratic form, so that the more efficient solvers for *Mixed-Integer Quadratic Programs* [25, 17, 4] can be used. Then, it is crucial, for practical implementation, to approximate the arrival cost with penalties of the type

$$\Gamma_T(z) = (z - \hat{x}(T|T))'\Psi_T(z - \hat{x}(T|T)) + \nu_T. \tag{13}$$

In the sequel we propose two algorithms to compute quadratic penalties such that the condition $\Gamma_T(z) \le \Xi_T(z)$ is fulfilled. Before describing the first one, we introduce a preliminary Lemma.

Hereafter, for sake of simplicity, we denote with $A_{x(t)}$ ($\mathcal{X}_{x(t)}$, $\phi_{x(t)}$) the matrix A_i (the set $\mathcal{X}_i$, the map ϕ_i) whose index i is uniquely defined by the relation $x(t) \in \mathcal{X}_i$.

Lemma 1. *Consider the optimization problem*

$$\beta^* = \min_{x\in\mathbb{X}} x'H_x x - 2x'h_x + \mu_x \tag{14}$$

Where H_i are positive-definite matrices and h_i are column vectors of suitable dimension. Then,

$$\beta^* \ge \min\left\{-h_i' H_i^{-1} h_i + \mu_i,\ i = 1,\dots,s\right\} \tag{15}$$

Proof. The thesis follows easily by relaxing the constraints in the following way

$$\begin{aligned}
\beta^* &= \min\left\{\min_{x\in\mathcal{X}_i} x'H_i x - 2x'h_i + \mu_i,\ i=1,\dots,s\right\} \ge \\
&\ge \min\left\{\min_{x\in\mathbb{R}} x'H_i x - 2x'h_i + \mu_i,\ i=1,\dots,s\right\} = \\
&= \min\left\{-h_i' H_i^{-1} h_i + \mu_i,\ i=1,\dots,s\right\}
\end{aligned}$$

□

5.1 Algorithm 1

Consider the time $k = T - M$. By using forward dynamic programming we can write the arrival cost as,

$$\Xi_T(\xi) = \min_{\substack{x(k+1), w, \\ x(T)=\xi}} J(k+1, T, w, v, x(k+1), \tilde{\Xi}_{k+1}) \text{ subj. to (3)}, \tag{16}$$

$$\tilde{\Xi}_{k+1}(z) = \min_{\substack{x(k), w \\ x(k+1)=z}} J(k, k+1, w, v, x(k), \Gamma_k) \text{ subj. to (3)}. \tag{17}$$

As a consequence of Remark 4 and equation (13), the initial penalty can be expressed as $\Gamma_k(z) = z'\Psi_k z - 2z'\psi_k + \mu_k$, where $\psi_k = \Psi_k \hat{x}(k|k)$ and $\mu_k = \nu_k + \hat{x}'(k|k)\Psi_k \hat{x}(k|k)$. Then, by relaxing the constraint $w \in \mathbb{W}$, we obtain

$$\tilde{\Xi}_{k+1}(z) \geq \min_{x(k)\in\mathbb{X}} \phi_{x(k)}(x(k), z)$$

where

$$\phi_{x(k)}(x(k), z) \triangleq x'(k)H_{x(k)}x(k) - 2x'(k)L_{x(k)} + N_{x(k)} \tag{18}$$

$$H_i = C_i'RC_i + A_i'QA_i + \Psi_k \tag{19}$$

$$L_i = C_i'R(y(k) - g_i) + A_i'Q(z - f_i) + \psi_k \tag{20}$$

$$N_i = \mu_k + (y(k) - g_i)'R(y(k) - g_i) + (z - f_i)'Q(z - f_i) \tag{21}$$

From Lemma 1, it follows that, $\forall z \in \mathbb{X}$

$$\min_{x(k)\in\mathbb{X}} \phi_{x(k)}(x(k), z) \geq \min\left\{-L_i'H_i^{-1}L_i + N_i,\ i = 1, \ldots, s\right\},$$

where, by direct calculation,

$$-L_i'H_i^{-1}L_i + N_i = z'\tilde{\Psi}_i z - 2z'\tilde{\psi}_i + \tilde{\mu}_i, \tag{22a}$$

$$\tilde{\Psi}_i = -QA_iH_i^{-1}A_i'Q + Q \tag{22b}$$

$$\tilde{\psi}_i = QA_iH_i^{-1}M_i + Qf_i \tag{22c}$$

$$\tilde{\mu}_i = -M_i'H_i^{-1}M_i + \mu_k + (y(k) - g_i)'R(y(k) - g_i) + f_i'Qf_i \tag{22d}$$

$$M_i = C_i'R(y(k) - g_i) - A_i'Qf_i + \psi_k \tag{22e}$$

Note that, from (19), the matrix H_i is positive definite and also $\tilde{\Psi}_i$ is positive-definite by the property of Riccati equations. Now, compute $\bar{\Psi}_{k+1} \in \mathbb{R}^{n\times n}$, $\bar{\psi}_{k+1} \in \mathbb{R}^{n\times 1}$, $\bar{\mu}_{k+1} \in \mathbb{R}$ such that

$$\begin{aligned} \Xi_{k+1}(z) &\triangleq z'\bar{\Psi}_{k+1}z - 2z'\bar{\psi}_{k+1} + \bar{\mu}_{k+1} \leq \\ &\leq \min\{z'\tilde{\Psi}_i z - 2z'\tilde{\psi}_i + \tilde{\mu}_i,\ i = 1, \ldots, s\}. \end{aligned} \tag{23}$$

It is apparent that $\Xi_{k+1}(z)$ is a lower bound to $\tilde{\Xi}_{k+1}(z)$. It follows that

$$\Xi_T(z) \geq \min_{\substack{x(k+1), w \\ x(T)=z}} J(k+1, T, w, v, x(k+1), \Xi_{k+1}) \text{ subj. to (3)}. \tag{24}$$

Now, we can proceed to the next step $k+1$. By again using forward dynamic programming, formula (24) can be written as

$$\Xi_T(\xi) \geq \min_{\substack{x(k+2),w,\\ x(T)=\xi}} J(k+2,T,w,v,x(k+2),\tilde{\Xi}_{k+2}) \text{ subj. to (3)}, \quad (25)$$

where

$$\tilde{\Xi}_{k+2}(z) = \min_{\substack{x(k+1),\\ x(k+2)=z}} J(k+1,k+2,w,v,x(k+1),\Xi_{k+1}) \text{ subj. to (3)}. \quad (26)$$

Note that (25) and (26) have the same form as (16)-(17) since Ξ_{k+1} is a quadratic function. Then, by applying the arguments described above in a recursive way until $k = T-1$ one obtains a global lower bound to the arrival cost.

Algorithm 1.

1. `At time step` T `set` $n = n_c + n_\ell$, $\bar{\Psi}_{T-M} = \Psi_{T-M}$, $\bar{\psi}_{T-M} = \Psi_{T-M}\hat{x}(T-M|T-M)$, $\bar{\mu}_{T-M} = \nu_{T-M} + \hat{x}'(T-M|T-M)\Psi_{T-M}\hat{x}(T-M|T-M)$
2. `for` $k = T-M, \ldots, T-1$

2.1. `for` $i = 1, \ldots, s$

$$H_{i,k} = C_i'RC_i + A_i'QA_i + \bar{\Psi}_k \quad (27a)$$

$$M_{i,k} = C_i'R(y(k)-g_i) - A_i'Qf_i + \bar{\psi}_k \quad (27b)$$

$$\tilde{\Psi}_{i,k} = -QA_iH_{i,k}^{-1}A_i'Q + Q \quad (27c)$$

$$\tilde{\psi}_{i,k} = QA_iH_{i,k}^{-1}M_{i,k} + Qf_i \quad (27d)$$

$$\tilde{\mu}_{i,k} = -M_{i,k}'H_{i,k}^{-1}M_{i,k} + \bar{\mu}_k + (y(k)-g_i)'R(y(k)-g_i) + f_i'Qf_i \quad (27e)$$

2.2. `if` $k < T-1$ `compute` $\bar{\Psi}_{k+1} \in \mathbb{R}^{n\times n}$, $\bar{\psi}_{k+1} \in \mathbb{R}^{n\times 1}$, $\bar{\mu}_{k+1} \in \mathbb{R}$ `such that`

$$\begin{aligned} \Xi_{k+1}(z) &\doteq z'\bar{\Psi}_{k+1}z - 2z'\bar{\psi}_{k+1} + \bar{\mu}_{k+1} \leq \\ &\leq \min\{z'\tilde{\Psi}_{i,k}z - 2z'\tilde{\psi}_{i,k} + \tilde{\mu}_{i,k},\ \ i = 1,\ldots,s\}. \end{aligned} \quad (28)$$

2.3. `if` $k = T-1$ `compute` Ψ_T, `and` ν_T `such that`

$$\begin{aligned} \Gamma_T(z) &\doteq (z-\hat{x}(T|T))'\Psi_T(z-\hat{x}(T|T)) + \nu_T \leq \\ &\leq \min\{z'\tilde{\Psi}_{i,T-1}z - 2z'\tilde{\psi}_{i,T-1} + \tilde{\mu}_{i,T-1},\ \ i = 1,\ldots,s\}. \end{aligned}$$

We did not mention yet how to compute the matrices $\bar{\Psi}$, $\bar{\psi}$ and the coefficient $\bar{\mu}$ in steps 2.2. and 2.3. We suggest a procedure based on Linear Matrix Inequalities (LMIs) [10]. In step 2.2., for a given set of design points $\{\bar{x}_1, \cdots, \bar{x}_r\}$, the following LMI problem has to be solved,

$$\max_{\left[\begin{smallmatrix}\bar{\Psi}_k & -\bar{\psi}_k \\ -\bar{\psi}_k' & \bar{\mu}_k\end{smallmatrix}\right]} \sum_{j=1}^{r} \begin{bmatrix} \bar{x}_j' & 1 \end{bmatrix} \begin{bmatrix} \bar{\Psi}_k & -\bar{\psi}_k \\ -\bar{\psi}_k' & \bar{\mu}_k \end{bmatrix} \begin{bmatrix} \bar{x}_j \\ 1 \end{bmatrix} \tag{29}$$

$$\text{subj. to} \quad \begin{bmatrix} \bar{\Psi}_k & -\bar{\psi}_k \\ -\bar{\psi}_k' & \bar{\mu}_k \end{bmatrix} \leq \begin{bmatrix} \tilde{\Psi}_{i,k-1} & -\tilde{\psi}_{i,k-1} \\ -\tilde{\psi}_{i,k-1}' & \tilde{\mu}_{i,k-1} \end{bmatrix}, \; i = 1, \ldots, s. \tag{30}$$

$$\bar{\Psi}_k \geq 0 \tag{31}$$

The constraints (30) ensure that the solution is a global lower bound to each given paraboloid given by $\tilde{\Psi}_{i,k}$, $\tilde{\psi}_{i,k}$ and $\tilde{\mu}_{i,k}$. On the other hand the cost functional (29) pushes the solution up as much as possible in the design points $\bar{x}_i$. In order to have a well-defined solution, we use a number of design points that is larger than the number of degrees of freedom of the n-dimensional quadratic form, i.e. $r \geq n(n+1)/2 + n + 1$.

Obviously, the solution of (30) depends on the specific design points. A sensible choice would be to include in the set of design points the minimum point of every paraboloid $z'\tilde{\Psi}_{i,k}z - 2\tilde{\psi}_{i,k}'z + \tilde{\mu}_{i,k}$ that has to be lower bounded. Indeed, this would give a lower bound that minimizes the distance between $\min_x \Gamma_T(x)$ and Θ_T^*.

As for step 2.3., we require that the lower bound achieves its minimum at the point $\hat{x}(T|T)$ so that the minimum point of the arrival cost and of Γ_T coincide. This can be computed by solving the LMI problem

$$\max_{\left[\begin{smallmatrix}\Psi_T & 0 \\ 0 & \mu_k\end{smallmatrix}\right]} \sum_{j=1}^{r} \begin{bmatrix} \bar{x}_j' - \hat{x}(T|T)' & 1 \end{bmatrix} \begin{bmatrix} \Psi_T & 0 \\ 0 & \mu_T \end{bmatrix} \begin{bmatrix} \bar{x}_j - \hat{x}(T|T) \\ 1 \end{bmatrix} \tag{32}$$

$$\text{subj. to} \quad \begin{bmatrix} \Psi_T & -\hat{x}(T|T)'\Psi_T \\ -\Psi_T\hat{x}(T|T) & \mu_T \end{bmatrix} \leq \begin{bmatrix} \tilde{\Psi}_{i,T-1} & -\tilde{\psi}_{i,T-1} \\ -\tilde{\psi}_{i,T-1}' & \tilde{\mu}_{i,T-1} \end{bmatrix}, \; i = 1, \ldots, s. \tag{33}$$

$$\Psi_T \geq 0 \tag{34}$$

Algorithm 1 involves several approximations in order to obtain Γ_T. They are highlighted next by summarizing the main steps of its derivation.

a. The arrival cost is split in the "one-step" arrival costs $\tilde{\Xi}_{k+1}(z)$ defined in (17).

b. By removing the constraints $w \in \mathbb{W}$, $\tilde{\Xi}_{k+1}(z)$ is lower-bounded by the lower profile of the set of functions $\min_{x(k)\in\mathcal{X}_i} \phi_i(x(k), z)$, $i = 1, \ldots, s$ where $\phi_i(x(k), z)$ are defined by (18).

c. By removing the constraint $x(k) \in \mathcal{X}_i$, each minimization problem in point (b) is explicitly solved via the Kalman filter updates given by (22b)-(22e) so obtaining the quadratic forms (22a).

d. The lower profile of the quadratic forms (22a) is then lower bounded by a single paraboloid in steps 2.2. and 2.3. of Algorithm 1.

Since, in Algorithm 1, neither the constraints nor the structure of the partition $\{\mathcal{X}_i\}_{i=1}^s$ are taken into account, the condition $\Gamma_T(z) \geq \Theta^*{}_T$ is rarely satisfied, especially if the number of regions s is large.

On the computational side, we note that Algorithm 1 requires only sM Kalman Filter updates (27a)-(27e) and the solution of M LMI problems with less than $\frac{(n+1)^2}{2}$ unknowns.

5.2 Algorithm 2

In the sequel we propose a second algorithm that produces better lower bounds to the arrival cost. We first state a general result about quadratic programs.

Lemma 2. *Let $\mathbb{X} \subset \mathbb{R}^n$ and $\mathcal{X} \subset \mathbb{R}^n$ be two bounded polyhedra, $\mathcal{X} \subseteq \mathbb{X}$ and*

$$\phi(x,z) \triangleq \begin{bmatrix} x' & z' \end{bmatrix} \begin{bmatrix} V_1 & V_2 \\ V_2' & V_3 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} q_1' & q_2' \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \mu$$

where $V_1 > 0$, $V_3 > 0$, $V_3 - V_2' V_1^{-1} V_2 > 0$, $x \in \mathcal{X}$ and $z \in \mathbb{X}$.

Consider also $\Phi(z) \triangleq \min_{x \in \mathcal{X}} \phi(x,z)$. Then,

$$\begin{aligned} \Phi(z) &\geq p(z) \\ p(z) &\triangleq (z - z^*)'(V_3 - V_2' V_1^{-1} V_2)(z - z^*) + \phi(x^*, z^*) \end{aligned}$$

where (x^*, z^*) is the minimum point of $\phi(x,z)$.

Proof. The proof is reported in Appendix B. □

Set again $k = T - M$. We follow the rationale of Algorithm 1 and split the arrival cost as in (16)-(17). The constraints $w \in \mathbb{W}$ are removed in order to obtain the inequality

$$\tilde{\Xi}_{k+1}(z) \geq \min_{x(k) \in \mathbb{X}} \phi_{x(k)}(x(k), z)$$

where $\phi_{x(k)}(x(k), z)$ is defined by (18)-(21). In matrix notation,

$$\phi_i(x,z) \triangleq \begin{bmatrix} x' & z' \end{bmatrix} \begin{bmatrix} H_i & -A_i'Q \\ -QA_i & Q \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} -2M_i' & -2f_i'Q \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + P_i \tag{35}$$

where H_i is given by (19) and

$$M_i \triangleq C_i'R(y(k) - g_i) - A_i'Qf_i + \psi_k \tag{36}$$

$$P_i \triangleq \mu_k + (y(k) - g_i)'R(y(k) - g_i) + f_i'Qf_i. \tag{37}$$

Note that the matrix

$$\tilde{\Psi}_i \triangleq Q - QA_iH_i^{-1}A_i'Q, \tag{38}$$

is positive definite by a standard property of Riccati equations. From Lemma 2, it follows

$$\tilde{\Xi}_{k+1}(z) \geq \min\{(z - z_i^*)'\tilde{\Psi}_i(z - z_i^*) + \phi_i(x_i^*, z_i^*),\ i = 1, \ldots, s\}$$

where

$$(x_i^*, z_i^*) = \arg \min_{\substack{x \in \mathcal{X} \\ z \in \mathbb{X}}} \phi_i(x,z)$$

Now compute $\bar{\Psi}_{k+1} \in \mathbb{R}^{n\times n}$, $\bar{\psi}_{k+1} \in \mathbb{R}^{n\times 1}$, $\bar{\mu}_{k+1} \in \mathbb{R}$ such that

$$\begin{aligned}\Xi_{k+1}(z) &\doteq z'\bar{\Psi}_{k+1}z - 2z'\bar{\psi}_{k+1} + \bar{\mu}_{k+1} \leq \\ &\leq \min\{(z - z^*_{i,k})'\tilde{\Psi}_{i,k}(z - z^*_{i,k}) + \phi_{i,k}(x^*_{i,k}, z^*_{i,k}),\ i = 1, \dots, s\}\end{aligned}$$

It is apparent that Ξ_{k+1} is a quadratic lower bound to $\tilde{\Xi}_{k+1}$. Therefore, the recursive argument used in (24)-(26) can be applied to obtain the initial penalties that satisfy, by construction, $\Gamma_T(z) \leq \Xi_T(z)$.

Algorithm 2.

1. At time step T set $n = n_c + n_\ell$, $\bar{\Psi}_{T-M} = \Psi_{T-M}$, $\bar{\psi}_{T-M} = \Psi_{T-M}\hat{x}(T - M|T - M)$, $\bar{\mu}_{T-M} = \nu_{T-M} + \hat{x}'(T - M|T - M)\Psi_{T-M}\hat{x}(T - M|T - M)$
2. for $k = T - M, \dots, T - 1$

2.1. for $i = 1, \dots, s$

$$H_{i,k} = C_i'RC_i + A_i'QA_i + \bar{\Psi}_k \tag{39a}$$
$$M_{i,k} = C_i'R(y(k) - g_i) - A_i'Qf_i + \bar{\psi}_k \tag{39b}$$
$$P_{i,k} = \bar{\mu}_k + (y(k) - g_i)'R(y(k) - g_i) + f_i'Qf_i \tag{39c}$$
$$\tilde{\Psi}_{i,k} = -QA_iH_{i,k}^{-1}A_i'Q + Q \tag{39d}$$
$$\phi_{i,k}(x, z) \triangleq \begin{bmatrix} x' & z' \end{bmatrix}\begin{bmatrix} H_{i,k} & -A_i'Q \\ -QA_i & Q \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} -2M_{i,k}' & -2f_i'Q \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix} + P_{i,k} \tag{39e}$$
$$(x^*_{i,k}, z^*_{i,k}) = \arg \min_{\substack{x \in \mathcal{X} \\ z \in \mathbb{X}}} \phi_{i,k}(x, z) \tag{39f}$$

2.2. if $k < T - 1$ compute $\bar{\Psi}_{k+1} \in \mathbb{R}^{n\times n}$, $\bar{\psi}_{k+1} \in \mathbb{R}^{n\times 1}$, $\bar{\mu}_{k+1} \in \mathbb{R}$ such that

$$\begin{aligned}\Xi_{k+1}(z) &\doteq z'\bar{\Psi}_{k+1}z - 2z'\bar{\psi}_{k+1} + \bar{\mu}_{k+1} \leq \\ &\leq \min\{(z - z^*_{i,k})'\tilde{\Psi}_{i,k}(z - z^*_{i,k}) + \phi_{i,k}(x^*_{i,k}, z^*_{i,k}),\ \ i = 1, \dots, s\}\end{aligned}$$

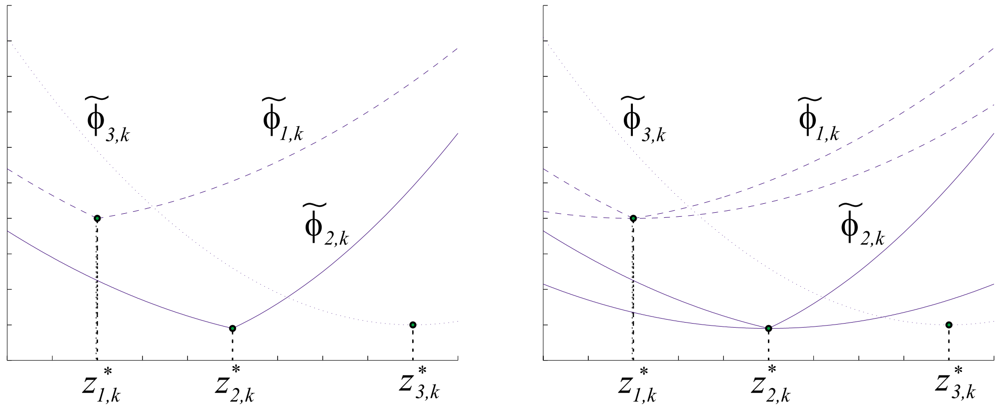
2.3. if $k = T - 1$ compute Ψ_T, and ν_T such that

$$\begin{aligned}\Gamma_T(z) &\triangleq (z - \hat{x}(T|T))'\Psi_T(z - \hat{x}(T|T)) + \nu_T \leq \\ &\leq \min\{(z - z^*_{i,T-1})'\tilde{\Psi}_{i,T-1}(z - z^*_{i,T-1}) + \phi_{i,T-1}(x^*_{i,T-1}, z^*_{i,T-1}),\ \ i = 1, \dots, s\}.\end{aligned}$$

As for Algorithm 1, we summarize the main steps behind the derivation of Algorithm 2.

a. By removing the constraints $w \in \mathbb{W}$, the one-step arrival cost $\tilde{\Xi}_{k+1}(z)$ is lower-bounded by $\underline{\phi}_k(z) = \min\{\tilde{\phi}_{i,k}(z),\ i = 1, \dots, s\}$, $\tilde{\phi}_{i,k}(z) = \min_{x(k)\in\mathcal{X}_i} \phi_{i,k}(x(k), z)$, where $\phi_{i,k}(x(k), z)$ are defined by (39e).

b. From Lemma 2, each $\tilde{\phi}_{i,k}$ is lower-bounded by the quadratic forms $(z - z^*_{i,k})'\tilde{\Psi}_{i,k}(z - z^*_{i,k}) + \phi_{i,k}(x^*_{i,k}, z^*_{i,k})$. Note that the global minima of these quadratic forms coincide with the global minima of each $\tilde{\phi}_{i,k}$. The typical situation is depicted in Figure 1.



(a) The functions $\tilde{\phi}_{i,k}(z)$ are represented with different textures. $\underline{\phi}_k(z)$ is given by the lower profile of these functions.

(b) Each map $\tilde{\phi}_{i,k}(z)$ is represented with the associated quadratic lower bound.

Figure 1: Lower bounds used in Algorithm 2

c. The lower profile of the quadratic forms in point (b) is finally lower bounded with a single paraboloid in steps 2.2. and 2.3. of Algorithm 2.

Algorithm 2 exploits the PWA form more than Algorithm 1. In fact, both the structure of the partition $\{\mathcal{X}_i\}_{i=1}^{s}$ and the constraints on the state are taken into account. Therefore, the penalty Γ_T is more likely to satisfy the condition $\Gamma_T \geq \Theta^*{}_T$. This increment in performance is paid for on the computational side. Indeed, compared to Algorithm 1, there is the additional cost of solving sM quadratic programs (39f).

6 Examples

In order to illustrate the difference between MHE without initial penalties and with the penalties of Algorithm 2, we conducted experiments using the piecewise linear system

$$\begin{aligned}
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(t+1) &= \begin{cases} A_1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(t) + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}(t) & \text{if } 0.2 \leq x_1(t) \leq 10, \\ A_2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(t) + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}(t) & \text{otherwise} \end{cases} \\
y(t) &= \begin{bmatrix} 0.0625 & -0.25 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(t) + v(t). \\
A_1 &= \begin{bmatrix} 0.5582 & 0.5582 \\ -0.5582 & 0.5582 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 1.1 \\ -1.1 & 0 \end{bmatrix}
\end{aligned} \tag{40}$$

The set of states is $\mathbb{X} = [-10, 10] \times [-10, 10]$ and the initial state $x(0)$ is chosen as $\begin{bmatrix} -3 & 2 \end{bmatrix}$. To what concerns the noise model, $w_1(t)$, $w_2(t)$ and $v(t)$ are assumed to be mutually independent and identically distributed. In particular the samples $w_1(t)$ and $w_2(t)$ are withdrawn from the

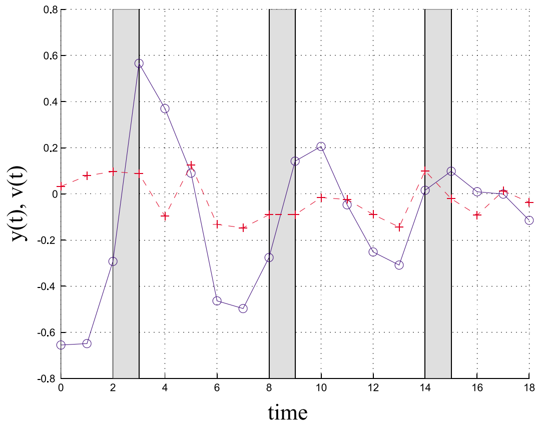


Figure 2: Simulation of system (40): Output trajectory $y(t)$ (circles) and noise samples $v(t)$ (crosses). In the white [gray] regions the dynamic A_1 [A_2] is active.

uniform distribution on $[-0.05, 0.05]$ whereas $v(t)$ are sampled from the uniform distribution on $[-0.3, 0.3]$. Then we have $\mathbb{W} = [-0.05, 0.05] \times [-0.05, 0.05]$.

To check the incremental observability of the system, we used Algorithm 2 reported in [3], in order to conclude that the system is incrementally observable in $\bar{T} = 2$ steps.

The output trajectory of the system along with the signal $v(t)$ are depicted in Figure 2.

For the MHE algorithm, we choose $Q = 1200 \cdot I_2$ and $R = 33.33$ which are the inverses of the noise covariances. The estimation horizon M is set equal to $\bar{T}$, i.e. the minimum allowed. The initial guess on $x(0)$ is $\bar{x} = \begin{bmatrix} 0 & 0 \end{bmatrix}'$ and the penalty $\Gamma_0(x)$ is initialized in Algorithm 2 as

$$\Gamma_0(x) = (x - \bar{x})' \begin{bmatrix} 0.005 & 0 \\ 0 & 0.005 \end{bmatrix} (x - \bar{x}). \tag{41}$$

The estimation results using either no initial penalty or Algorithm 2 are reported in Figure 3. The estimates obtained with both strategies are very similar at the beginning because the signal to noise ratio of the output is large and the uncertainty on $\bar{x}$ associated with (41) is large. Intuitively this means that filtering is not fundamental because the information carried by the output samples is not too much corrupted by noise. Conversely, for $t > 8$ the signal to noise ratio of the output becomes smaller and smaller as can be seen by visual inspection in Figure 2. Then, the filtering action due to the introduction of the initial penalty becomes relevant and this explains the better performance of the MHE scheme exploiting Algorithm 2. The eigenvalues of the matrices Ψ_t, $t = 0, \dots, 18$ are drawn in Figure 4 and their increasing behaviour indicates how Algorithm 2 takes into account the past data neglected by using an estimation horizon of fixed size. On the other hand, the estimation scheme without initial penalties does not use the information carried by the past samples and smoothing can be achieved only with a large estimation horizon. In our example, the short horizon $M = 2$ (that makes the optimization problem (11) computationally more tractable) prevents from having such benefits.

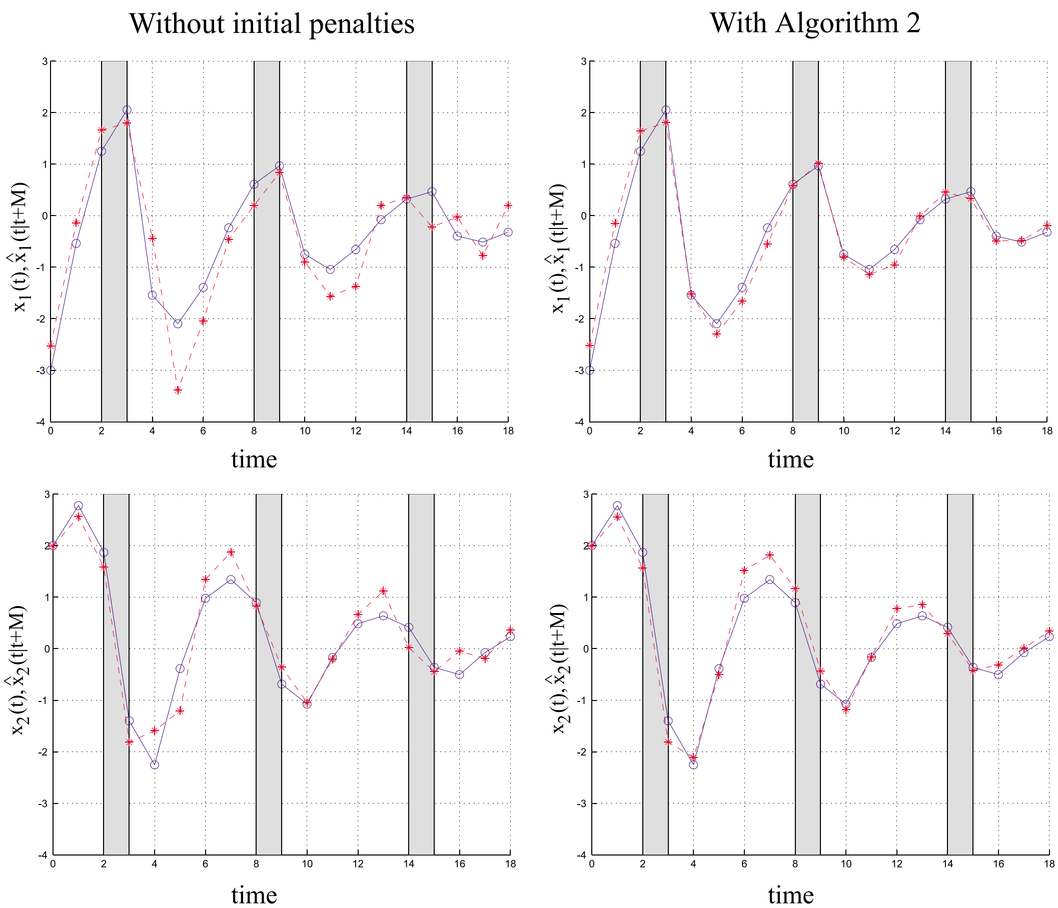


Figure 3: True state trajectory $x(t)$ (circles) and smoothed estimates $\hat{x}(t|t+M)$ (stars) obtained without initial penalties (left side) and with initial penalties (right side). In the white [gray] regions the dynamic A_1 [A_2] is active.

7 Conclusions

In this paper we considered MHE for hybrid systems in the PWA (or equivalently MLD) form. We provided sufficient conditions to guarantee asymptotic convergence of MHE. We also discussed the main issues in the practical implementation of MHE and provided two algorithms, of different computational demands and performance, to calculate initial penalties that aim at achieving good filtering properties in the presence of noise.

Acknowledgments

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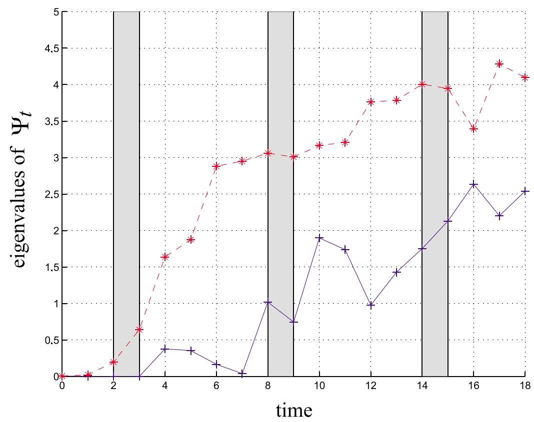


Figure 4: Maximum (stars) and minimum (crosses) eigenvalues of the matrices Ψ_t, $t = 0, \ldots, 18$ characterizing the initial penalty. In the white [gray] regions the dynamic A_1 [A_2] is active.

Appendix A

As in Section 5 we denote with $A_{x(t)}$ ($\mathcal{X}_{x(t)}$, $\phi_{x(t)}$) the matrix A_i (the set $\mathcal{X}_i$, the map ϕ_i) whose index i is uniquely defined by the relation $x(t) \in \mathcal{X}_i$. Before proving the main theorem we need some preliminary results.

Lemma 3. *Let Assumptions 3 and 4 hold. Then, $\forall T \geq 1$, and for all $x_0 \in \mathbb{X}$ such that Assumption 2 is fulfilled, it holds $\Xi_T(x_\Sigma(T, x_0)) \leq \Gamma_0(x_0)$. Moreover, $\Theta^*{}_T \leq \Gamma_0(x_0)$, $\forall T \geq M$.*

Proof. We prove the first inequality proceeding by induction. For $T \in \{1, \ldots, M\}$ it holds

$$\Xi_T(x_\Sigma(T, x_0)) = \min_{\substack{x(0),\, w \\ x(T) = x_\Sigma(T, x_0)}} J(0, T, w, v, x(0), \Gamma_0) \text{ subj. to (3).}$$

It is apparent that $\Xi_T(x_\Sigma(T, x_0)) \leq \Gamma_0(x_0)$, since the solution $x(0) = x_0$, $w(k) = 0$, $v(k) = 0$, $k = 0, \ldots, T-1$ is feasible.

For $T > M$, assume that $\Xi_T(x_\Sigma(T, x_0)) \leq \Gamma_0(x_0)$. Then, at the step $T + M$, it holds

$$\Xi_{T+M}(x_\Sigma(T+M, x_0)) = \min_{\substack{x(T),\, w \\ x(T+M) = x_\Sigma(T+M, x_0)}} J(T, T+M, w, v, x(T), \Gamma_T) \text{ subj. to (3),} \tag{42}$$

and the inequality $\Xi_{T+M}(x_\Sigma(T+M, x_0)) \leq \Gamma_T(x_\Sigma(T, x_0))$ follows again from optimality since $w(k) = 0$, $v(k) = 0$ and $x(T) = x_\Sigma(T, x_0)$ is feasible. From Assumption 4, we also have $\Gamma_T(x_\Sigma(T, x_0)) \leq \Xi_T(x_\Sigma(T, x_0))$ and from the induction assumption that $\Xi_T(x_\Sigma(T, x_0)) \leq \Gamma_0(x_0)$ the first part is proved.

To what concerns the inequality $\Theta^*{}_T \leq \Gamma_0(x_0)$, for $T = M$ we have $\Gamma_0(x_0) \geq \Theta^*_M$ because the trajectory $x(0) = x_0$, $w(t) = 0$, $v(t) = 0$ is feasible for (7). For $T > M$, from Assumption 4 one has $\Xi_T(x_\Sigma(T, x_0)) \geq \Theta^*_T$ and the thesis follows from the inequality $\Xi_T(x_\Sigma(T, x_0)) \leq \Gamma_0(x_0)$ proved in the first part. $\square$

Lemma 4. *Let Assumptions 3 and 4 hold and define*

$$a_T = \max_{i \in \{0, \dots, M-1\}} \{\|\hat{w}(T+i|T+M)\|_Q, \|\hat{v}(T+i|T+M)\|_R\}.$$

Then, $\lim_{T \to +\infty} a_T = 0$.

Proof. We follow the rationale of Proposition 3.9 in [28]. By definition

$$\Theta^*_{T+M} - \Theta^*_T = J(T, T+M, \hat{w}(\cdot|T+M), \hat{v}(\cdot|T+M), \hat{x}(T|T+M), \Gamma_T) - \Theta^*_T \geq 0$$

where the last inequality follows from Assumption 4. Then, the subsequence Θ^*_{T+kM}, $k = 0, 1, \dots$ is increasing as $k \to \infty$. From Lemma 3, it is also upper bounded and then converges. On the other hand, since $\Theta^*_{T+(k-1)M} \leq \Gamma_{T+(k-1)M}(x)$ (see Assumption 4), from the definition of Θ^*_{T+kM} we have

$$\Theta^*_{T+kM} \geq \sum_{t=T+(k-1)M}^{T+kM-1} \|\hat{v}(t|T+kM)\|^2_R + \|\hat{w}(t|T+kM)\|^2_Q + \Theta^*_{T+(k-1)M}$$

and convergence of Θ^*_{T+kM} implies that the sum $\sum_{t=T+(k-1)M}^{T+kM-1} \|\hat{v}(t|T+kM)\|^2_R + \|\hat{w}(t|T+kM)\|^2_Q$ goes to zero as $k \to \infty$. Since the same fact holds for every $T \in \{0, \dots, M-1\}$ the thesis follows. □

Lemma 5. *Consider a time horizon $\sigma \in \mathbb{N}^+$ and an initial state $\tilde{x} \in \mathbb{X}$, such that $x_\Sigma(t, \tilde{x}) \notin \Delta$, $t = 0, \dots, \sigma - 1$. Then $\forall \rho > 0$, $\exists \epsilon > 0$ such that if $\|w(t)\|_Q < \epsilon$, it holds*

$$\sum_{t=0}^{\sigma-1} \|y_\Sigma(t, \tilde{x}) - y_{\hat{\Sigma}}(t, \tilde{x}, w(t))\|_R < \rho \tag{43}$$

Proof. Let $\mathcal{D}_0 = \{\tilde{x}\}$. We first discuss the problem of choosing ϵ such that, for all $i \in \{0, \dots, \sigma - 1\}$, the following conditions simultaneously hold

$$\mathcal{D}_{i+1} \triangleq \left\{x \in \mathbb{X} : x \in A_{x_\Sigma(i,\tilde{x})}\mathcal{D}_i + B_{x_\Sigma(i,\tilde{x})}w + f_{x_\Sigma(i,\tilde{x})}, \forall w \in \mathbb{W}, \|w\|_Q < \epsilon\right\} \tag{44}$$

$$\mathcal{D}_i \subseteq \mathcal{X}_{x_\Sigma(i,\tilde{x})} \tag{45}$$

$$\text{diam}_R\left(C_{x_\Sigma(i,\tilde{x})}\mathcal{D}_i\right) \leq \frac{\rho}{\sigma} \tag{46}$$

where $\text{diam}_R(\mathcal{Y})$ denotes the maximum diameter of the bounded set $\mathcal{Y} \subset \mathbb{R}^{p_c+p_\ell}$ in the $\|\cdot\|_R$ norm [2].

Consider the first step, i.e. $i = 0$. The set $\mathcal{D}_1$ it is just the bounded set $\{A_{\tilde{x}}\tilde{x} + B_{\tilde{x}}w + f_{\tilde{x}}, \|w\|_Q < \epsilon \, \forall w \in \mathbb{W}\}$. By using the assumption that $\tilde{x}$ does not belong to the edge set, it is possible to find an $\epsilon_1 > 0$ s.t. condition (45) holds $\forall \epsilon < \epsilon_1$. It is also apparent that an $\epsilon_2 \leq \epsilon_1$ can be found s.t. condition (46) also holds $\forall \epsilon < \epsilon_2$. Note that condition (45) ensures that, at the next time instant, the evolution of the set $\mathcal{D}_i$ is described by the update law (44). This follows from the fact that $\mathcal{D}_i$ is included into a single region $\mathcal{X}_i$. Since, for $\epsilon \to 0$, every set $\mathcal{D}_i$, $i \in \{0, \dots, \sigma - 1\}$ shrinks to the point $x_\Sigma(i, \tilde{x})$, one can recursively apply the above rationale to ensure that there exists a single $\epsilon > 0$ satisfying (44)–(46), $\forall i \in \{0, \dots, \sigma - 1\}$. We outline that

[2] In a formal way, $\text{diam}_R(\mathcal{Y}) = \sup_{y_1, y_2 \in \mathcal{Y}} \|y_1 - y_2\|_R$

the sets $\mathcal{D}_i$ collect all the possible states reached from $\tilde{x}$ in i steps by the means of an input $w(i)$ s.t. $\|w(i)\|_Q < \epsilon$. Then,

$$\|y_\Sigma(i,\tilde{x}) - y_{\hat{\Sigma}}(i,\tilde{x},w(t))\|_R \leq \text{diam}_R \left(C_{x_\Sigma(i,\tilde{x})}\mathcal{D}_i\right) \leq \frac{\rho}{\sigma}$$

and the thesis follows. □

Proof of Theorem 1. The proof proceeds by contradiction. For a fixed $\tau \in \{\bar{T}, \dots M\}$, assume that τ-convergence does not hold. This means that $\exists \mu > 0$ such that $\forall \tilde{T}\ \exists T > \tilde{T}$ that yields

$$\|\hat{x}(T-\tau|T-1) - x_\Sigma(T-\tau, x_0)\|_S > \mu \tag{47}$$

where the norm $\|\cdot\|_S$ is the same used in (9) to check the incremental observability of the system. We show that an integer T can be chosen such that the following conditions simultaneously hold:

a. for $\rho = \frac{w_{RS}\mu}{2}$ and $\epsilon > 0$ it holds

$$\begin{aligned} \|\hat{w}(t|T)\|_Q < \epsilon,\ \forall t \in \{T-M, \dots, T-1\} \\ \sum_{t=T-M}^{T-1} \|\hat{v}(t|T)\|_R < w_{RS}\mu/2. \end{aligned} \tag{48}$$

b. $x_\Sigma(t, \hat{x}(T-M|T)) \notin \Delta$, $\forall t \in \{T-M, \dots, T-1\}$

c. $\|\hat{x}(T-\tau|T) - x_\Sigma(T-\tau, x_0)\|_S > \mu$

Such a T can be always found because

1. Lemma 4 and Lemma 5 guarantee that, for ϵ fixed, $\exists T_1$ s.t. $\forall T > T_1$ point (a) holds.
2. By Assumption 5, $\exists T_2$ s.t. $\forall T > T_2$ point (b) holds.
3. From (47), there exists $T_3 > max\{T_1, T_2\}$ s.t. $\|\hat{x}(T_3-\tau|T_3) - x_\Sigma(T_3-\tau, x_0)\|_S > \mu$.

Then $T = T_3$ satisfies all the requirements (a)-(c).

Note that in point (a) the parameter ϵ is free. By exploiting point (b) we can apply Lemma 5 with $\tilde{x} = \hat{x}(T-M|T)$, $\rho = \frac{w_{RS}\mu}{2}$ and $\sigma = M$ to choose ϵ such that the inequality(43) holds true. From Lemma 5 it also follows that

$$\sum_{t=T-M}^{T-1} \|y_{\hat{\Sigma}}(t, \hat{x}(T-M|T), 0, 0) - y_{\hat{\Sigma}}(t, \hat{x}(T-M|T), \hat{w}(\cdot|T), 0)\|_R < \rho = \frac{w_{RS}\mu}{2} \tag{49}$$

For sake of clarity, in (49) and in the next formulas, we denote the output trajectory of the system Σ with $y_{\hat{\Sigma}}(t, x(\tau), 0, 0)$. By the inverse triangle inequality, one has

$$
\begin{aligned}
\sum_{t=T-M}^{T-1} \|\hat{v}(t|T)\|_R &= \sum_{t=T-M}^{T-1} \| y_{\hat{\Sigma}}(t, x(T-M), 0, 0) - y_{\hat{\Sigma}}\left(t, \hat{x}(T-M|T), \hat{w}(\cdot|T), 0\right) \|_R \geq \\
&\geq \sum_{t=T-M}^{T-1} \| y_{\hat{\Sigma}}(t, x(T-M), 0, 0) - y_{\hat{\Sigma}}\left(t, \hat{x}(T-M|T), 0, 0\right) \|_R + \\
&- \sum_{t=T-M}^{T-1} \| y_{\hat{\Sigma}}\left(t, \hat{x}(T-M|T), 0, 0\right) - y_{\hat{\Sigma}}\left(t, \hat{x}(T-M|T), \hat{w}(\cdot|T), 0\right) \|_R \geq \\
&\geq \sum_{t=T-\tau}^{T-\tau+\bar{T}-1} \| y_{\hat{\Sigma}}(t, x(T-M), 0, 0) - y_{\hat{\Sigma}}\left(t, \hat{x}(T-M|T), 0, 0\right) \|_R + \\
&- \sum_{t=T-M}^{T-1} \| y_{\hat{\Sigma}}\left(t, \hat{x}(T-M|T), 0, 0\right) - y_{\hat{\Sigma}}\left(t, \hat{x}(T-M|T), \hat{w}(\cdot|T), 0\right) \|_R \geq \\
&\geq w_{RS} \|x(T-\tau) - \hat{x}(T-\tau|T)\|_S - \frac{w_{RS}\mu}{2}
\end{aligned}
$$

where the last inequality follows from the definition of incremental observability and from formula (49). Finally, by using condition (c), we have

$$
\sum_{t=T-M}^{T-1} \|\hat{v}(t|T)\|_R \geq w_{RS} \|x(T-\tau) - \hat{x}(T-\tau|T)\|_S - \frac{w_{RS}\mu}{2} \geq \frac{w_{RS}\mu}{2} \tag{50}
$$

that is in contradiction with (48). □

Appendix B

In this Appendix, the functions $\Phi(z)$, $\phi(x, z)$ and $p(z)$, the point (x^*, z^*), the matrices V_1, V_2, V_3, q_1, q_2 and the sets $\mathcal{X}$ and $\mathbb{X}$ are defined as in Lemma 2. The following result was derived in [8, 15].

Lemma 6. *The function $\Phi(z) : \mathbb{X} \mapsto \mathbb{R}$ is continuous, convex and there exists a finite polyhedral partition $\{\mathcal{C}_i\}_{i=1}^{\bar{s}}$ of $\mathbb{X}$ such that $\Phi(z)|_{\mathcal{C}_i}$ is a quadratic form, $\forall i \in \{1, \dots, \bar{s}\}$.*

By using the results of [8, 15] it is also possible to characterize the set $\mathcal{C}_1$ as

$$
\begin{aligned}
\mathcal{C}_1 &\triangleq \left\{ z \in \mathbb{X} : x_z^* \notin \partial\mathcal{X} \right\}, \\
x_z^* &\triangleq \arg\min_{x \in \mathcal{X}} \phi(x, z)
\end{aligned}
$$

where $\partial\mathcal{X}$ denotes the boundary of $\mathcal{X}$. In other words, $\mathcal{C}_1$ is the set of the parameters z for which $\Phi(z)$ coincides with the solution of an unconstrained minimization problem with respect to the variable x. We remark that $\mathcal{C}_1$ could be empty.

Lemma 7. *Let $\Phi(z)|_{\mathcal{C}_i} = z'\tilde{V}_i z + \tilde{v}_i' z + \tilde{\nu}_i$. Then,*

$$
\tilde{V}_1 = V_3 - V_2' V_1^{-1} V_2, \ \textit{if} \ \ \mathcal{C}_1 \neq \emptyset, \tag{51}
$$

$$
\tilde{V}_i > V_3 - V_2' V_1^{-1} V_2, \ \ i \neq 1. \tag{52}
$$

Proof. If $\mathcal{C}_1 \neq \emptyset$ and $z \in \mathcal{C}_1$, we have that $\Phi(z)|_{\mathcal{C}_1} = z'\tilde{V}_1 z + \tilde{v}_1' z + \tilde{\nu}_1$ with $\tilde{V}_1 = V_3 - V_2' V_1^{-1} V_2$. This is due to the fact that $z \in \mathcal{C}_1$ implies that $x_z^* \notin \partial\mathcal{X}$ that means the constraints are not active. Therefore $\Phi(z)$ can be computed relying on the explicit formula for the solution of an unconstrained quadratic program, so obtaining the expression of $\tilde{V}_1$.

Otherwise, assume that $z \in \mathcal{C}_i$, $i \neq 1$. This implies that $x_z^* \in \partial\mathcal{X}$. Note that $\mathcal{X}$ is a polyhedron and then the set of the n_{cs} active constraints can be represented by the equation $Rx = \rho$, for suitable choices of $R \in \mathbb{R}^{n_{cs} \times n}$ and $\rho \in \mathbb{R}^{n_{cs} \times 1}$. By using the vector of multipliers $\lambda = [\lambda_1, \cdots, \lambda_{n_{cs}}]$ we can define the Lagrangian

$$\begin{aligned} \mathcal{L} &\triangleq \phi(x,z) + \lambda(Rx - s), \\ \nabla\mathcal{L}_x &= 2V_1 x + 2V_2 z + q_1 + R'\lambda', \\ \nabla\mathcal{L}_\lambda &= Rx - s. \end{aligned}$$

By imposing the optimality conditions $\nabla\mathcal{L}_x = 0$ and $\nabla\mathcal{L}_\lambda = 0$ we have

$$\begin{aligned} \lambda &= -\left[\rho' + (2z'V_2 + q_1')\frac{1}{2}V_1^{-1}R'\right]\left(\frac{1}{2}RV_1^{-1}R'\right)^{-1} \\ x^* &= -\frac{1}{2}V_1^{-1}(2V_2'z + q_1 + R'\lambda). \end{aligned}$$

By direct calculation, it follows that the Hessian matrix of $\Phi(z)$ is

$$\tilde{V}_i = V_3 - V_2'V_1^{-1}V_2 + V_2'V_1^{-1}R'RV_1^{-1}R'RV_1^{-1}V_2,$$

from which it is apparent that $\tilde{V}_i > V_3 - V_2'V_1^{-1}V_2$. □

Lemma 8. *Let $\mathcal{C}_1 \neq \emptyset$ and (x^*, z^*) such that $z^* \in \mathcal{C}_1$. Then $\Phi(z) \geq p(z)$, $\forall z \in \mathbb{X}$.*

Proof. Since $z^* \in \mathcal{C}_1$, then $p(z) = \min_{x \in \mathbb{R}^n} \phi(x,z)$. This is the same minimization defining $\Phi(z)$ apart from the fact that the constraints $x \in \mathcal{X}$ are relaxed. □

Lemma 9. *Let $z^* \in \mathcal{C}_i$, $i \neq 1$. Then $\Phi(z) > p(z)$, $\forall z \in \mathbb{X} - \{z^*\}$ and $\Phi(z^*) = p(z^*)$.*

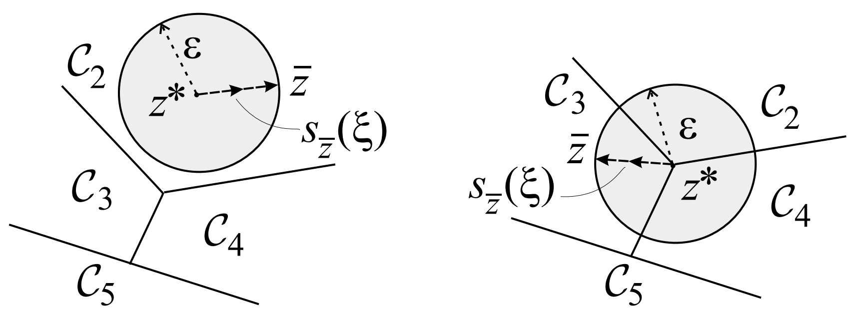
Proof. All along the proof we consider the function $V(z) \triangleq \Phi(z) - p(z)$. From the definition of $p(z)$, it follows that $\Phi(z^*) = p(z^*)$ and then $V(z^*) = 0$. Let Δ be the edge set (see Definition 3) induced by the polyhedral partition $\{\mathcal{C}_i\}_{i=1}^{\bar{s}}$ defined in Lemma 6 and $\Phi(z)|_{\mathcal{C}_i} = z'\tilde{V}_i z + \tilde{v}_i' z + \tilde{\nu}_i$. Define also $\tilde{V}_1 = V_3 - V_2'V_1^{-1}V_2$, from which we obtain

$$V(z)|_{\mathcal{C}_i} = z'(\tilde{V}_i - \tilde{V}_1)z + (\tilde{v}_i' + 2z^*\tilde{V}_1)z + \text{constant}$$

From Lemma 6, $V(z)$ is continuous and piecewise quadratic. Moreover it is also convex, because the Hessian matrix of $V(z)|_{\mathcal{C}_i/\Delta}$ is $\tilde{V}_i - \tilde{V}_1$, $\forall i \in \{2, \ldots, \bar{s}\}$, that is positive definite, in view of Lemma 7.

Hereafter the proof proceeds in two steps: First, we show that $V(z)$ is positive in a neighborhood of z^*. Then, we prove, by contradiction, that $V(z) \geq 0$ on the overall set $\mathbb{X}$.

Assume that $z^* \notin \Delta$. Then we can construct an ϵ-ball $B(z^*, \epsilon)$ that does not intersect Δ. Otherwise, if $z^* \in \Delta$ we choose $B(z^*, \epsilon)$ such that it intersects the minimum number of sets $\mathcal{C}_i$.



(a) The point z^* does not belong to the edge set.

(b) The point z^* belongs to the edge set.

Figure 5: ϵ-balls in the proof of Lemma 9

Both the cases are depicted in Figure 5. Let $\bar{z}$ be a point on $\partial B(z^*, \epsilon)$ (the boundary of the ball) and $s_{\bar{z}}(\xi) \triangleq \xi(\bar{z} - z^*) + z^*$, $\xi \in (0, 1)$. The oriented segment $s_{\bar{z}}(\xi)$ is depicted in Figure 5. Note that $V(s_{\bar{z}}(\xi))$ is always a parabola. By direct calculation,

$$\frac{\partial^2 V(s_{\bar{z}}(\xi))}{\partial \xi^2} = 2(\bar{z} - z^*)'(\tilde{V}_i - \tilde{V}_1)(\bar{z} - z^*) > 0, \tag{53}$$

where i is the index of the region $\mathcal{C}_i$ such that $\bar{z} \in \mathcal{C}_i$. The inequality in (53) follows from Lemma 7 because, by assumption, $z \notin \mathcal{C}_1$.

Consider now the first derivative of $V(s_{\bar{z}}(\xi))$. Since z^* is a global minimum of $\Phi(z)$ we have

$$b \triangleq \lim_{\xi \to 0} \frac{\partial V(s_{\bar{z}}(\xi))}{\partial \xi} \geq 0.$$

Then, by using the fact that $V(z^*) = 0$, it holds $V(s_{\bar{z}}(\xi)) = \xi^2(\bar{z} - z^*)'(\tilde{V}_i - \tilde{V}_1)(\bar{z} - z^*) + \xi b > 0$, $\forall \xi \in (0, 1)$. Since $\forall z \in B(z^*, \epsilon)$ it is possible to choose $\bar{z}$ such that $z \in s_{\bar{z}}((0, 1))$, we have that $V(z) > 0$, $\forall z \in B(z^*, \epsilon)/\{z^*\}$.

Now we prove that $V(z) \geq 0$, $\forall z \in \mathbb{X}$. Consider the level set $\mathcal{V} \triangleq \{z \in \mathbb{X} : V(z) \leq 0\}$. Since $V(z)$ is continuous and convex, $\mathcal{V}$ is a closed and convex set containing z^*. By contradiction, assume that $V(\tilde{z}) < 0$ for some $\tilde{z} \neq z^*$. Then $\tilde{z} \in \mathcal{V}$ and, from the convexity of $\mathcal{V}$,

$$\begin{aligned} V(\tilde{s}_{\tilde{z}}(\xi)) &< 0, \ \forall \xi \in (0, 1) \\ \tilde{s}_{\tilde{z}}(\xi) &= \xi(\tilde{z} - z^*) + z^*. \end{aligned}$$

since the segment $\tilde{s}_{\tilde{z}}((0, 1))$ intersects every ball $B(z^*, \epsilon)$, we have that there exists $z \in \tilde{s}_{\tilde{z}}((0, 1)) \bigcap B(z^*, \epsilon)$ such that $V(z) < 0$. But, from the first part of the proof, for the same point z it holds $V(z) > 0$, a contradiction. □

Proof of Lemma 2. If $z^* \in \mathcal{C}_1$ the thesis is proven by Lemma 8. If $z^* \in \mathcal{C}_i$, $i \neq 1$, the result follows from Lemma 9. □

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