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# A stabilizing model-based predictive control algorithm for nonlinear systems☆

L. Magni*, G. De Nicolao, L. Magnani, R. Scattolini

*Dipartimento di Informatica e Sistemistica, Università di Pavia, via Ferrata 1, 27100 Pavia, Italy*

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*Using distinct prediction and control horizons, nonlinear model-based predictive control can guarantee: (i) computational efficiency, (ii) enlargement of the stability domain and (iii) local optimality.*

**Abstract**

Predictive control of nonlinear systems subject to state and input constraints is considered. Given an auxiliary linear control law, a good nonlinear receding-horizon controller should (i) be computationally feasible, (ii) enlarge the stability region of the auxiliary controller, and (iii) approximate the optimal nonlinear infinite-horizon controller in a neighbourhood of the equilibrium. The proposed scheme achieves these objectives by using a prediction horizon longer than the control one in the finite-horizon cost function. This means that optimization is carried out only with respect to the first few input moves whereas the state movement is predicted (and penalized) over a longer horizon where the remaining input moves are computed using the auxiliary linear control law. Closed-loop stability is ensured by means of a penalty on the terminal state which is a computable approximation of the infinite-horizon cost associated with the auxiliary controller. As an illustrative example, the predictive control of a highly nonlinear chemical reactor is discussed. 

*Keywords:* Model predictive control; Receding-horizon control; Nonlinear control, Output admissible set, Performance

## 1. Introduction

In this paper, the state-feedback control of discrete-time nonlinear systems subject to state and input constraints is considered. In principle, a conceptually elegant solution would be given by the optimal infinite-horizon (IH) controller, obtained through the minimization of an IH cost function subject to the state and input constraints (Keerthi & Gilbert, 1988). On the other hand, it is apparent that the practical implementation of such a control law poses formidable computational problems since it involves a nonlinear optimization in an infinite-dimensional decision space. Hence, the commonly adopted solution is to use a control law designed on the basis of the linearized dynamics around the desired equilibrium. However, in view of the system nonlinearity and the presence of constraints, both the performance and the stability region of the controller may be unsatisfactory.

Predictive controllers are based on the receding-horizon (RH) methodology that offers a powerful approach to the design of state feedback controllers for constrained systems (Garcia, Prett, & Morari, 1989), (Mayne, Rawlings, Rao, & Scokaert, 2000). In particular, the main advantage with respect to IH optimal control is that the control input is computed by solving a finite-horizon optimization problem (where constraints are explicitly taken into account). Although predictive controllers of the first generation did not guarantee closed-loop stability even in the linear case (Bitmead, Gevers, & Wertz, 1990), by now there are several predictive control schemes with guaranteed stability for nonlinear systems (Keerthi & Gilbert, 1988; Chen & Shaw, 1982; Mayne & Michalska, 1990; Michalska & Mayne,

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* Corresponding author. Fax: + 390-382-505373.

*E-mail address:* magni@conpro.unipv.it (L. Magni).

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1993; Yang & Polak, 1993; Parisini & Zoppoli, 1995; Chen & Allgöwer, 1998; De Nicolao, Magni, & Scattolini, 1998; Magni & Sepulchre, 1997). In general, stability is enforced by means of suitable penalties and constraints on the state at the end of the finite optimization horizon (Mayne et al., 2000). Most methods use an auxiliary linear control law in order to derive the terminal penalties and constraints.

A nonlinear predictive controller is preferable to the direct use of the auxiliary linear controller only if it enjoys some properties. First of all, the solution of the finite-horizon optimization problem should be computationally affordable, meaning that the number of decision variables should not grow too much. The second requirement is that the stability region should be at least as large as that of the linear controller (enlargement property). Finally, one would like the RH control law to be close to the IH optimal control law at least in a neighborhood of the equilibrium (local optimality).

These properties could be enforced by constraining the terminal state to belong to the stability region of the auxiliary linear controller, and adding a penalty on the terminal state equal to the IH cost incurred by the application of the auxiliary control law to the nonlinear system (De Nicolao et al., 1998). Unfortunately, the stability region of the auxiliary controller is hardly computable in practice and the exact evaluation of the IH cost involves the simulation of the closed-loop nonlinear system over an infinite horizon. The control schemes available in the literature circumvent these problems in various ways. For instance, the terminal state is constrained to belong to an inner bound of the stability region for the auxiliary controller (usually, a level set of a quadratic Lyapunov function) (Michalska & Mayne, 1993), and a suitable quadratic terminal penalty is adopted (Parisini & Zoppoli, 1995; Chen & Allgöwer, 1998). As a drawback, however, the enlargement and local optimality properties can be recovered only at the cost of a possibly substantial increase of the optimization horizon, which directly affects the number of decision variables.

In the present paper, we develop a new RH control scheme that enjoys all the desired properties without becoming computationally prohibitive. The scheme is based on two main ideas. First, by working out suitable bounds, it is shown how to truncate the series expressing the IH cost associated with the auxiliary linear control law without losing stability. Second, the use of a long (prediction) horizon in the cost function is made possible by dividing the horizon in two parts. The inputs associated with the first part (control horizon) are the only free decision variables, whereas the subsequent inputs are obtained through the auxiliary linear control law. In this way, the dimension of the decision space depends only on the number of "free" input variables belonging to the control horizon which can be kept relatively short. Although the use of distinct control and prediction horizons has a long history in linear predictive control (Clarke, Mothadi, & Tuffs, 1987), in the nonlinear case the only attempt at using a prediction horizon longer than the control one was so far limited to locally open-loop stable systems using the trivial constant-control strategy as auxiliary controller (Chen & Allgöwer, 1997; Zheng, 2000). When applicable, the constant-control auxiliary strategy is indeed computationally efficient but it does not meet the local optimality requirement.

## 2. Problem statement

Consider the nonlinear discrete-time dynamic system

$$x(k+1) = f(x(k), u(k)), \quad x(t) = \bar{x}, \quad k \geqslant t, \tag{1}$$

where $k$ is the discrete time index, $x(k) \in R^n$, $u(k) \in R^m$, $f(\cdot,\cdot) \in C^2$ and $f(0,0) = 0$. The state and control variables are required to fulfill the following constraints:

$$x(k) \in X, \quad u(k) \in U, \quad k \geqslant t, \tag{2}$$

where $X$ and $U$ are compact subsets of $R^n$ and $R^m$, respectively, both containing the origin as an interior point. In order to design a state-feedback control law $u = \kappa(x)$ for (1), one may consider the minimization with respect to $u(\cdot)$ of the IH cost function

$$J_{\text{IH}}(\bar{x}, u(\cdot)) = \sum_{k=t}^{\infty} x(k)'Qx(k) + u(k)'Ru(k) \tag{3}$$

subject to (1) and (2). In (3) $Q$ and $R$ are positive definite weighting matrices.

**Definition 1.** Given a control law $u = \kappa(x)$, the term output admissible set (Keerthi & Gilbert, 1988), referred to the closed-loop formed by (1) joined with $u(k) = \kappa(x(k))$, denotes an invariant set $\bar{X}$ which is a domain of attraction of the origin and such that $\bar{x} \in \bar{X}$ implies that $x(k) \in X$ and $\kappa(x(k)) \in U$, $k \geqslant t$.

Let $X^{\text{IH}}$ be the set of states $\bar{x}$ such that the IH problem is solvable. If $u = \kappa^{\text{IH}}(x)$ denotes the optimal IH control law, then $X^{\text{IH}}$ is an output admissible set for $x(k+1) = f(x(k), \kappa^{\text{IH}}(x(k)))$ (Keerthi & Gilbert, 1988). In general, the IH nonlinear optimal control problem is computationally intractable since it involves an infinite number of decision variables. Nevertheless, it constitutes a touchstone for suboptimal approaches. For instance, it is desirable to design a regulator whose output admissible set approximates $X^{\text{IH}}$ as much as possible. The simplest way to obtain an easy-to-compute suboptimal solution is to resort to linearization techniques. Letting $A = (\partial f/\partial x)(0,0)$, $B = (\partial f/\partial u)(0,0)$, the linearized dynamics is

$$x(k+1) = Ax(k) + Bu(k), \quad x(t) = \bar{x}, \quad k \geqslant t. \tag{4}$$

**Assumption A1.** *The pair $(A, B)$ is stabilizable.*

Hereafter, a gain $K$ will be said to be stabilizing if $A_{cl} := A + BK$ is stable. Moreover, given a stabilizing linear control law

$$u = Kx, \tag{5}$$

we denote by $\Omega(K)$ the *output admissible set* associated with the closed-loop dynamics

$$x(k+1) = f(x(k), Kx(k)), \quad x(t) = \bar{x}, \quad k \geqslant t. \tag{6}$$

In the following, $\bar{\Omega}(K)$ will denote a *closed* output admissible set that coincides with the maximum output admissible set, if exists, or is sufficiently close to the supremum of the output admissible sets. In general, it is not possible to compute the supremum (or the maximum) of the output admissible sets. However, an ellipsoidal output admissible set (i.e. an inner bound of $\bar{\Omega}(K)$) can be obtained as a by-product of the following lemma. Hereafter, $\varphi_L(k, t, \bar{x}, K)$, $k \geqslant t$, indicates the solution $x(k)$ of (6) and, given a matrix $Q$, $\lambda_{\min}(Q)$ and $\lambda_{\max}(Q)$ are the minimum and maximum eigenvalue of $Q$, respectively.

**Lemma 1.** *Assume that K is stabilizing. Let $\tilde{Q}$ be a positive definite matrix, and $\gamma$ a real positive scalar such that $\gamma < \lambda_{\min}(\tilde{Q})$. Let $\Pi$ be the unique symmetric positive definite solution of the following Lyapunov equation*:

$$A'_{cl} \Pi A_{cl} - \Pi + \tilde{Q} = 0. \tag{7}$$

*Then, there exists a constant $c \in (0, \infty)$ specifying a neighborhood $\Omega_c(K)$ of the origin of the form $\Omega_c(K) = \{x \in \Re^n \mid x'\Pi x \leqslant c\}$ such that*

(i) $x \in X$, $Kx \in U$, *for all* $x \in \Omega_c(K)$;
(ii) $\forall x \in \Omega_c(K)$, $\quad f(x, Kx)'\Pi f(x, Kx) - x'\Pi x \leqslant -\gamma x'x$ *that is $V_L(x) = x'\Pi x$ is a Lyapunov function for the nonlinear closed-loop system* (6);
(iii) $\forall \bar{x} \in \Omega_c(K)$, $\quad \|\varphi_L(k, t, \bar{x}, K)\| \leqslant a e^{-b(k-t)} \|\bar{x}\|$, $\quad \forall k \geqslant t$, *with*

$$a = \frac{\lambda_{\max}(\Pi)}{\lambda_{\min}(\Pi)}, \qquad b = \ln\left(\frac{\lambda_{\max}(\Pi)}{\lambda_{\max}(\Pi) - \gamma}\right) > 0. \tag{8}$$

Note that, in view of (ii), the set $\Omega_c(K)$ is invariant under (6) so that, since constraints fulfillment is ensured by (i), $\Omega_c(K)$ is an output admissible set for the closed-loop formed by (1) joined with linear state feedback $u(k) = Kx(k)$. For what concerns the practical computation of the constant $c$, the interested reader is referred to (Polak, Mayne, & Stimmler, 1984). Under Assumption A1, the solution of the minimization of (3) subject to the linearized dynamics (4) leads to the control law

$$u = K^{LQ} x \tag{9}$$

where $K^{LQ} = (R + B'PB)^{-1} B'PA$ and $P$ is the (unique) positive definite solution of the algebraic Riccati equation

$$P = A'PA + Q - A'PB(R + B'PB)^{-1} B'PA. \tag{10}$$

If the linear control law $u(k) = K^{LQ} x(k)$ is applied to the nonlinear system (1), the origin will be stabilized with a nonzero-measurable domain of attraction. On the other hand, the extent of the associated supremum output admissible set may be unsatisfactory and even more so the extent of the inner bound provided by $\Omega_c(K^{LQ})$. The previous discussion justifies the search for sub-optimal regulator schemes that yield a nonlinear control law $\kappa(x)$ enjoying the following properties:

(i) *Performance/ complexity trade-off*: by suitably tuning the design parameters of the regulation scheme it should be possible to choose a compromise between an arbitrarily good approximation of the optimal (but computationally intractable) IH controller $\kappa^{IH}$ and the computationally cheap (but largely suboptimal) linear controller $u = Kx$.
(ii) *Enlargement property*: the output admissible set associated with $\kappa(x)$ should be larger than the output admissible set $\bar{\Omega}(K)$ associated with the auxiliary controller (5).
(iii) *Local optimality*: close to the origin the control law should behave as the infinite-horizon one: $d\kappa(x)/dx|_{x=0} = d\kappa^{IH}(x)/dx|_{x=0} = K^{LQ}$.

An effective strategy for designing suboptimal controllers is to resort to the RH strategy. However, none of the RH schemes currently available is completely satisfactory with respect to properties (i)–(iii).

## 3. Receding horizon control algorithm

In order to introduce the new algorithm, a finite-horizon optimization problem is first defined. Define, $u_{t_1,t_2} := [u(t_1)\ u(t_1+1) \ldots u(t_2)]$, $t_2 \geqslant t_1$.

**Finite horizon optimal control problem (FHOCP).** Given a stabilizing gain $K$, consider the set $\Omega_c(K)$ and the positive real numbers $a$, $b$ defined in Lemma 1. Then, given the positive integers $N_c$ (*control horizon*) and $N_p$ (*prediction horizon*), $N_c \leqslant N_p$, the positive definite matrices $Q$, $R$, and the real number $\rho$, $0 < \rho < 1$, minimize, with respect to $u_{t,t+N_c-1}$, the performance index

$$J(\bar{x}, u_{t,t+N_c-1}, N_c, N_p) = \sum_{k=t}^{t+N_p-1} \{x(k)'Qx(k) + u(k)'Ru(k)\} + V_f(x(t+N_p), \bar{x}) \tag{11}$$

subject to (i) the state dynamics (1); (ii) the constraints (2); (iii) the auxiliary control law $u(k) = Kx(k)$, $k \in [t+N_c, t+N_p-1]$; and (iv) the terminal state constraint $x(t+N_p) \in \Omega_c(K)$.

Recalling that $\varphi_L(k,t,\bar{x},K)$, $k \geqslant t$, denotes the solution $x(k)$ of (6), the terminal state penalty $V_f$ is defined as

$$V_f(x(t+N_p,\bar{x})) = \sum_{k=t+N_p}^{t+N_p+M-1} \|\varphi_L(k,t+N_p,x(t+N_p),K)\|^2_{Q+K'RK}, \quad (12)$$

where $M$ is the smallest positive integer such that

$$\frac{a}{1-e^{-b}}\|\varphi_L(t+N_p+M,t+N_p,x(t+N_p),K)\|^2 \times \lambda_{\max}(Q+K'RK) < \rho\bar{x}'Q\bar{x}. \quad (13)$$

For $M = \infty$, the penalty (12) coincides with the IH cost associated with the auxiliary control law $u = Kx$. Since $M$ is finite, (12) is just a computable approximation of such an IH cost. Inequality (13) is a technical condition on $M$, that will be used in the proof of closed-loop stability (Theorem 2), and whose scope is to ensure that the approximation is accurate enough. It is now possible to define the new *stabilizing nonlinear receding horizon* (SNRH) *control law:* at every time instant $t$, define $\bar{x} = x(t)$ and find the optimal control sequence $u^o_{t,t+N_c-1}$ by solving the FHOCP. Then, apply the control $u(t) = \kappa^{RH}(\bar{x})$, where $\kappa^{RH}(\bar{x}) = u^o_{t,t}(\bar{x})$ is the first column of $u^o_{t,t+N_c-1}$. In practice, the SNRH algorithm consists of the following steps.

*Off-line*

(1) Choose $Q$, $R$, $N_c$, $N_p$, and $\rho$.
(2) Find $K$ such that $A+BK$ is asymptotically stable (for instance $K = K^{LQ}$ defined in (9)).
(3) Choose $\tilde{Q}$ and $\gamma$ such that $\gamma < \lambda_{\min}(\tilde{Q})$ and find the solution $\Pi$ of the Lyapunov equation (7). In particular, if $K = K^{LQ}$, and $\tilde{Q} = Q + K^{LQ'}RK^{LQ}$, then $\Pi = P$, where $P$ is the solution of the *ARE* (10).
(4) According to Lemma 1, determine $c$ corresponding to a region $\Omega_c(K) \in X$ such that $f(x,Kx)'\Pi f(x,Kx) - x'\Pi x \leqslant -\gamma x'x$ and $Kx \in U$, $\forall x \in \Omega_c(K) = \{x \in \Re^n \colon x'\Pi x \leqslant c\}$.
(5) Calculate $a$ and $b$ according to (8).

*On-line*: At each step, solve the FHOCP and determine the current control variable.

The SNRH algorithm defines a nonlinear control law $u = \kappa^{RH}(x)$ that, applied to system (1), yields the closed-loop system

$$x(k+1) = f(x(k)),\ \kappa^{RH}(x(k)),\ x(t) = \bar{x},\ k \geqslant t \quad (14)$$

whose stability is established in the following result.

**Theorem 2.** *Let $X^o(N_c,N_p)$ be the set of states $\bar{x}$ such that there exists a control sequence $u^o_{t,t+N_c-1}$ that solves the FHOCP. Then, the SNRH control algorithm applied to* (1) *exponentially stabilizes the origin with output admissible set $X^o(N_c,N_p)$.*

The method proposed in this paper can be interpreted as posing a (nonquadratic) terminal penalty at time $t+N_c$, which is obtained through an approximation of the cost-to-go of the auxiliary controller applied to the nonlinear plant. According to this point of view, the constraints on the input $u_{t,t+N_c-1}$ must account for the need of bringing $x(t+N_p)$ inside the ellipsoidal invariant set $\Omega_c$. Compared to methods that use a quadratic terminal penalty at time $t+N_c$ with the constraint that $x(t+N_c) \in \Omega_c$, there are two possible advantages. First of all the constraints of our method are less restrictive (for given $N_c$ and $\Omega_c$). Second the nonquadratic terminal penalty, differently from the quadratic one, takes into account the nonlinear dynamics of the plant with possible performance benefits, see e.g. Section 4.3.

## 4. Properties of the SNRH algorithm

In this section, with reference to the three issues highlighted at the end of Section 2, the properties of the SNRH algorithm are discussed.

### 4.1. Performance/complexity trade-off

The algorithm has two design parameters $N_c$ and $N_p$ that directly affect the computational complexity. As a matter of fact the complexity mainly depends on $N_c$ because it is proportional to the number of decision variables. Conversely, for a fixed $N_c$, an increase of $N_p$ involves integration of system (6) over a longer interval and the fulfillment of a larger number of input and state constraints but does not affect the dimensionality of the optimization space. It is remarkable that the algorithm is well defined and guarantees stability even for $N_c = 1$. On the other hand, it is ideally possible to approximate the optimal IH controller arbitrarily well by increasing the value $N_c$. An objective way to assess performance would be to measure the value of the IH cost (3) when a controller is applied to (1). In order to avoid the infinite-horizon simulation of the closed-loop system (6) or (14), one can compute an upper bound of the IH performance as shown below for both the auxiliary and the SNRH control law.

**Theorem 3.** *Let $J^K_{IH}(\bar{x})$ be equal to* (3) *subject to* (6) (*in other words, $J^K_{IH}(\bar{x})$ is the IH cost associated with the linear control law* (5)) *Then, for any $\rho$, $0 < \rho < 1$, it results that $J^K_{IH}(\bar{x}) \leqslant \bar{J}^K_{IH}(\bar{x}) := V_f(\bar{x},\bar{x}) + \rho\bar{x}'Q\bar{x}$, where $V_f$ is defined in* (12) *and* (13). *Moreover*, $\lim_{\rho\to 0} V_f(\bar{x},\bar{x}) = J^K_{IH}(\bar{x})$.

**Theorem 4.** *Let $J^{RH}_{IH}(\bar{x},N_c,N_p)$ be equal to* (3) *subject to* (14) (*in other words $J^{RH}_{IH}$ is the IH cost associated with the* *nonlinear* RH *control law* $u = \kappa^{RH}(x)$) *and* $J^{o}(\bar{x}, N_c, N_p)$ *be equal to* $J(\bar{x}, u^{o}_{t,t+N_c-1}, N_c, N_p)$ (*i.e.* $J^{o}$ *is the optimal value of the* FH *cost* (11)). *Then, it results that* $J^{RH}_{IH}(\bar{x}, N_c, N_p) \leqslant \bar{J}^{RH}_{IH}(\bar{x}, N_c, N_p) := \lim_{\rho \to 0}(J^{o}(\bar{x}, N_c, N_p) + \rho \bar{x}' Q \bar{x})$ *where* $\rho$ *is the positive real number appearing in the definition of the FHOCP.*

It is unfortunate that the upper bound involves the limit for $\rho$ tending to zero. However from (13) it is possible to see that an exponential decrease of $\rho$ requires only a linear increase of $M$. Therefore, good approximations of the limit may be obtainable with relatively small values of $\rho$. It is hardly possible to prove that the $J^{RH}_{IH}$ is a monotone non increasing function of $N_c$. Nevertheless, the next theorem shows that there is a monotonicity relationship between the performance bounds.

**Theorem 5.** *For a given* $N_p \leqslant M(\bar{x}, \bar{x})$, *assume that* $N_c < N_p$. *Then,* $\bar{J}^{RH}_{IH}(\bar{x}, N_c + 1, N_p) \leqslant \bar{J}^{RH}_{IH}(\bar{x}, N_c, N_p) \leqslant J^{K}_{IH}$.

As already observed, in general it cannot be guaranteed that an increase of $N_c$ yields an automatic decrease of $J^{RH}_{IH}$. Nevertheless, when $N_c$ increases, the FHOCP tends to an IH minimization problem and the actual IH performance of the SNRH control scheme must eventually decrease because it converges to the IH cost achieved by the IH nonlinear optimal controller $\kappa^{IH}$.

### 4.2. Enlargement property

The advantage of having introduced $N_p$ is that it is possible to enlarge the stability region without increasing $N_c$.

**Theorem 6.** *Let* $X^{o}(N_c, N_p)$ *be the output admissible set associated with the closed-loop system* (14). *Then* (i) $X^{o}(N_c, N_p) \supseteq \Omega_c(K)$, $\forall N_c, N_p$; (ii) $X^{o}(N_c, N_p + 1) \supseteq X^{o}(N_c, N_p)$, $\forall N_c, N_p$; (iii) *there exists a finite* $\bar{N}_p$ *such that* $X^{o}(N_c, \bar{N}_p) \supseteq \bar{\Omega}(K)$, $\forall N_c$, *where* $\bar{\Omega}(K)$ *is any closed output admissible set associated with* $u = Kx$ (*see Section* 2).

Point (ii) of the previous result shows that increasing $N_p$ results in a direct improvement of the extent of the output admissible set. More importantly, point (iii) shows that it is possible (with an arbitrary $N_c$) to achieve an output admissible set which is not smaller than that guaranteed by the linear controller. Remarkably, without introducing two different horizons, the only way to reach the output admissible set of the linear controller is by suitably increasing the control horizon $N_c$ with obvious computational drawbacks. Note also that, in general, by increasing only $N_p$ it is not possible to reach the output admissible set $X^{IH}$ achievable with the optimal IH control law. To do this also the control horizon $N_c$ must be increased. Another parameter that affects the size of the output admissible set is $\gamma$. More precisely a smaller $\gamma$ will produce a larger $\Omega_c$ and consequently a larger $X^{o}(N_c, N_p)$ (see Theorem 6(i)). On the other hand, in view of (8), a smaller $\gamma$ yields a value of $b$ closer to zero and this requires a larger $M$ (see (13)).

### 4.3. Local optimality

A somewhat undesirable feature of RH control is that even in the nominal case (no disturbances), the open-loop predicted state movement (i.e. the one computed in the solution of the FHOCP) is in general different from the actual closed-loop state movement. On the other hand, it is well known that the two state movements do coincide only when an infinite horizon optimal controller is applied. A less demanding but more realistic requirement is that the distance between the predicted and actual state movements tends to zero as the initial state $\bar{x}$ gets closer to the origin. This is just what is obtained by imposing the local optimality property defined in Section 2. In order to assess the local discrepancy between the SNRH control law and the optimal IH one, in the following lemma we compute the linearization of the SNRH control law $u = \kappa^{RH}(x)$ around the origin.

**Lemma 7.** *The SNRH control law is differentiable around the origin and* $\partial \kappa^{RH}(x)/\partial x|_{x=0} = K^{RH} = -(R + B'\Sigma(N_c)B)^{-1} B'\Sigma(N_c)A$ *where* $\Sigma(N_c)$ *is obtained by integrating the Lyapunov equation* $S(k+1) = A'_{cl} S(k) A_{cl} + Q$ *in* $k \in [0, N_p + M - N_c]$ *with the initial condition* $S(0) = 0$ *and the Riccati equation*

$$\Sigma(k+1) = A'\Sigma(k)A + Q - A'\Sigma(k)B(R + B'\Sigma(k)B)^{-1}B'\Sigma(k)A$$

*in* $k \in [0, N_c]$ *with the initial condition* $\Sigma(0) = S(N_p + M - N_c)$.

**Theorem 8.** *For any given* $N_c$, *if* $K = K^{LQ}$, *then* $\lim_{N_p \to \infty} K^{RH} = K^{LQ}$.

This theorem shows that the SNRH controller can approximate the local optimality condition with arbitrary precision by properly increasing $N_p$.

## 5. Illustrative example

In this section, the new SNRH algorithm is applied to the highly nonlinear model of a continuous stirred tank reactor (CSTR), see (Seborg, Edgar, & Mellichamp, 1989, page 5). Assuming constant liquid volume, the CSTR for an exothermic, irreversible reaction, A → B, is described by the following dynamic model based on a component balance for reactant A and an energy balance:

$$\dot{C}_A = \frac{q}{V}(C_{Af} - C_A) - k_0 \exp\left(-\frac{E}{RT}\right)C_A,$$

$$\dot{T} = \frac{q}{V}(T_f - T) + \frac{(-\Delta H)}{\rho C_p}k_0 \exp\left(-\frac{E}{RT}\right)C_A + \frac{UA}{V\rho C_p}(T_c - T), \tag{15}$$

where $C_A$ is the concentration of A in the reactor, $T$ is the reactor temperature, and $T_c$ is the temperature of the coolant stream. The constraints are $280\ \text{K} \leqslant T_c \leqslant 370\ \text{K}$, $280\ \text{K} \leqslant T \leqslant 370\ \text{K}$, $0 \leqslant C_A \leqslant 1$ mol/l. The objective is to regulate $C_A$ and $T$ by manipulating $T_c$. The nominal operating conditions, which correspond to an unstable equilibrium $C_A^{eq} = 0.5$ mol/l, $T^{eq} = 350$ K, $T_c^{eq} = 300$ K are: $q = 100$ l/min, $T_f = 350$ K, $V = 100$ l, $\rho = 1000$ g/l, $C_p = 0.239$ J/g K, $\Delta H = -5 \times 10^4$ J/mol, $E/R = 8750$ K, $k_0 = 7.2 \times 10^{10}\ \text{min}^{-1}$, $UA = 5 \times 10^4$ J/min K. The open-loop responses for $\pm 5$ K step changes in $T_c$, reported in Fig. 1, demonstrate that the reactor exhibits highly nonlinear behavior in this operating regime. The nonlinear discrete-time state-space model (1) of system (15) can be obtained by defining the state vector $x = [C_A - C_A^{eq}, T - T^{eq}]'$, the manipulated input $u = T_c - T_c^{eq}$ and by discretizing equations (15) with sampling period $\Delta t = 0.03$ min. With reference to the discrete-time linearized system (4), letting $Q = diag(1/0.5, 1/350)$ and $R = 1/300$, the stabilizing $LQ$ control gain $K^{LQ}$ is obtained. For several pairs $(\tilde{Q}, \gamma)$, the solution $\Pi$ of (7) with $K = K^{LQ}$ was computed, and the parameter $c$ was optimized so as to maximize the extent of the invariant region $\Omega_c(K^{LQ})$ defined in Lemma 1. The best result was obtained in correspondence of $\tilde{Q} = 0.05I$ and $\gamma = 0.01$, yielding $c = 0.0915$. The SNRH control law was synthesized as described in Section 3 using the $LQ$ control law as auxiliary controller, and letting $Q = diag(1/0.5, 1/350)$ and $R = 1/300$, $N_c = 3$, $N_p = 75$, $\rho = 0.1$. The optimizations required by the algorithm were performed using the MatLab Optimization Toolbox.

### 5.1. Simulation results

Starting from the three different initial states reported in Table 1, the closed-loop system (6) was simulated and the resulting trajectories were plotted in Fig. 2, where also the region $\Omega_c(K^{LQ})$ is depicted. In Table 1, for each experiment, the number of sample times needed to reach the terminal region $\Omega_c(K^{LQ})$, and the value of the infinite-horizon cost $J_{IH}^K(x(0))$ defined in Theorem 3 are reported. It is interesting to note that all the initial points of

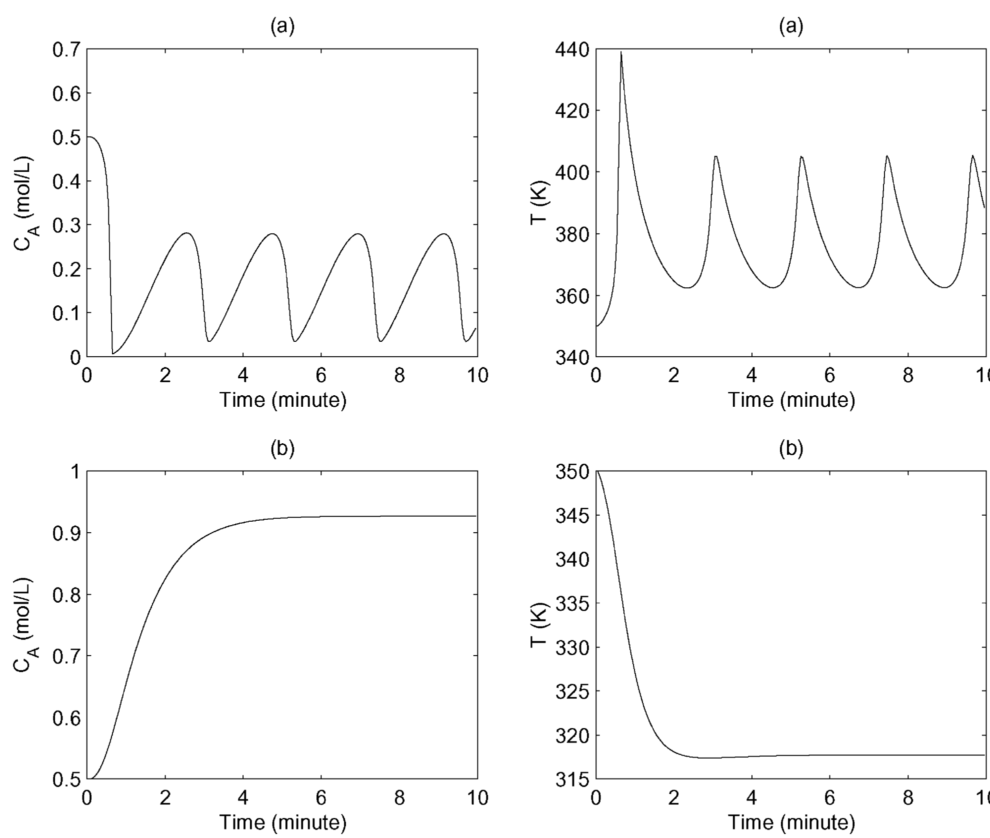


Fig. 1. Open-loop responses for $+5$ K (a) and $-5$ K (b) step changes in $T_c$.

these experiments belong to an output admissible set $\bar{\Omega}(K^{LQ})$ which is much larger than the ellipssoidal inner bound $\Omega_c(K^{LQ})$. In fact, in these examples the state and control constraints are never violated. However, starting from $[C_A(0)\ T(0)]' = [0.7\ 350.5]'$ the control constraint is violated, see Fig. 3, and therefore such a state is external to any output admissible set for $K^{LQ}$.

The same simulation experiments performed with the $LQ$ control law were made also with the SNRH control algorithm. In view of the number of samples needed by the $LQ$ controller to reach the terminal region $\Omega_c(K^{LQ})$, we chose a long prediction horizon ($N_p = 75$), whereas a very short control horizon ($N_c = 3$) was selected to maintain a low computational effort. The resulting closed-loop trajectories are plotted in Fig. 2. In Table 2, the flops, as well as the values of the infinite-horizon performance $J_{IH}^{RH}$ and its upper bound $\bar{J}_{IH}^{RH}$, both defined in Theorem 4, are reported. Notably, in all cases but one $\bar{J}_{IH}^{RH}$ (computed just after the first optimization at time 0) is very close to the true IH performance $J_{IH}^{RH}$ of the RH control law. This means that the difference between the (open-loop) state trajectory predicted at time 0 and the actual closed-loop trajectory is relatively small. As observed in Section 4.3, this is a symptom of the closeness of the SNRH controller to the optimal IH one. A comparison between Tables 1 and 2 shows that the SNRH control law improves on the closed-loop performance. In order to assess the advantage of using two distinct control and prediction horizons $N_c$ and $N_p$, we applied also a nonlinear RH (NRH) regulator with $N_p = N_c = 15$ and a quadratic terminal penalty, see (Chen & Allgöwer, 1998), (Mayne et al., 2000). The resulting closed-loop trajectories are plotted in Fig. 2. In Table 3, the values of the IH performance and the flops are reported. By comparing Tables 2 and 3, in all cases the SNRH scheme requires a significantly smaller amount of computations and its performances are generally superior except for Experiment 4 where there is a negligible deterioration of the performance. Moreover, by comparing Tables 1 and 3 it is seen that in Experiment 1 the performance of the NRH regulator is worse than that of the $LQ$ regulator. The transients in Experiment 4 are reported in Fig. 3 for all the three controllers (recall that with the unconstrained $LQ$ controller the control variable $T_c$ violates the constraints).

Table 1
$LQ$ experiments

| Experiment | $[C_A(0)\ T(0)]'$ | Samples to reach $\Omega_c(K^{LQ})$ | $J_{IH}^{K}(x(0))$ |
|---|---|---|---|
| 1 | [0.3 363]′ | 48 | 9.8874 |
| 2 | [0.3 335]′ | 20 | 59.6827 |
| 3 | [0.6 335]′ | 75 | 20.3420 |

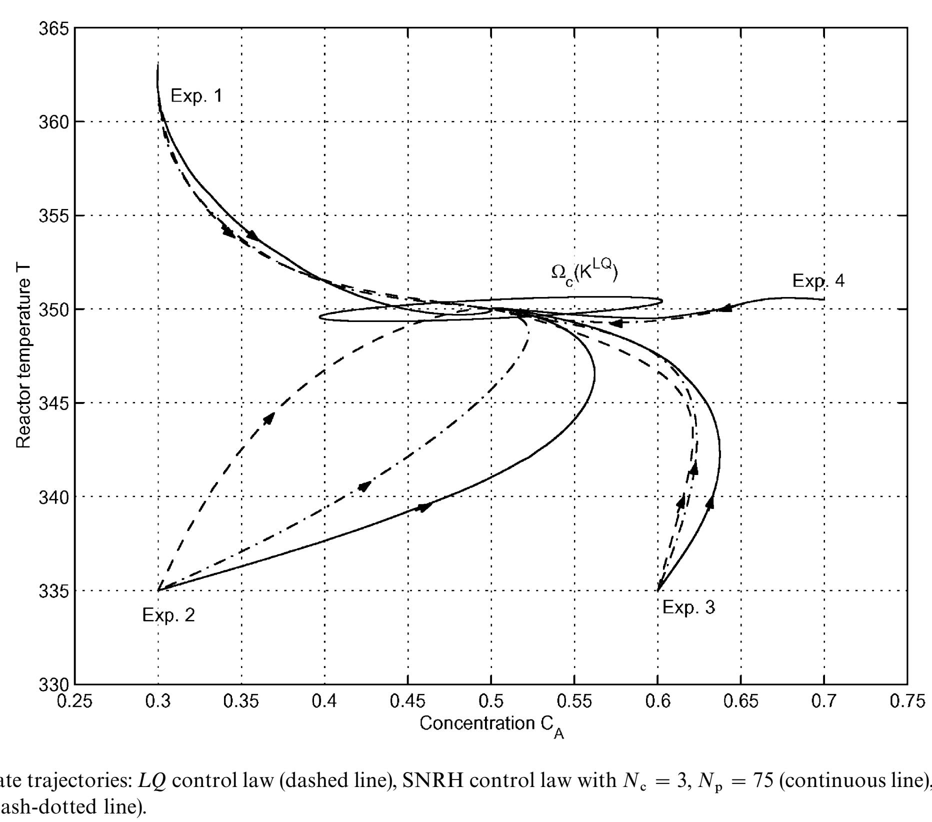


Fig. 2. Closed-loop state trajectories: $LQ$ control law (dashed line), SNRH control law with $N_c = 3$, $N_p = 75$ (continuous line), and NRH control law with $N_c = N_p = 15$ (dash-dotted line).

Table 2
SNRH experiments with $N_c = 3$ and $N_p = 75$

| Experiment | $[C_A(0)\ T(0)]'$ | $J_{IH}^{RH}(x(0), 3, 75)$ | $\bar{J}_{IH}^{RH}(x(0), 3, 75)$ | Flops |
|---|---|---|---|---|
| 1 | [0.3 363]′ | 9.3547 | 9.7150 | $2.8418 \times 10^8$ |
| 2 | [0.3 335]′ | 41.6681 | 52.3820 | $3.6592 \times 10^8$ |
| 3 | [0.6 335]′ | 19.0880 | 19.4300 | $4.1187 \times 10^8$ |
| 4 | [0.7 350.5]′ | 13.2135 | 13.3334 | $3.7017 \times 10^8$ |

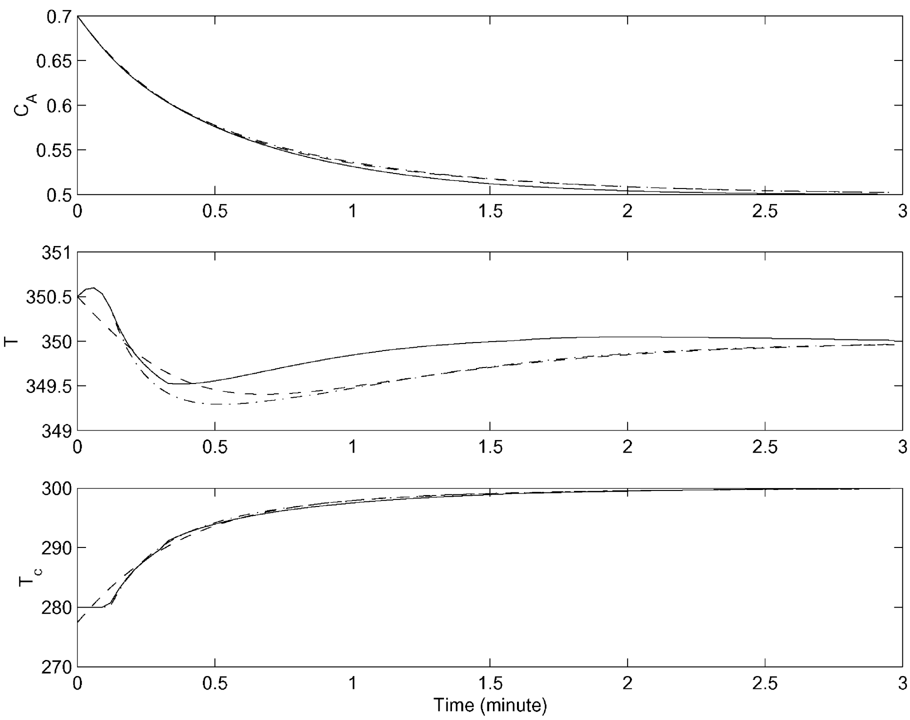


Fig. 3. Experiment 4: closed-loop responses with the $LQ$ control law (dashed line), the SNRH control law with $N_c = 3$ and $N_p = 75$ (continuous line), and the NRH control law with $N_c = N_p = 15$ (dash-dotted line).

Table 3
NRH experiments with $N_c = N_p = 15$

| Experiment | $[C_A(0)\ T(0)]'$ | $J_{IH}^{RH}(x(0), 15, 15)$ | Flops |
|---|---|---|---|
| 1 | [0.3 363]′ | 10.2105 | $4.3725 \times 10^8$ |
| 2 | [0.3 335]′ | 42.6423 | $5.1299 \times 10^8$ |
| 3 | [0.6 335]′ | 19.6796 | $6.3381 \times 10^8$ |
| 4 | [0.7 350.5]′ | 13.1397 | $4.4623 \times 10^8$ |

## 6. Conclusions

In the novel nonlinear RH control scheme proposed in this paper, the introduction of the notion of prediction and control horizons allows one to enlarge the size of the domain of attraction without prohibitively increasing the computational burden. For all other existing methods the enlargement can be obtained only by lengthening the control horizon with a consequent increase of the number of decision variables. Previously, it was not even possible to guarantee that the domain of attraction of the nonlinear RH controller is larger than that of the auxiliary local linear controller while our new method enjoys this important property (Theorem 6). Finally, the optimal infinite horizon behavior can be recovered, at least locally, by acting only on the prediction horizon. Again, other methods can achieve such a recovery only through a computationally expensive increase of the control horizon. The simulation experiments confirm the good properties of the new algorithm and show its advantages over $LQ$ control and RH schemes that do not apply distinct prediction and control horizons.

## Appendix. Proofs of lemmas and theorems

**Proof of Lemma 1.** (i) The existence of a constant $c_1 \in (0, \infty)$, such that $x \in X$, $Kx \in U$, for all $x \in \Omega_{c_1}(K)$ is proven in (Michalska & Mayne, 1993). Thus, let $c \in (0, c_1)$.

(ii) Letting $\Phi(x) = f(x, Kx) - A_{cl}x$, the inequality

$$f(x,Kx)'\Pi f(x,Kx) - x'\Pi x \leqslant -\gamma x'x \quad \forall x \in \Omega_c(K) \quad \text{(A.1)}$$

is equivalent to

$$2\Phi(x)'\Pi A_{cl}x + \Phi(x)'\Pi\Phi(x) + x'A_{cl}\Pi A_{cl}x - x'\Pi x \leqslant -\gamma x'x. \quad \text{(A.2)}$$

From Eq. (7) it is easy to see that inequality (A.2) is equivalent to

$$x'\tilde{Q}x - \gamma x'x \geqslant 2\Phi(x)'\Pi A_{cl}x + \Phi(x)'\Pi\Phi(x). \quad \text{(A.3)}$$

Now, define $L_r = \sup_{x \in B_r} \|\Phi(x)\|/\|x\|$ where $B_r = \{x : \|x\| \leqslant r\}$ ($L_r$ exists finite because $f \in C^2$). Then, $\forall x \in B_r$ (A.3) is satisfied, if

$$(\lambda_{\min}(\tilde{Q}) - \gamma)\|x\|^2 \geqslant \{2L_r\|\Pi\|\|A_{cl}\| + L_r^2\|\Pi\|\}\|x\|^2. \quad \text{(A.4)}$$

By the definition of $\gamma$, it holds that $\lambda_{\min}(\tilde{Q}) - \gamma > 0$. Then, since $L_r \to 0$ as $r \to 0$, there exists $c \in (0, c_1)$ such that inequality (A.4) holds $\forall x \in \Omega_c(K)$, which implies that inequality (A.1) holds as well. Moreover, from (A.1) it follows that, $\forall x \in \Omega_c(K)$, $V_L(f(x,Kx)) \leqslant V_L(x)$ so that, $x \in \Omega_c(K)$ implies $f(x,Kx) \in \Omega_c(K)$ ($\Omega_c(K)$ is an invariant set for system (1) controlled by the linear feedback $u = Kx$). In conclusion, $V_L = x'\Pi x$ is a Lyapunov function satisfying, $\forall x \in \Omega_c(K)$, $\alpha\|x\|^2 \leqslant V_L(x) \leqslant \beta\|x\|^2$, $V_L(x) - V_L(f(x,Kx)) \geqslant \gamma\|x\|^2$, with $\alpha = \lambda_{\min}(\Pi)$ and $\beta = \lambda_{\max}(\Pi)$.

(iii) In view of the previous point and the standard Lyapunov stability theorem, the origin is an exponentially stable equilibrium point and there exist positive numbers $a, b$ such that, for all $\bar{x} \in \Omega_c(K)$,

$$\|\varphi_L(k,t,\bar{x},K)\|^2 \leqslant a e^{-b(k-t)}\|\bar{x}\|^2, \quad \forall k \geqslant t. \quad \text{(A.5)}$$

From (7) it follows that $\Pi \geqslant \tilde{Q}$ and therefore $\gamma < \lambda_{\min}(\tilde{Q}) \leqslant \lambda_{\max}(\Pi) = \beta$. Hence, $\gamma < \beta$. Observing that $-\|x\|^2 \leqslant -V_L(x)/\beta$, it follows that $V_L(\varphi_L(t+1,t,\bar{x},K)) \leqslant V_L(\bar{x}) - \gamma\|\bar{x}\|^2 \leqslant (1-\gamma/\beta)V_L(\bar{x})$. Then, $\alpha\|\varphi_L(k,t,\bar{x},K)\|^2 \leqslant V_L(\varphi_L(k,t,\bar{x},K)) \leqslant (1-\gamma/\beta)^{(k-t)} V_L(\bar{x}) \leqslant (1-\gamma/\beta)^{(k-t)}\beta\|\bar{x}\|^2$ so that $\|\varphi_L(k,t,\bar{x},K)\|^2 \leqslant (\beta/\alpha)(1-\gamma/\beta)^{(k-t)}\|\bar{x}\|^2 = (\beta/\alpha)\, e^{[\ln(1-\gamma/\beta)](k-t)}\|\bar{x}\|^2$ and consequently $a = \beta/\alpha$, $b = \ln(\beta/(\beta-\gamma))$.

**Proof of Theorem 2.** Let $\tilde{J}(\bar{x}, u_{t,t+N_c-1}, N_c, N_p)$ be the cost functional (11) with $V_f(\cdot,\cdot)$ defined as $V_f(\tilde{x},\bar{x}) = \sum_{k=t}^{\infty}\{\varphi_L(k,t,\tilde{x},K)'(Q+K'RK)\varphi_L(k,t,\tilde{x},K)\}$. Recall that $J(\bar{x}, u_{t,t+N_c-1}, N_c, N_p)$ is the cost functional formed by (11) and (12), and $u^o_{t,t+N_c-1}(\bar{x})$ is the associated optimal solution. Let $\tilde{u}^o_{t,t+N_c-1}(\bar{x})$ be the optimal solution of the optimization problem with cost function $\tilde{J}$ and terminal region $\Omega_c(K)$. Furthermore, define $\zeta = f(x, \kappa(x))$, $\tilde{J}^o(\bar{x}, N_c, N_p) = \tilde{J}(\bar{x}, \tilde{u}^o_{t,t+N_c-1}, N_c, N_p)$. In order to prove that the SNRH control law

$$u = \kappa^{RH}(x) = u^o_{t,t}(x) \quad \text{(A.6)}$$

stabilizes the origin of (1) with output admissible set $X^o(N_c, N_p)$, it will be shown that, $\forall x \in X^o(N_c, N_p)$, $V(x) := \tilde{J}^o(x, N_c, N_p)$ is a Lyapunov function for the closed-loop system (14). To this aim, the main point is to prove that

$$\tilde{J}^o(\zeta, N_c, N_p) \leqslant \tilde{J}^o(x, N_c, N_p) - x'\Gamma x, \quad \forall x \in X^o(N_c, N_p), \quad \text{(A.7)}$$

where $\Gamma$ is a positive definite matrix. The keystone of the proof is the monotonicity property

$$\tilde{J}^o(\bar{x}, N_c, N_p) \leqslant \tilde{J}^o(\bar{x}, N_c - 1, N_p). \quad \text{(A.8)}$$

For this purpose, let $\tilde{u}^o_{t,t+N_c-2}$ be the optimal solution of the FHOCP where the cost function to be minimized is $\tilde{J}(\bar{x}, u_{t,t+N_c-2}, N_c - 1, N_p)$. It is clear that $\bar{u}_{t,t+N_c-1} = [\tilde{u}^o_{t,t+N_c-2}\ Kx]$ is an admissible solution for the FHOCP where the cost function to be minimized is $\tilde{J}(\bar{x}, u_{t,t+N_c-1}, N_c, N_p)$ so that, by optimality, the monotonicity property follows. To demonstrate (A.7) we have to show that

$$\tilde{J}^o(\zeta, N_c - 1, N_p) \leqslant \tilde{J}^o(x, N_c, N_p) - x'\Gamma x. \quad \text{(A.9)}$$

By definition, a sequence $u_{t,t+N_c-1}$ is admissible for the FHOCP with cost function $J$ and terminal constraints $\Omega_c(K)$ iff it is admissible for the optimization problem with cost function $\tilde{J}$ and terminal region $\Omega_c(K)$. By optimality arguments, and since in $J$ the series is truncated after a finite number of terms,

$$J(\bar{x}, u^o_{t,t+N_c-1}, N_c, N_p) \leqslant J(\bar{x}, \tilde{u}^o_{t,t+N_c-1}, N_c, N_p) \leqslant \tilde{J}^o(\bar{x}, N_c, N_p). \quad \text{(A.10)}$$

Letting

$$\varepsilon_V(M) := \tilde{J}(\bar{x}, u^o_{t,t+N_c-1}, N_c, N_p) - J(\bar{x}, u^o_{t,t+N_c-1}, N_c, N_p) \quad \text{(A.11)}$$

and $\tilde{x} := \varphi_L(t+N_p, t+N_c, \varphi(t+N_c, t, \bar{x}, u^o_{t,t+N_c-1}), K)$, where $\varphi(k, t, \bar{x}, u_{t,k-1})$ is the solution $x(k)$ of (1) for $k \geqslant t$, with initial state $\bar{x}$ and subject to the control sequence $u_{t,k-1}$, we have that $\varepsilon_V(M) \leqslant \sum_{k=t+N_p+M}^{\infty}\|\varphi_L(k, t+N_p, \tilde{x}, K)\|^2\lambda_{\max}(Q+K'RK)$. Since $\tilde{x} \in \Omega_c(K)$, from Lemma 1(iii) it follows $\varepsilon_V(M) \leqslant a\|\varphi_L(t+N_p+M, t+N_p, \tilde{x}, K)\|^2 \times \lambda_{\max}(Q+K'RK)\sum_{k=t+N_p+M}^{\infty} e^{-b(k-(t+N_p+M))}$. Observe that, the series in the above expression can be written as $\sum_{i=0}^{\infty}(e^{-b})^i = 1/(1-e^{-b})$. Then, from (13), it follows that

$$\varepsilon_V(M) \leqslant \rho\bar{x}'Q\bar{x}. \tag{A.12}$$

In view of (A.10)–(A.12)

$$\tilde{J}(\bar{x}, u^o_{t,t+N_c-1}, N_c, N_p) = J(\bar{x}, u^o_{t,t+N_c-1}, N_c, N_p) + \varepsilon_V(M) \leqslant \tilde{J}^o(\bar{x}, N_c, N_p) + \rho\bar{x}'Q\bar{x}. \tag{A.13}$$

Now, consider the following inequalities:

$$\tilde{J}^o(\zeta, N_c-1, N_p) \leqslant \tilde{J}(\zeta, u^o_{t+1,t+N_c-1}, N_c-1, N_p) = \tilde{J}(\bar{x}, u^o_{t,t+N_c-1}, N_c, N_p) - \bar{x}'Q\bar{x} - u^{o'}_{t,t}Ru^o_{t,t}. \tag{A.14}$$

Then, in view of (A.13) and (A.14) $\tilde{J}^o(\zeta, N_c-1, N_p) \leqslant \tilde{J}^o(\bar{x}, N_c, N_p) - (1-\rho)\bar{x}'Q\bar{x}$ which proves (A.9) with $\Gamma = (1-\rho)Q > 0$ and hence (A.7).

**Proof of Theorem 3.** Using the same notation as in the proof of Theorem 2, it suffices to observe that $J^K_{IH}(\bar{x}) = V_f(\bar{x},\bar{x}) + \varepsilon_V(M) \leqslant V_f(\bar{x},\bar{x}) + \rho\bar{x}'Q\bar{x}$. The limit follows from the fact that $\lim_{\rho\to 0}\varepsilon_V(M) = 0$.

**Proof of Theorem 4.** Using the same notation as in the proof of Theorem 2, and defining $\varphi^{RH}(k,t,\bar{x})$ as the solution $x(k)$ of (14), observe that

$$\begin{aligned}
J^o(\bar{x}, N_c, N_p) + \rho\bar{x}'Q\bar{x}
&\geqslant \tilde{J}^o(\bar{x}, N_c, N_p) \geqslant \tilde{J}^o(\varphi^{RH}(t+1,t,\bar{x}), N_c, N_p) \\
&\quad + \bar{x}'\Gamma\bar{x} + \kappa^{RH}(\bar{x})'R\kappa^{RH}(\bar{x}) \\
&\geqslant \tilde{J}^o(\varphi^{RH}(t+2,t,\bar{x}), N_c, N_p) \\
&\quad + \sum_{k=t}^{t+1} \varphi^{RH}(k,t,\bar{x})'\Gamma\varphi^{RH}(k,t,\bar{x}) \\
&\quad + \kappa^{RH}(\varphi^{RH}(k,t,\bar{x}))'R\kappa^{RH}(\varphi^{RH}(k,t,\bar{x})) \\
&\geqslant \cdots \geqslant \sum_{k=t}^{\infty} \varphi^{RH}(k,t,\bar{x})'\Gamma\varphi^{RH}(k,t,\bar{x}) \\
&\quad + \kappa^{RH}(\varphi^{RH}(k,t,\bar{x}))'R\kappa^{RH}(\varphi^{RH}(k,t,\bar{x})).
\end{aligned}$$

Then, by taking the limit as $\rho \to 0$, the last term in the chain of inequalities tends to $J^{RH}_{IH}(\bar{x}, N_c, N_p)$.

**Proof of Theorem 5.** Using the same notation as in the proof of Theorem 2, it suffices to observe that $\bar{J}^{RH}_{IH}(\bar{x}, N_c, N_p) = \lim_{\rho\to 0}(J^o(\bar{x}, N_c, N_p) + \rho\bar{x}'Q\bar{x}) = \tilde{J}^o(\bar{x}, N_c, N_p)$ and then from (A.8) it immediately follows that $\bar{J}^{RH}_{IH}(\bar{x}, N_c+1, N_p) \leqslant \bar{J}^{RH}_{IH}(\bar{x}, N_c, N_p)$. Moreover, from Theorem 3 and the definition of the FHOCP it follows that $\bar{J}^{RH}_{IH}(\bar{x}, 1, N_p) \leqslant J^K_{IH}$.

**Proof of Theorem 6.** (i) Given a state $\bar{x} \in \Omega_c(K)$, the control sequence obtained using the linear control law $u = Kx$ is a feasible solution for the FHOCP, $\forall N_c, N_p$ so that $\bar{x} \in X^o(N_c, N_p)$. (ii) Given a state $\bar{x} \in X^o(N_c, N_p)$, let $u_{t,t+N_c-1}$ be an associated feasible control sequence. Since $\Omega_c(K)$ is an invariant set for the linear control law (5), $u_{t,t+N_c-1}$ is a feasible solution also for the FHOCP with prediction horizon $N_p+1$, so that $\bar{x} \in X^{RH}(N_c, N_p+1)$. (iii) If $\bar{x} \in \bar{\Omega}(K)$, since $\bar{\Omega}(K)$ is a closed set, the linear control law (5) steers $\bar{x}$ into the terminal inequality set $\Omega_c(K)$ in a finite number of steps $\bar{N}_p$. Then, $\forall N_p > \bar{N}_p$ there always exists a feasible solution for the FHOCP, and therefore $\bar{x} \in X^{RH}(N_c, N_p)$.

**Proof of Lemma 7.** Since the linearization of the RH control law is evaluated in a neighborhood of the origin, we can neglect the state and control constraints (2). First it is shown that the control law is differentiable in the origin. Letting $\bar{u} := u_{t,t+N_c-1}$, re-write (11) subject to (1) as

$$J(\bar{x}, \bar{u}) = \bar{x}'W\bar{x} + \bar{x}'Y\bar{u} + \bar{u}'Z\bar{u} + h(\bar{x}, \bar{u}), \tag{A.15}$$

where $W$, $Y$, $Z$, derive from the terms of $J$ of order less than three and of $f(x,u)$ of order less than two. The function $h(\bar{x}, \bar{u})$, conversely, contains all the higher order terms.

The optimal control sequence $\bar{u}^o := u^o_{t,t+N_c-1}$, that is the minimum of (A.15), satisfies the necessary conditions $\partial J(\bar{x}, \bar{u}^o)/\partial\bar{u} = 0$, $\partial^2 J(\bar{x}, \bar{u}^o)/\partial\bar{u}^2 > 0$ that is

$$\alpha(\bar{x}, \bar{u}^o) := \bar{x}'Y + 2\bar{u}^{o'}Z + \partial h(\bar{x}, \bar{u}^o)/\partial\bar{u} = 0, \tag{A.16}$$

$$\partial\alpha(\bar{x}, \bar{u}^o)/\partial\bar{u} = 2Z + \partial^2 h(\bar{x}, \bar{u}^o)/\partial\bar{u}^2 > 0. \tag{A.17}$$

Since a term of order $n$ has the first $n-1$ derivatives equal to zero in the origin, inequality (A.17) is satisfied in a neighborhood of the origin because $R > 0$ implies $Z > 0$. The proof of the theorem is then obtained by applying Lemmas 9 and 10 (see the end of appendix), to the function (A.16). In fact $h(x, \bar{u})$, being the composition of a finite number of $C^2$ functions, is a $C^2$ function and then also $\alpha(\bar{x}, \bar{u}) \in C^1$. Moreover, $f(0,0) = 0$, implies that $h(0,0) = 0$. Finally, $\partial h(\bar{x}, \bar{u})/\partial\bar{u}|_{\bar{x}=0,\bar{u}=0} = 0$ because $h(x, \bar{u})$ is of order greater than two so that $\alpha(0,0) = 0$. Then, Lemmas 9 and 10, guarantee that, by an appropriate choice of the sets $\Gamma$ and $\Xi$, there exists one and only one solution $\bar{u} = \beta(\bar{x})$ such that (A.16) is satisfied with $\beta(\bar{x}) \in C^1$. Obviously, also the RH control law $u = \kappa(x)$ that is the first column of $\beta(\bar{x})$ is a $C^1$ function of $x$. The linearization of $\kappa(x)$ around the origin is now computed.

Imposing $\alpha(x, \beta(x)) = 0$, we have that

$$\frac{\partial\alpha(x,\beta(x))}{\partial x} = \frac{\partial\alpha(x,\beta(x))}{\partial x} + \frac{\partial\alpha(x,\beta(x))}{\partial u}\;\frac{\partial\beta(x)}{\partial x} = 0$$

and then

$$\frac{\partial\beta}{\partial x} = -\left[2Z + \frac{\partial^2 h}{\partial u^2}\right]^{-1}\left[Y + \frac{\partial}{\partial x}\left[\frac{\partial h}{\partial u}\right]\right]$$

Since $h$ is $o(x^2)$, then

$$\left.\frac{\partial \beta}{\partial x}\right|_{x=0} = -\frac{1}{2}Z^{-1}Y$$

and

$$\left.\frac{\partial \kappa}{\partial x}\right|_{x=0} = -[I0 \cdots 0]\frac{1}{2}Z^{-1}Y \tag{A.18}$$

that represents the linearization of the control law around the origin.

The rest of the proof follows simply noting that (A.18) coincide with the linear RH gain relative to the linearized system (4) subject to the cost functional (11). This linear RH control law has the expression $u = \kappa^{\mathrm{RH}}x$ (Bitmead et al., 1990).

**Proof of Theorem 8.** For $N_{\mathrm{p}}$ tending to infinity, the solution $S(k)$ of the difference Lyapunov equation in Lemma 7 converges to the unique constant equilibrium $\bar{S} = P$, where $P$ is the solution of the algebraic Riccati equation (10). Then, $\Sigma(0) = P$ implies $\Sigma(k) = P$, $\forall k \geqslant 0$ so that $K^{\mathrm{RH}} = K^{LQ}$.

The following two lemmas, used in the proof of Lemma 7, are reported without proof, which consists in an application of the implicit function theorem.

**Lemma 9.** *Let $\Omega$ be an open subset of $\Re^n \times \Re^m$ and $\alpha(x,u): \Omega \to \Re^n$ be a $C^1$ function of its arguments. Let, moreover, $(x_{\mathrm{o}}, u_{\mathrm{o}})$ be a point of $\Omega$ such that $\alpha(x_{\mathrm{o}}, u_{\mathrm{o}}) = 0$ and $(\partial\alpha/\partial u)(x_{\mathrm{o}}, u_{\mathrm{o}})$ be nonsingular. Then there exists an open neighborhood $\Gamma$ of $x_{\mathrm{o}}$ and an open neighborhood $\Xi$ of $u_{\mathrm{o}}$ such that, $\forall x \in \Gamma$, equation*

$$\alpha(x,u) = 0 \tag{A.19}$$

*has, in $\Xi$, one and only one solution $u = \beta(x)$. Finally, it is possible to choose $\Gamma$ and $\Xi$ such that the function $\beta$, is continuous.*

**Lemma 10.** *Let $\Omega$ be an open subset of $\Re^n \times \Re^m$, $\alpha: \Omega \to \Re^n$ be a $C^k$ function with $k \geqslant 1$, $\Gamma$ be an open subset of $\Re^n$ and $\beta: \Gamma \to \Re^m$ be a continuous function whose graphic is included in $\Omega$, such that $\alpha(x, \beta(x)) = 0$, $\forall x \in \Gamma$, and let $\Gamma' = \{x \in \Gamma : \det(\partial\alpha/\partial u)(x, \beta(x)) \neq 0\}$. Then $\Gamma'$ is an open subset of $\Re^n$ and $\beta$ is $C^k$ in $\Gamma'$.*

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**Lalo Magni** was born in Bormio (SO), Italy, in 1971. He received the Laurea degree (summa cum laude) in Computer Engineering from the University of Pavia, Italy, in 1994. He got his Ph.D. in Electronic and Computer Engineering in 1998. Currently he is Assistant Professor at the University of Pavia. During October 1996 to February 1997 and in March 1998 he was at CESAME, Universitè Catholique de Louvain, Louvain La Neuve (Belgium). During October to November 1997 he was at the University of Twente with the System and Control Group in the Faculty of Applied Mathematics. His current research interests include nonlinear control, predictive control, receding-horizon control, robust control and fault detection.

**Giuseppe De Nicolao** was born in Padova, Italy, on April 24, 1962. He received the Doctor degree in Electronic Engineering ("Laurea") from the Politecnico of Milano, Italy, in 1986. During 1987 to 1988 he was with the Biomathematics and Biostatistics Unit of the Institute of Pharmacological Researches "Mario Negri", Milano. During 1988 to 1992 he was a researcher of the Centro di Teoria dei Sistemi (Milano) of the

Italian National Research Council (CNR). He has been a visiting fellow at the Department of Systems Engineering of the Australian National University, Canberra (Australia) in 1991. He has been Associate Professor from 1992 to 2000, and Full Professor since 2000 in the Dipartimento di Informatica e Sistemistica at Università di Pavia, Italy. In 1998 he was a plenary speaker at the workshop on "Nonlinear model predictive control: Assessment and future directions for research", Ascona, Switzerland. His research interests include predictive and receding-horizon control, optimal and robust prediction and filtering, Bayesian learning and neural networks, linear periodic systems, deconvolution techniques, modeling and simulation of endocrine systems. On these subjects he has authored or coauthored more than 60 journal papers. Giuseppe De Nicolao Serves as an Associate Editor of the IEEE Transactions on Automatic Control.

**Lorenza Magnani** was born in Pavia, Italy, in 1974. She received the Laurea degree in Electrical Engineering from the University of Pavia, Italy, in 1998. She is presently Ph.D. student in Electronic and Computer Engineering at the University of Pavia, Italy. During November 2000 to February 2001 she was at the Systems, Signals and Control Department of the University of Twente, The Netherlands. Her research interests include receding-horizon control and hybrid systems.

**Riccardo Scattolini** was born in Milano, Italy, in 1956. He received the Laurea degree in Electrical Engineering from the Politecnico di Milano. Since 1982 he has held various research positions at the Politecnico di Milano and at the University of Pavia, where he is presently Full Professor of Process Control. He also spent one year working in industry on the simulation and control of chemical plants. During the academic year 1984/1985 he was with the Department of Engineering Science, Oxford University. In 1991 he was awarded the Heaviside Premium of the Institution of Electrical Engineers. During his academic career he also received various national awards for his research activity. He has been member of the IFAC Technical Committee on Automotive Control and of several Program Committees of International Conferences. His current research interests include predictive control of nonlinear systems and identification and fault detection of complex dynamic systems; the application of identification and digital control techniques to industrial problems. His research is witnessed by about 50 papers that have appeared in the main international journals of the field.