On the stabilization of nonlinear discrete-time systems with output feedback

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SUMMARY
This paper considers output-feedback control of nonlinear discrete-time systems using a nonlinear state-feedback controller combined with a nonlinear observer. Under suitable assumptions, either asymptotic stability or exponential stability of the closed-loop system are established. These results have a general validity and are essential in many advanced nonlinear control techniques, such as model predictive control of nonlinear discrete-time systems, where many algorithms for the solution of the state-feedback control problem have been proposed in recent years. Copyright © 2004 John Wiley & Sons, Ltd.

KEY WORDS: nonlinear systems; nonlinear control; output feedback; stability; nonlinear model prediction control

1. INTRODUCTION

This paper considers dynamic output feedback control of nonlinear discrete-time systems. Output feedback regulators are usually obtained by coupling a nonlinear state-feedback regulator [1] with a nonlinear observer [2, 3], such as the popular extended Kalman filter. For linear systems, it is well known that if a stabilizing control law is applied to the state estimate provided by a stable observer, the overall output feedback closed-loop system is stable [4]. For nonlinear systems whose linearization is stabilizable and detectable, the same arguments hold only locally, while the problem of extending local results is still largely open and of great interest. In fact, the use of observers for state recovery is essential in many advanced control techniques, such as in nonlinear model predictive control (see References [5, 6]) where many algorithms for the solution of the state-feedback control problem have been proposed in recent years.

For nonlinear continuous time systems, the stability properties provided by dynamic output feedback regulators have been analysed in References [1, 7]. Specifically, in Reference [1] only local and global results are derived while in Reference [7] \(\delta\)-regional results rely on the

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continuity of the state movement with respect to time, and as such they cannot be trivially extended to the discrete time case. Differently from local results, in \( \delta \)-regional results stability is guaranteed inside a compact ball of radius \( \delta \), whose value can be computed. In global results, stability is required in the whole space: this can be either an advantage or a drawback, since in real applications constraints in the state and control spaces are often present so that global results do not apply and one has to resort to \( \delta \)-regional results.

The design of an output feedback control law can also be performed by explicitly considering the problem of state reconstruction in the design of the feedback regulator. Along this line, in References [8, 9] it has been proposed to synthesize a robust stabilizing state-feedback control law based on the knowledge of a bound on the observer error; conversely in Reference [10] the approach taken is to synthesize a high gain observer ‘sufficiently fast’ with respect to the dynamics of the state feedback controlled system. In so doing, it is possible to achieve ‘semi-globally practically stable’ systems, but conservativeness must be introduced to rigorously prove the property. Specifically, robustness of the state feedback with respect to the estimation error may be achieved at the cost of a reduction of the overall performance, while with high gain observers there might be no explicit way to design the regulator. In Reference [11], a sampled scheme with a ‘sufficiently fast’ observer and a possible discontinuous state-feedback control law is considered. The state feedback control law must be robust in the face of measurement disturbance (the estimation error) and the action of an artificial input, necessary for the observation.

In this paper, the analysis problem is considered for discrete time systems. By extending the preliminary contributions reported in Reference [12], two main results are achieved. First, conditions on the nonlinear state-feedback regulator and the observer, under which the closed-loop system is \( \delta \)-regionally (or globally) asymptotically stable (Theorem 6) or \( \delta \)-regionally (or globally) exponentially stable (Theorem 7), are provided. Second, the assumption on the detectability of the linearized system is relaxed (Theorem 6). It is also shown (Corollary 9) that, with a ‘sufficiently fast’ observer, the size of the region of attraction essentially depends on the design of the state-feedback. This establishes an interesting link with the results reported in Reference [10], although the implementation in terms of sampled data systems considered in that paper allows one to circumvent the intrinsic limits due to the pure discrete time framework analysed here. These discrete time results are not available in the literature, since in Reference [13] only local results are derived.

The symbol \( \| \cdot \| \) denotes the Euclidean norm in \( \mathbb{R}^n \) (where the dimension \( n \) follows from the context); \( B_r \) denotes the compact ball of radius \( r \), i.e. \( B_r = \{ x \in \mathbb{R}^n : \| x \| \leq r \} \). A function \( W : \mathbb{R} \rightarrow \mathbb{R}_+ \) is said to belong to class \( K_{\infty} \) if: (a) it is continuous, (b) \( W(s) = 0 \Leftrightarrow s = 0 \), (c) it is non-decreasing, (d) \( W(s) \rightarrow \infty \) when \( s \rightarrow \infty \). For ease of reference some stability definitions are recalled in the appendix.

2. PROBLEM FORMULATION

Consider the nonlinear discrete-time dynamic system

\[
\begin{align*}
    x(k+1) &= f(x(k), u(k)), \quad x(t) = \bar{x}, \quad k \geq t \\
    y(k) &= h(x(k))
\end{align*}
\]  

(1)

(2)

where \( x \in \mathbb{R}^n \) is the state, \( y \in \mathbb{R}^m \) is the output, \( u \in \mathbb{R}^m \) is the input and \( f(0,0) = 0, h(0) = 0 \).
Assumption 1

\( f(\cdot, \cdot) \) and \( h(\cdot) \) are Lipschitz functions in \( B_{\xi_1} \subset \mathbb{R}^{n+m} \) and \( B_{\xi_2} \subset \mathbb{R}^n \), respectively, with Lipschitz constants \( L_f \) and \( L_h \), respectively.

Associated with system (1) let

\[ u = \kappa(x) \]  \hspace{1cm} (3)

with \( \kappa(0) = 0 \), be a state-feedback control law satisfying the following assumption.

Assumption 2

The state feedback control law (3) is Lipschitz with Lipschitz constant \( L_\kappa \) in a neighbourhood \( B_\rho \) of the origin, and the origin of the closed-loop system (1), (3) is an exponentially stable equilibrium point having a Lipschitz Lyapunov function \( V(\cdot) \in C^1 \), with Lipschitz constant \( L_V \). Moreover, there exist positive constants \( \rho, a, b, \) and \( c \) such that, for all \( x \in B_\rho \),

\[ a\|x\| \leq V(x) \leq b\|x\| \]  \hspace{1cm} (4)

\[ V(f(x, \kappa(x))) - V(x) \leq -c\|x\| \]  \hspace{1cm} (5)

Consider now a state observer for system (1)–(2)

\[ \dot{x}(k + 1) = g(\hat{x}(k), y(k), u(k)), \quad \dot{\hat{x}}(t) = \bar{x}, \quad k \geq t \]  \hspace{1cm} (6)

with \( g(0, 0, 0) = 0 \).

Assumption 3

Observer (6) is Lipschitz with Lipschitz constant \( L_r \) in a neighbourhood \( B_{r_x} \times B_{r_y} \times B_{r_u} \) of \((\hat{x}, y, u) = (0, 0, 0)\).

Assumption 4

Observer (6) is a ‘weak observer’ [14] with an associated function \( V_E(\cdot, \cdot) \in C^1 \) such that there exist functions \( a_E, b_E, c_E \) of class \( K_\infty \) satisfying

\[ a_E(\|x - \hat{x}\|) \leq V_E(x, \hat{x}) \leq b_E(\|x - \hat{x}\|), \quad \forall (x, \hat{x}) \in B_{r_x} \times B_{r_x} \]  \hspace{1cm} (7)

\[ \Delta V_E(x, \hat{x}) \leq -c_E(\|x - \hat{x}\|), \quad \forall (x, \hat{x}, u) \in B_{r_x} \times B_{r_x} \times B_{r_u} \]  \hspace{1cm} (8)

where

\[ \Delta V_E(x, \hat{x}) := V_E(f(x(k), u(k)), g(\hat{x}(k), h(x(k)), u(k))) - V_E(x(k), \hat{x}(k)) \]

Assumption 5

Observer (6) is a ‘weak exponential observer’ [14] with an associated function \( V_E(\cdot, \cdot) \in C^1 \) such that there exist functions \( a_E(\sigma) = a_E\sigma, b_E(\sigma) = b_E\sigma \) and \( c_E(\sigma) = c_E\sigma \) satisfying (7)–(8), with \( a_E, b_E, \) and \( c_E \) positive constants.

An output feedback regulator for system (1)–(2) is obtained by joining the state-feedback control law (3) to observer (6)

\[ \dot{\hat{x}}(k + 1) = g(\hat{x}(k), y(k), \kappa(\hat{x}(k))), \quad \dot{\hat{x}}(t) = \bar{x} \]  \hspace{1cm} (9)
\[ u(k) = \kappa(\hat{x}(k)) \]  

(10)

We are now in a position to state the main results of the paper.

**Theorem 6**

Assume Assumptions 1–4 hold. Then, the origin in \( \mathbb{R}^n \times \mathbb{R}^n \) is a \( \tilde{\delta} \)-regionally asymptotically stable (see appendix for the formal definition) equilibrium point of the closed-loop system (1)–(2), (9)–(10), where \( \tilde{\delta} \) is defined in (11). Moreover if Assumptions 1–4 are globally satisfied, then the origin in \( \mathbb{R}^n \times \mathbb{R}^n \) is a globally asymptotically stable equilibrium point of the closed-loop system (1)–(2), (9)–(10).

**Proof**

Define \( \bar{\delta} = \min \{ \xi_1, \xi_2, \rho, r_\xi, r_y, r_\mu \} \).

**Stability:** In the first part of the proof, given \( x, 0 < x \leq 1 \) and letting \( \bar{r} := x \bar{\delta} \), we compute \( \bar{\delta} \) such that if \( [x(t), \dot{x}(t)] \in B_{\bar{\delta}}, \) then \( [x(k), \dot{x}(k)] \in B_{\bar{\delta}} \) for all \( k \geq t \) (this proves stability of the equilibrium, see Definition 1 in the appendix).

Define the following quantities where \( \alpha_i \) are positive constants, \( 0 < \alpha_i \leq 1 \) (the notation \( \alpha_j : < c \) means that \( \alpha_j \) is defined as a quantity strictly less than \( c \)):

\[
\begin{align*}
\alpha_0 &:= x / 2 \\
\varepsilon &:= \alpha_0 \bar{r} \\
\alpha_1 &:= \min(1, 1 / L_\delta) \\
\alpha' &:= \alpha_1 \bar{r} \\
\alpha_2 &:= \min(\alpha_0, \alpha_1 / L_\delta, \alpha_1 / (L_f + L_L), \alpha_1 / (L_g + L_L)) \\
\alpha_3 &:= (\alpha / b) \alpha_2 \\
\delta' &:= \alpha_3 \bar{r} \\
\alpha_{4_i} &:= \alpha_1 - \alpha_2, \quad \alpha_{4_2} = \alpha_1 / L_\delta - \alpha_2 \\
\alpha_{4_3} &= \alpha_1 / L_f L_\delta - \alpha_2 (1 + L_\delta)/L_\delta, \quad \alpha_{4_4} = \alpha_1 / (L_g + L_L) - \alpha_2 \\
\alpha_{4_5} &= (\alpha / L_f L_\delta) \alpha_2 - (b / L_f L_L) \alpha_3, \quad \alpha_{4_6} = (c / L_f L_\delta) \alpha_3 \\
\alpha_{4_7} &= \alpha_0 - \alpha_2 \\
\alpha_{4_8} &= (\alpha_1 - (L_f + L_f L_\delta) \alpha_2) / (L_f L_\delta + L_g + L_L) \\
\mu_i &:= \alpha_{4_i} \bar{r}, \quad 1 \leq i \leq 8 \\
\alpha_{4_9} &:= \min(\alpha_{4_i}), \quad 1 \leq i \leq 8 \\
\mu &:= \alpha_{4_9} \bar{r} \\
\mu' &\text{ such that } b_E(\mu') \leq a_E(\mu) \\
\delta_\mu &:= \min(\delta', \mu') \\
\delta &:= \mu' - \delta_\mu
\end{align*}
\]
\[
\begin{align*}
\delta' &:= \min(\delta_\varepsilon, \delta_\varepsilon) \\
\varepsilon'' &:= \varepsilon' + \mu
\end{align*}
\] (11)

From the definitions of the above quantities it is immediate to derive some inequalities that will be useful in the sequel. In particular, since \(a_E(x) \leq b_E(x)\), then \(\mu' \leq \mu = \varepsilon'' - \varepsilon' < \varepsilon'' < r'\). Moreover, \(\mu < \mu_1, \delta' < \varepsilon' < r'\).

By the definition of \(\delta\), if \([x, \hat{x}] \in B_e\) then \(\hat{x} \in B_\delta\). In the following, we will show by induction that, given \(x(t) \in B_\delta\) and \(\hat{x}(t) \in B_\delta\), then \(\hat{x}(k) \in B_\delta\) and \(\hat{x}(k) \in B_\delta\), \(\forall k > t\). More precisely, we show that given \(x(t) \in B_\delta\), \(\hat{x}(t) \in B_\delta\), and \(x(k) \in B_\delta\) and \(\hat{x}(k) \in B_\delta\) for \(k = t, \ldots, t + i\), then we have \(x(t + i + 1) \in B_\delta\), \(\hat{x}(t + i + 1) \in B_\delta\) (to start the induction, note that \(\delta_\varepsilon < \delta' < \varepsilon', \delta_\varepsilon < \mu' < \mu = \varepsilon'' < \varepsilon'\)).

First, note that if \(x(t) \in B_e\) and \(\hat{x}(t) \in B_e\), then \(u(k) | x(t) \in B_e\) and \(v(k) \in B_e\) for \(k = t, \ldots, t + i\). Then we can use (7)–(8) to derive the following inequalities for \(k = t + 1, \ldots, t + i\):

\[
\begin{align*}
a_E(||x(k) - \hat{x}(k)||) &\leq V_E(x(k), \hat{x}(k)) \\
&\leq V_E(x(k - 1), \hat{x}(k - 1)) - c_E(||x(k) - \hat{x}(k)||) \\
&\leq V_E(x(t), \hat{x}(t)) - c_E(||x(k) - \hat{x}(k)||) - \ldots \\
&\quad - c_E(||x(t) - \hat{x}(t)||) \\
&\leq b_E(||x(t) - \hat{x}(t)||) - c_E(||x(t) - \hat{x}(t)||) \\
&\leq b_E(||x(t) - \hat{x}(t)||)
\end{align*}
\] (12)

From the definition of \(\delta_\varepsilon\) and \(\delta_\varepsilon\) it follows that if \(x(t) \in B_\delta\) and \(\hat{x}(t) \in B_\delta\) then \((\hat{x}(t) - x(t)) \in B_\mu\); in view of (13) and the definition of \(\mu'\),

\[
a_E(||x(k) - \hat{x}(k)||) \leq b_E(||x(t) - \hat{x}(t)||) \leq b_E(\mu') \leq a_E(\mu)
\]

and then

\[
||x(k) - \hat{x}(k)|| \leq \mu, \quad k = t, \ldots, t + i
\] (14)

From Assumptions 1 and 2, (14) and \(\mu < \varepsilon_4, \bar{r}\) the following inequalities follow:

\[
\begin{align*}
||x(t + i + 1)|| &= ||f(x(t + i), \kappa(\hat{x}(t + i)))|| \\
&\leq ||f(x(t + i), \kappa(\hat{x}(t + i))) + f(x(t + i), \kappa(\hat{x}(t + i))) - f(x(t + i), \kappa(\hat{x}(t + i)))|| \\
&\quad + ||f(x(t + i), \kappa(\hat{x}(t + i))) - f(x(t + i), \kappa(\hat{x}(t + i)))|| \\
&\leq (L_f + L_f \varepsilon) ||x(t + i)|| + L_f L_n ||x(t + i)|| \\
&\leq (L_f + L_f \varepsilon + L_f L_n) \varepsilon' + L_f L_n \varepsilon \varepsilon_4, \bar{r} \leq r'
\end{align*}
\]
From Assumption 3 and \( \mu < \mu_b \) we have that (recall that \( x(t) \in B_{\bar{r}}, k = t, \ldots, t+i \))
\[
\|\hat{x}(t+i+1)\| = \|x(t+i+1)\| + \|\hat{x}(t+i)\| - \|x(t+i+1)\| \\
\leq \|x(t+i+1)\| + \|\hat{x}(t+i) - x(t+i)\| \\
\leq (L_f + L_k \|x(t+i)\| + L_f L_k \|\hat{x}(t+i) - x(t+i)\| \\
+ \|g(\hat{x}(t+i), h(\hat{x}(t+i)), \kappa(x(t+i)) \\
- g(x(t+i), h(x(t+i)), \kappa(x(t+i)) \\
\leq (L_f + L_k \|x(t+i)\| \\
+ (L_f L_k + L_g + L_g L_k) \|\hat{x}(t+i) - x(t+i)\| \\
\leq (L_f + L_g L_k) \hat{x} + (L_f L_k + L_g + L_g L_k) \mu \\
\leq r'
\]

Having shown that \( \|\hat{x}(t+i+1)\| \leq r' \) and \( \|x(t+i+1)\| \leq r' \) then (4), (5), (7) and (8) hold for \( k = t, \ldots, t+i+1 \).

Now in view of Assumptions 1 and 2 and (5) we obtain
\[
V(x(t+i+1)) \\
= V(f(x(t+i), \kappa(\hat{x}(t+i)))) \\
\leq V(f(x(t+i), \kappa(\hat{x}(t+i)))) \\
+ L_k \|f(\hat{x}(t+i), \kappa(\hat{x}(t+i))) - f(x(t+i), \kappa(x(t+i)))\| \\
\leq V(f(x(t+i), \kappa(\hat{x}(t+i)))) + L_k L_f L_k \|\hat{x}(t+i) - x(t+i)\| \\
\leq V(x(t+i)) - c \|x(t+i)\| + L_k L_f L_k \|\hat{x}(t+i) - x(t+i)\| \\
\leq (15)
\]
and then from (14)
\[
V(x(t+i+1)) - V(x(t+i)) \leq - c \|x(t+i)\| + L_k L_f L_k \mu \\
\leq (16)
\]

Two distinct cases arise: either \( \|x(t+i)\| \leq \delta' \) or \( \|x(t+i)\| > \delta' \).

We first deal with the former possibility. Since \( \|x(t+i)\| \leq \delta' \), in view of (4) and (16)
\[
a \|x(t+i+1)\| \leq V(x(t+i+1)) \leq V(x(t+i)) + L_k L_f L_k \mu \\
\leq b \|x(t+i)\| + L_k L_f L_k \mu \leq b \delta' + L_k L_f L_k \mu
\]

Then \( x(t+i+1) \in B_{\bar{r}} \) since \( \mu \leq \mu_b \).

In the second case, we first identify all the integers \( l_1, l_2, \ldots, l_n < t+i \), such that \( \|x(l_j)\| \leq \delta' \) for \( j = 1, 2, \ldots, n \) (at worst, we get \( n = 1 \) with \( l_1 = 0 \) as \( \|x(l)\| \leq \delta < \delta' \)). It follows that \( \|x(k)\| > \delta' \) \( \forall k = l_n + 1, \ldots, t+i \); then, for all such values of \( k \), from (16) we have
\[
V(x(k+1)) \leq V(x(k)) - c \delta' + L_k L_f L_k \mu
\]
and, since \( \mu \leq \mu_b \), then \( V(x(k+1)) \leq V(x(k)) \) for all \( k = l_n + 1, l_n + 2, \ldots, t+i \). Therefore,
\[
V(x(t+i+1)) \leq V(x(t+i)) \leq V(x(t+i-1)) \leq \cdots \leq V(x(l_n+1))
\]}
From (16), (4) and \( \|x(t_n)\| \leq \delta' \) it follows that
\[
a \|x(t+i+1)\| \leq V(x(t+i+1)) \leq V(x(t_n+i+1))
\]
\[
\leq V(x(t_n)) + L_V L_f L_{k_\delta} \mu \leq b \|x(t_n)\| + L_V L_f L_{k_\delta} \mu
\]
and then \( \mu \leq \mu_5 \), implies \( \|x(t+i+1)\| < \delta' \).

If \( x(t) \in B_{\delta_5}, \hat{x}(t) \in B_{\delta_5} \), then, as seen before, we have \( (\hat{x}(t) - x(t)) \in B_{\delta'} \). Then, having shown that \( x(t+i+1) \in B_{\delta'} \) and \( \hat{x}(t+i+1) \in B_{\delta'} \), one can proceed as in (13)–(14) to obtain \( \|\hat{x}(t+i+1) - x(t+i+1)\| \leq \mu \).

By induction, finally, we find that if \( x(t) \in B_{\delta_5}, \hat{x}(t) \in B_{\delta_5} \Rightarrow x(k) \in B_{\delta'} \subseteq B_{\delta} \) and \( \hat{x}(k) \in B_{\delta'} \subseteq B_{\delta} \), \( \forall k \geq t \), so proving stability. Moreover \( \|u(k)\| = \|\kappa(\hat{x}(k))\| < L_{k_\delta} \delta'' < r' \) for all \( k \geq t \).

**Attractiveness:** We show that, \( \delta_5 \) and \( \delta_5 \) defined above are such that for each \( \varepsilon > 0 \), there exists an integer \( N(\varepsilon) \) such that if \( x(t) \in B_{\delta_5}, \hat{x}(t) \in B_{\delta_5} \), then \( \|x(t+i)\| \leq \varepsilon \) and \( \|\hat{x}(t+i)\| \leq \varepsilon \) for all \( i \geq N(\varepsilon) \) (see Definition 5 in the appendix).

If \( x(t) \in B_{\delta_5}, \hat{x}(t) \in B_{\delta_5} \), then \( x(k) \in B_{\delta'} \) and \( \hat{x}(k) \in B_{\delta'} \) for all \( i \geq 0 \) and from (15) we have that
\[
V(x(t+i+1)) - V(x(t+i)) \leq -c \|x(t+i)\| + L_V L_f L_{k_\delta} \|x(t+i) - \hat{x}(t+i)\|
\]
\[
\leq - \frac{c}{b} V(x(t+i)) + L_V L_f L_{k_\delta} \|x(t+i) - \hat{x}(t+i)\|
\]
Note that from (4)–(5) we have that
\[
0 < V(f(x, \kappa(x))) \leq V(x) - c \|x\|
\]
\[
\leq b \|x\| - c \|x\| = (b - c) \|x\|, \quad \forall x \neq 0
\]
so that \( 0 < 1 - (c/b) < 1 \). Letting \( \xi := 1 - (c/b) \), then,
\[
V(x(t+i+1)) \leq \xi V(x(t+i)) + L_V L_f L_{k_\delta} \|x(t+i) - \hat{x}(t+i)\|
\]
Repeatedly using this relationship leads to
\[
V(x(t+n)) \leq \xi^n V(x(t+i)) + \xi^{n-1} L_V L_f L_{k_\delta} \|x(t+i) - \hat{x}(t+i)\| + \cdots
\]
\[
+ L_V L_f L_{k_\delta} \|x(t+n-1) - \hat{x}(t+n-1)\|
\]
From Assumption 4 and the stability proof there exists \( \delta'' \) such that
\[
\|x(t+i) - \hat{x}(t+i)\| \leq \min \left( (1 - \xi) \varepsilon, \left( \left( 1 - \xi \right) \delta \right) / \left( 4 L_V L_f L_{k_\delta} \right), \frac{\varepsilon}{2} \right)
\]
for all \( i \geq i', \forall x(t) \in B_{\delta_5}, \forall \hat{x}(t) \in B_{\delta_5} \). From (19) this leads to
\[
V(x(t+i''+n)) \leq \xi^n V(x(t+i'')) + \sum_{i=0}^{n-1} \xi^i \left( 1 - \xi \right) \delta \varepsilon \left( 4 L_V L_f L_{k_\delta} \right)
\]
\[
= \xi^n V(x(t+i'')) + \left( 1 - \xi^n \right) \delta \varepsilon \left( 4 L_V L_f L_{k_\delta} \right)
\]
\[
\leq \xi^n V(x(t+i'')) + \frac{\delta \varepsilon}{4}
\]
Further, choose \( n' \) such that \( \xi'' V(x(t + i'')) \leq a_E/4 \); then, \( \xi'' V(x(t + i'')) \leq a_E/4 \) for all \( n \geq n' \), since \( \xi < 1 \). From (21)

\[
V(x(t + i'' + n)) \leq \frac{a_E}{2}
\]

for all \( n \geq n' \) and thus, from (4),

\[
\|x(t + i)\| \leq \frac{e_c}{2}
\]

for all \( i \geq N(e_c) \), where \( N(e_c) = i'' + n' \). Moreover from (20) and (22) it follows

\[
\|\dot{x}(t + i)\| \leq e_c
\]

for all \( i \geq N(e_c) \), so that asymptotic convergence of \( x \) and \( \dot{x} \) follow. Finally, combining stability and attractiveness for \( [x, \dot{x}] \in B_\delta, \bar{\delta} \)-regional asymptotic stability of the origin is proven (see Definition 5).

**Global asymptotic stability:** We show that for all pairs of positive numbers \( M, e_\epsilon \), there exists a finite integer \( N(M, e_\epsilon) \) such that, if \( [x, \dot{x}] \in B_M, \|x(t + i)\| \leq e_\epsilon \) and \( \|\dot{x}(t + i)\| \leq e_\epsilon \) for all \( i \geq N(e_\epsilon) \). This is proven by following the attractiveness analysis with \( \bar{\delta} = M, \mu' = 2M, \delta_x = \delta_\epsilon = M \) and \( \bar{r} \) such that \( \bar{r} > \mu'/2 \) (so that \( \delta' > \mu' \)) and \( b_E(\mu') < a_E(2\delta_\epsilon) \). Note that if Assumptions 1–4 are globally satisfied it is always possible to find a value of \( \bar{r} \) satisfying these conditions, because \( a_E \) is a function of class \( K_\infty \).

**Theorem 7**

Assume that Assumptions 1–5 hold. Then, the origin in \( \mathbb{R}^n \times \mathbb{R}^n \) is a \( \bar{\delta} \)-regionally exponentially stable (see appendix for the formal definition) equilibrium point of the closed-loop system (1)–(2), (9)–(10), where \( \bar{\delta} \) is defined in (11). Moreover if Assumptions 1, 2, 3, 5 are globally satisfied, then the origin in \( \mathbb{R}^n \times \mathbb{R}^n \) is a globally exponentially stable equilibrium point of the closed-loop system (1)–(2), (9)–(10).

**Proof**

The details are identical to those of the proof of Theorem 6, until we arrive at (18). In the present case \( a_E(\sigma), b_E(\sigma) \) and \( c_E(\sigma) \) are of the form \( a_E(\sigma) = a_E\sigma^n, b_E(\sigma) = b_E\sigma^n, c_E(\sigma) = c_E\sigma^n \) and the following inequalities hold:

\[
a_E\|x(t + i) - \dot{x}(t + i)\|^n \\
\leq V_E(x(t + i), \dot{x}(t + i)) \\
\leq V_E(x(t + i - 1), \dot{x}(t + i - 1)) - c_E\|x(t + i - 1) - \dot{x}(t + i - 1)\|^n \\
\leq \left(1 - \frac{c_E}{b_E}\right) V_E(x(t + i - 1), \dot{x}(t + i - 1)) \leq \cdots \leq \left(1 - \frac{c_E}{b_E}\right)^i V_E(x(t), \dot{x}(t)) \\
\leq \left(1 - \frac{c_E}{b_E}\right)^i b_E\|x(t) - \dot{x}(t)\|^n \quad i \geq 0
\]

Then,

\[
\|x(t + i) - \dot{x}(t + i)\| \leq \left(1 - \frac{c_E}{b_E}\right)^{i/n} \left(\frac{b_E}{a_E}\right)^{1/n} \|x(t) - \dot{x}(t)\|
\]

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with $0 < \xi_E < 1$, $\xi_E := (1 - c_E/b_E)^{1/n}$ (similar to 17) and thus, repeatedly using (18), we obtain

$$V(x(t + i)) \leq \xi^i V(x(t)) + L_V L_f L_n \left( \frac{b_E}{a_E} \right)^{1/n} \sum_{j=0}^{i-1} \xi^{i-j-1} \xi^n \|\hat{x}(t) - x(t)\|$$

$$\leq \xi^i V(x(t)) + L_V L_f L_n \left( \frac{b_E}{a_E} \right)^{1/n} \xi^{-1} \tilde{\xi}^n \|\hat{x}(t) - x(t)\|$$

$$\leq \xi^i V(x(t)) + L_V L_f L_n \left( \frac{b_E}{a_E} \right)^{1/n} \xi^{-1} \tilde{\xi}^n \|\hat{x}(t) - x(t)\|$$

where $\tilde{\xi} = \max\{\tilde{\xi}_E, \xi\}. Define \( \tilde{\xi}' < 1 \) and $\tilde{\xi}'' < 1$ such that $\xi = \tilde{\xi}' \tilde{\xi}''$. Let $K = -1/(e \log(\tilde{\xi}''))$. Then, $\forall i \geq 0$, $i \tilde{\xi}'' \leq K$ so that

$$V(x(t + i)) \leq \xi^i V(x(t)) + L_V L_f L_n \left( \frac{b_E}{a_E} \right)^{1/n} \tilde{\xi}^{-1} K \tilde{\xi}'' \|\hat{x}(t) - x(t)\|$$

and, in view of (4), (observe that $\tilde{\xi}'' \geq \xi$)

$$\|x(t + i)\| \leq \xi^i \left( \frac{b}{a} \right) \|x(t)\| + \left( \frac{1}{a} \right) L_V L_f L_n \left( \frac{b_E}{a_E} \right)^{1/n} \xi^{-1} K \tilde{\xi}'' \|\hat{x}(t) - x(t)\|$$

$$\leq \xi^i \left( \frac{b}{a} \right) \|x(t)\| + \left( \frac{1}{a} \right) L_V L_f L_n \left( \frac{b_E}{a_E} \right)^{1/n} \xi^{-1} K \tilde{\xi}'' \|\hat{x}(t)\| + \|x(t)\|$$

$$\leq K \tilde{\xi}'' \|x(t)\| + \left( \frac{1}{a} \right) L_V L_f L_n \left( \frac{b_E}{a_E} \right)^{1/n} \xi^{-1} \tilde{\xi}'' \|\hat{x}(t)\|$$

$$\leq K \tilde{\xi}'' \|x(t)\| + \|\hat{x}(t)\|$$

where $K \tilde{\xi} = 2 \max\{(b/a), (1/a)L_V L_f L_n(b_E/a_E)^{1/n} \tilde{\xi}^{-1} K\}$. In view of (23),

$$\|\hat{x}(t + i)\| - \|x(t + i)\| \leq \xi^i \left( \frac{b_E}{a_E} \right)^{1/n} \|\hat{x}(t)\| + \|x(t)\|$$

and then (note that $\tilde{\xi}'' \geq \xi$)

$$\|x(t + i)\| + \|\hat{x}(t + i)\| = (\|\hat{x}(t + i)\| - \|x(t + i)\|) + 2\|x(t + i)\|$$

$$\leq K_F \tilde{\xi}'' \left( \|x(t)\| + \|\hat{x}(t)\| \right)$$

for all $i \geq 0$, where $K_F = 2 \max\{2K, (b_E/a_E)^{1/n}\}$. Finally note that, if Assumptions 1, 2, 3, 5 are globally satisfied, then $K_F$ and $\tilde{\xi}''$ are independent of $[x(t), \hat{x}(t)]$, implying that the origin is globally exponentially stable.

The proof of Theorem 6 shows that the size of the region of attraction of the output feedback closed-loop system depends on the parameter $\tilde{\delta}$ defined in (11), which in turn is a non-trivial function of the parameters of the system, the control law and the observer. Concerning the latter, the parameters of interest are $a_E, b_E$ and $c_E$ defined in Assumption 4, and it is easy to see that the speed of convergence of the observer is proportional to $c_E$. It is now proven that with a sufficiently 'fast' observer, the parameter $\tilde{\delta}$ essentially depends on the state-feedback.
**Assumption 8**

For any $0 < \varepsilon_E < 1$, the observer (6) can be designed so that $\forall x \in B_{c_E}, \forall \hat{x} \in B_{\delta_E}$

$$b_E(||x - \hat{x}||) - c_E(||x - \hat{x}||) \leq \varepsilon_E a_E(||x - \hat{x}||)$$

(24)

If Assumption 8 holds the observer can be tuned in such a way that $c_E$ tends to $b_E$. In this case, the observer error goes to zero in only one step.

**Corollary 9**

Let the assumptions of Theorem 6 and Assumption 8 hold. Then, (a) the $\delta$-region depends only on the state-feedback closed-loop system and on the initial estimation error; (b) the speed of convergence of the closed-loop system does not depend on the observer.

**Proof**

**Stability:** The details are identical to those of the proof of Theorem 6 until we arrive at (12). From the definition of $\delta_E$ and $\delta_E$ it follows that if $x(t) \in B_{\delta_E}$ and $\hat{x}(t) \in B_{\delta_E}$ then $(\hat{x}(t) - x(t)) \in B_{\mu}$. Then, from Assumption 8 and (12) it follows that there exists $0 < \varepsilon_E < 1$ such that

$$||x(k) - \hat{x}(k)|| \leq \varepsilon_E ||x(t) - \hat{x}(t)||, \quad k = t + 1, \ldots, t + i$$

(25)

Then, letting $\mu' := \mu$ (instead of taking $\mu'$ such that $b_E(\mu') \leq a_E(\mu)$) it follows that (14) is satisfied. This means that with a fast observer the worst estimation error occurs at time $t$.

**Attractiveness:** For any $\varepsilon_c > 0$ and any $\varepsilon_E > 0$, one can select $\varepsilon_E, 0 < \varepsilon_E < 1$ such that from (25) it follows that $\forall x(t) \in B_{\varepsilon_c}, \forall \hat{x}(t) \in B_{\varepsilon_c}, ||x(t + \delta) - \hat{x}(t + \delta)||$ satisfies:

$$||x(t + \delta) - \hat{x}(t + \delta)|| < \min \left( \frac{(1 - \zeta) a_{c_E}}{(1 - \xi^{2})} \frac{\varepsilon_c E}{\varepsilon_E}, \varepsilon_c \varepsilon_E \right), \quad \delta > 0$$

Then, from (19)

$$V(x(t + n)) \leq \zeta^n V(x(t + 1)) + \sum_{i=0}^{n-1} \zeta^i L_{V} L_{f} L_{x} ||x(t - i + n) - \hat{x}(t - i + n)||$$

$$\leq \zeta^n V(x(t + 1)) + \sum_{i=0}^{n-1} \zeta^i \frac{(1 - \zeta)}{(1 - \xi^{2})} a_{c_E} \varepsilon_E$$

$$= \zeta^n V(x(t + 1)) + a_{c_E} \varepsilon_E$$

Further, choose $n'$ such that $\zeta^n V(x(t + 1)) \leq a_{c_E}$; then, since $\zeta < 1$, there exists $\varepsilon_E, 0 < \varepsilon_E < 0.5$ such that $\zeta^n V(x(t + 1)) \leq a_{c_E} (1 - 2 \varepsilon_E)$ for all $n > n'$. Thus

$$V(x(t + n)) \leq a_{c_E} (1 - 2 \varepsilon_E) + a_{c_E} \varepsilon_E$$

$$\leq a_{c_E} (1 - 2 \varepsilon_E) + a_{c_E} \varepsilon_E = a_{c_E} (1 - \varepsilon_E')$$

and

$$||x(t + \delta)|| \leq \varepsilon_c (1 - \varepsilon_E')$$

$$||\hat{x}(t + \delta)|| \leq ||\hat{x}(t + \delta) - x(t + \delta)|| + ||x(t + \delta)|| \leq \varepsilon_c$$

for all $t \geq N (\varepsilon_c)$, where $N(\varepsilon_c) = 1 + n'$ depends only by the full state-feedback system. Then (a) the $\delta$-region does not depend on $a_E$ and $b_E$ but depends only on the state-feedback closed-loop

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system and the initial estimation error; (b) the speed of convergence of the closed-loop system does not depend on the observer.

Remark 1
The result of Corollary 9 is very similar to those presented in [10] for sampled systems. In that paper it has been shown that with a ‘fast’ observer and a sufficiently small sampling period it is even possible to reach any subregion of the domain of attraction guaranteed by the state feedback control law. The achievement in Reference [10] of a stronger result is due to the possibility of arbitrarily reducing the sampling period. On the contrary, in the discrete-time framework analysed here, it is not possible to reduce the effect of the estimation error in the first step even with an ‘infinitely fast’ observer.

3. CONCLUSIONS

Stability properties of discrete-time output feedback controllers composed by a stabilizing control law and a stable observer have been established. Regional results are relevant because real applications are often characterized by the presence of constraints on state and control variables. Some of the assumptions on the state-feedback controller and the observer can be difficult to check or be satisfied only conservatively, in which case the guaranteed region of attraction may become small. Thus, future research will focus on either relaxing these conditions, or satisfying them via a suitable design of both the state-feedback control law (e.g. via nonlinear model predictive control) and the nonlinear observer.

APPENDIX A

Consider the unforced system
\[ x(k + 1) = f_u(x(k)), \quad x(t) = \bar{x}, \quad k \geq t \]  
(A1)
where it is assumed that \( f_u(0) = 0 \),

Definition 1
The equilibrium \( x = 0 \) of system (A1) is stable if, \( \forall \varepsilon > 0 \), there exists \( \delta = \delta(\varepsilon) \) such that
\[ ||\bar{x}|| < \delta(\varepsilon) \Rightarrow ||x(k)|| < \varepsilon, \quad \forall k \geq t \]

Definition 2
The equilibrium \( x = 0 \) of system (A1) is attractive if there exists a number \( M > 0 \) such that for each \( \varepsilon > 0 \) there exists \( N = N(\varepsilon) \) such that
\[ ||\bar{x}|| < M \Rightarrow ||x(k)|| < \varepsilon, \quad \forall k \geq t + N(\varepsilon) \]

Definition 3
The equilibrium \( x = 0 \) of system (A1) is asymptotically stable if it is stable and attractive.
**Definition 4**
The equilibrium $x = 0$ of system (A1) is exponentially stable if there exist constants $M, a > 0$ and $\rho < 1$ such that

$$||\tilde{x}|| < M \Rightarrow ||x(k)|| < a||\tilde{x}||\rho^{(k-t)}, \ \forall k \geq t$$

**Definition 5**
The equilibrium $x = 0$ of system (A1) is $\delta$-regionally asymptotically stable if (i) it is stable and (ii) for each $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that

$$||\tilde{x}|| < \delta \Rightarrow ||x(k)|| < \varepsilon, \ \forall k \geq t + N(\varepsilon)$$

**Definition 6**
The equilibrium $x = 0$ of system (A1) is $\delta$-regionally exponentially stable if there exist constants $a > 0$ and $\rho < 1$ such that

$$||\tilde{x}|| < \delta \Rightarrow ||x(k)|| < a||\tilde{x}||\rho^{(k-t)}, \ \forall k \geq t$$

**Definition 7**
The equilibrium $x = 0$ of system (A1) is globally asymptotically stable if (i) it is stable and (ii) for each pair of positive numbers $M, \varepsilon > 0$, there exists a finite number $N = N(M, \varepsilon)$ such that

$$||\tilde{x}|| < M \Rightarrow ||x(k)|| < \varepsilon, \ \forall k \geq t + N(M, \varepsilon)$$

**Definition 8**
The equilibrium $x = 0$ of system (A1) is globally exponentially stable if there exist constants $a > 0$ and $\rho < 1$ such that

$$||x(k)|| < a||\tilde{x}||\rho^{(k-t)}, \forall k \geq t, \ \forall \tilde{x} \in \mathbb{R}^n$$

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