PERGAMON

Annual Reviews in Control 28 (2004) 1–11

Annual Reviews in Control

www.elsevier.com/locate/arcontrol

Stabilizing model predictive control of nonlinear continuous time systems

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Received 20 October 2003; received in revised form 20 January 2004; accepted 24 January 2004

Abstract

This paper surveys some of the main design strategies of nonlinear model predictive control (*MPC*). The system under control, the performance index to be minimized and the state and control constraints to be fulfilled are defined in the continuous time. The considered algorithms are analyzed and compared in terms of stability, performance and implementation issues. In particular, it is shown that the solution of the optimization problem underlying the *MPC* formulation calls for (a) a suitable parametrization of the control variable, (b) the use of a suitable discretization of time, that is of a "sampled" control law and, (c) the numerical integration of the system over the considered prediction horizon. In turn, these implementation aspects are such that many theoretical results concerning stability have to be critically evaluated. In order to cope with these problems, two different methods guaranteeing stability are presented. One of them is used to global stabilize a pendulum.

Keywords: Model predictive control; Nonlinear continuous time systems; Sampled data systems

1. Introduction

The extraordinary industrial success of Model predictive control (*MPC*) techniques based on linear plant models, see e.g. the survey paper of Qin and Badgwell (1996), motivates the development of *MPC* algorithms for nonlinear systems. Nowadays there are many theoretical results, see Mayne, Rawlings, Rao, and Scokaert (2000) and Magni (2003), as well as industrial applications, see Qin and Badgwell (2000), which witness that *MPC* for nonlinear systems is going to have a diffusion and popularity similar to the one achieved by *MPC* algorithms for linear systems.

MPC methods for nonlinear systems are developed by assuming that the plant under control is either described by a continuous-time model, see Mayne and Michalska (1990), Michalska and Mayne (1993), Chen and Allgöwer (1998), Magni and Sepulchre (1997), Jadbabaie and Hauser (2001), and Jadbabaie, Primbs, and Hauser (2001), or by a discrete time one, see Keerthi and Gilbert (1988), De Nicolao, Magni, and Scattolini (1998), and Magni, De Nicolao, Magnani, and Scattolini (2001). A continuous time representation is much more natural, since the plant model is usually derived by resorting to first principles equations, but it results in a more difficult development of the *MPC* control law, which in principle calls for the solution of a functional optimization problem. As a matter of fact, the performance index to be minimized is defined in a continuous time setting and the overall optimization procedure is assumed to be continuously repeated after any vanishingly small sampling time, which often turns out to be a computationally intractable task. On the contrary, *MPC* algorithms based on a discrete time system representation are computationally simpler, but require the discretization of the model equations, so that they rely from the very beginning on an approximate system representation. Moreover, the performance index to be minimized as well as the state constraints only consider the system behavior in the sampling instants, so ignoring the intersample behavior, which in some cases could be significant in the evaluation of the control performance.

In this paper, the hybrid nature of sampled data control systems is fully considered. The plant under control, the state and control constraints and the performance index to be minimized are described in continuous time, while the manipulated signals are allowed to change at fixed and uniformly distributed sampling times. It is shown that a proper

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1367-5788/$ – see front matter
doi:10.1016/j.arcontrol.2004.01.001

choice of the terminal penalty and terminal inequality constraint is necessary in order to guarantee closed stability. Among the other possibilities, the stabilizing methods proposed in Keerthi and Gilbert (1988), Mayne and Michalska (1990), Michalska and Mayne (1993), Chen and Allgöwer (1998), De Nicolao et al. (1998) and Magni et al. (2001) are presented and compared in term of performance, enlargement properties and computational issues. In the second part of the paper, the effects of two approximation which must be introduced in the numerical solution of the optimization problem are considered: namely the parametrization of the input profile and the numerical integration of the system over the prediction horizon. Irrespective of the adopted algorithm, the parametrization of the input profile is required to limit the number of the optimization variables. In turn it forces the use of a congruent auxiliary control law, that is of a control law producing a signal compatible with the adopted parametrization. As for the numerical integration over the future prediction horizon, it must be performed at any sampling time for the on line solution of the optimization problem. However, in so doing the state constraints, originally posed in the continuous time, can only be checked at the integration time instants. Although the integration step can be definitely smaller than the sampling time, this means that a priori there is not any guarantee that these state constraints are fulfilled everywhere. When these approximations are considered, the *MPC* algorithms previously considered do not guarantee stability and constraints satisfaction, therefore two different nontrivial schemes are suggested in order to recover closed-loop stability (Magni & Scattolini, 2002; Magni, Scattolini, & Åström, 2002). One of them is used in the final section of the paper to globally stabilize a pendulum as well as to improve the performance and the stability region of the nonlinear energy control proposed in Åström and Furuta (2000).

2. Problem statement and preliminary results

Consider a plant P described by the nonlinear continuous-time dynamic system

$$\dot{x}(t) = f(x(t), u(t)), \quad t \geq 0,\ x(0) = x_0 \tag{1}$$

where $x \in R^n$ is the state, $u \in R^m$ is the input, $f(0, 0) = 0$ and $f(\cdot, \cdot)$ is a C^1 function of its arguments. The state and control variables are restricted to fulfill the following constraints:

$$x(t) \in X, \qquad u(t) \in U, \quad t \geq 0 \tag{2}$$

where X and U are compact subsets of R^n and R^m, respectively, both containing the origin as an interior point. The solution of (1) from the initial time $\bar{t}$ and initial state $x(\bar{t})$ for a control signal $u(\cdot)$ is denoted by $\varphi(t, \bar{t}, x(\bar{t}), u(\cdot))$.

Define by T_s a suitable sampling period and let $t_k = kT_s$, k is nonnegative integer, be the sampling instants; the goal is to determine a "sampled" feedback control law:

$$u(t) \equiv \kappa(t, x(t_k)), \quad \kappa(t, 0) = 0, \quad t \in [t_k, t_{k+1}) \tag{3}$$

which asymptotically stabilizes the origin of the associated closed-loop system.

The description of the hold mechanism implicit in (3) calls for a state augmentation. Letting $x_c := [x'\ x_1']' \in R^{2n}$, the closed-loop systems (1)–(3) is

$$\dot{x}_c(t) = \begin{bmatrix} f(x(t), \kappa(t, x_1(t))) \\ 0_{n,1} \end{bmatrix}, \quad t \in [t_k, t_{k+1}) \tag{4}$$

$$x_c(t_k) = \begin{bmatrix} x(t_k^-) \\ x(t_k^-) \end{bmatrix}$$

and its solution from the initial time $\bar{t}$ and initial state $x_c(\bar{t})$ is denoted by

$$\varphi_c(t, \bar{t}, x_c(\bar{t})) = \begin{bmatrix} \varphi_c^x(t, \bar{t}, x_c(\bar{t})) \\ \varphi_c^{x_1}(t, \bar{t}, x_c(\bar{t})) \end{bmatrix}$$

$$\varphi_c^x \in R^n, \qquad \varphi_c^{x_1} \in R^n$$

With reference to the closed-loop system (4), define the following sets.

Definition 1. A sampled output admissible set associated to (4) is a set $\Gamma_s^c(\kappa) \in R^n$ such that for all $x \in \Gamma_s^c(\kappa)$, $\varphi_c^x(t_{k+1}, t_k, [x'\ x']') \in \Gamma_s^c(\kappa)$, $\varphi_c^x(t, t_k, [x'\ x']') \in X$, $\kappa(t, \varphi_c^{x_1}(\tau, t, x_c)) \in U$, $t \in [t_k, t_{k+1})$,$\lim_{t\to\infty} \|\varphi_c^x(t, \bar{t}, x_c(\bar{t})))\| = 0$. In other words, $\Gamma_s^c(\kappa)$ is a state invariant set, associated to the closed-loop system (4), defined at the sampling instants t_k and such that (i) the state and control constraints (2) are satisfied in all the future continuous-time instants, (ii) the regulation problem is asymptotically solved. The (unique) maximal sampled output admissible set $X_s^c(\kappa)$ is defined as the union of all sampled output admissible sets.

Definition 2. An output admissible set associated to (4) is a set $\Gamma^c(t, \kappa) \in R^{2n}$ such that for all $x_c \in \Gamma^c(t, \kappa)$, $\varphi_c^x(t_k, t, x_c) \in X_s^c(\kappa)$, where t_k is the closest sampling time in the future, $\varphi_c^x(\tau, t, x_c) \in X$, $\kappa(t, \varphi_c^{x_1}(\tau, t, x_c)) \in U$, $\tau \in [t, t_k)$. In other words, $\Gamma^c(t, \kappa)$ is a set, defined at any continuous-time instant t, of states of the closed-loop system (4) such that (i) the state of (1) at the closest sampling time in the future belongs to $X_s^c(\kappa)$ and (ii) the state and control constraints (2) are satisfied in all the future continuous-time instants. The (unique) maximal output admissible set $X^c(t, \kappa)$ is defined as the union of all output admissible sets.

The regulation problem can now be formally stated as the problem of finding a sampled control law (3) such that its maximal output admissible set is nonempty. Such a control law will be called feasible hereafter. Besides, one can also wish to find the control law (3) with the largest maximal output admissible set X^c and which minimizes the Infinite-Horizon *IH* cost function:

$$J_{IH}(x(t_k), u(\cdot)) = \int_{t_k}^{\infty} \{\|x(\tau)\|_Q^2 + \|u(\tau)\|_R^2\}\, d\tau \tag{5}$$

subject to (1) and (2). In (5) Q and R are positive definite weighting matrices. Let X^{IH} be the set of states x such that the *IH* problem is solvable, $u^{IH}(\cdot)$ is the optimal solution and $\kappa^{IH}(t, x(t_k)) = u^{IH}(t)$, $t \in [t_k, t_{k+1})$*is* the optimal *IH* control law, so X^{IH} is the maximal output admissible set for (4) with $\kappa(\cdot, \cdot) = \kappa^{IH}(\cdot, \cdot)$. In general, the *IH* nonlinear optimal control problem is computationally intractable since minimization must be performed with respect to functions. Nevertheless, it constitutes a touchstone for suboptimal approaches yielding to nonlinear control laws $\kappa(t, x(t_k))$ with the following properties:

(i) performance/complexity trade-off: by suitably tuning the design parameters of the control synthesis algorithm, it should be possible to obtain a fair compromise between an arbitrarily good approximation of the optimal (but computationally intractable) *IH* controller κ^{IH} and a computationally cheap feasible control law;
(ii) enlargement property: the maximal output admissible set associated with $\kappa(\cdot, \cdot)$ should be larger than the maximal output admissible set of a possible already known feasible control law.

An effective strategy to design suboptimal controllers based on the *MPC* strategy will be presented in the following Section 3.

3. Model predictive control

In order to introduce the *MPC* algorithm, a finite-horizon optimization problem is first defined. Let $u_{t_1,t_2}: [t_1, t_2] \to R^m$ be a finite time control signal.

3.1. Finite horizon optimal control problem (FHOCP[1])

Given X_f, a compact subset of R^n containing the origin, a sampling time T_s, a prediction horizon N_p, two positive definite matrices Q and R, a penalty function $V_f(\cdot): R^n \to R$, at every sampling time instant t_k, minimize, with respect to $u_{t_k,t_{k+N_p}}$, the performance index

$$J_{FH}(x_{t_k}, u_{t_k,t_{k+N_p}}, N_p) = \int_{t_k}^{t_{k+N_p}} \{\|x(\tau)\|_Q^2 + \|u(\tau)\|_R^2\}\, d\tau + V_f(\varphi(t_{k+N_p}, t_k, x(t_k), u_{t_k,t_{k+N_p}})) \quad (6)$$

The minimization of (6) must be performed under the following constraints:

(i) the state dynamics (1) with $x(t_k) = x_{t_k}$;
(ii) the constraints (2), $t \in [t_k, t_{k+N_p})$;
(iii) the terminal state constraint $x(t_{k+N_p}) \in X_f$.

According to the Receding Horizon approach, the state-feedback *MPC* control law is derived by solving *FHOCP*[1] at every sampling time instant t_k, and applying the control signal $u(t) = u^o_{t_k,t_{k+1}}$, $t \in [t_k, t_{k+1})$ where $u^o_{t_k,t_{k+1}}$ is the first part of the optimal signal $u^o_{t_k,t_{k+N_p}}$. In so doing, one implicitly defines the sampled state-feedback control law

$$u(t) = \kappa^{RH}(t, x(t_k)), \quad t \in [t_k, t_{k+1}) \quad (7)$$

In order to establish the properties of the control law (7), first let

$$\varphi^{RH}(t, \bar{t}, x_c(\bar{t})) = \begin{bmatrix} \varphi_x^{RH}(t, \bar{t}, x_c(\bar{t})) \\ \varphi_{x_1}^{RH}(t, \bar{t}, x_c(\bar{t})) \end{bmatrix}$$

$$\varphi_x^{RH} \in R^n, \quad \varphi_{x_1}^{RH} \in R^n$$

be the solution of (4) with $\kappa(\cdot, \cdot) = \kappa^{RH}(\cdot, \cdot)$. Then, define the following sets.

Definition 3. Let $X_s^0(N_p) \in R^n$ be the set of states x_{t_k} of system (1) at the sampling times t_k such that there exists a feasible control sequence $u_{t_k,t_{k+N_p}}$ for *FHOCP*[1].

Definition 4. Let $X^0(t, N_p) \in R^{2n}$ be the set of states x_c such that for all $x_c(t) \in X^0(t, N_p)$, $\varphi_x^{RH}(t_k, t, x_c) \in X_s^0(N_p)$, where t_k is the closest sampling time in the future, $\varphi_x^{RH}(\tau, t, x_c) \in X$, $\kappa^{RH}(\tau, \varphi_{x_1}^{RH}(\tau, t, x_c)) \in U$, $\tau \in [t, t_k)$.

In order to guarantee the stability of the *MPC* closed-loop system, the terminal set X_f and the terminal cost function V_f introduced in the *FHOCP*[1] must be properly chosen.

Assumption 1. There exist an auxiliary control law $\kappa_f(x)$, a terminal set X_f and a terminal penalty V_f such that, letting $\varphi_f(t, \bar{t}, x(\bar{t}))$ the solution of the closed-loop system:

$$\dot{x}(t) = f(x(t), \kappa_f(x(t))), \quad (8)$$

from the initial time $\bar{t}$ and initial state $x(\bar{t})$, the following conditions hold:

- $X_f \subset X$, X_f closed, $0 \in X_f$;
- $\kappa_f(x) \in U$, $\forall x \in X_f$;
- X_f is positively invariant for (8);
- $V_f(\cdot): R^n \to R$ is such that $\forall x(t_k) \in X_f$

$$V_f(\varphi_f(t_{k+1}, t_k, x(t_k))) - V_f(x(t_k)) \le -\int_{t_k}^{t_{k+1}} \{\|\varphi_f(\tau, t_k, x(t_k))\|_Q^2 + \|\kappa_f(\varphi_f(\tau, t_k, x(t_k)))\|_R^2\}\, d\tau \quad (9)$$

Note that in Assumption 1, at this stage the auxiliary control law $\kappa_f(x)$ is not required to be a "sampled" control law because, as it will be clarified below in the description of some well known *MPC* algorithms, it is never applied to the systems but it is only used in simulation in order to obtain the terminal set and the terminal penalty.

Theorem 1. *Given an auxiliary control law κ_f, a terminal set X_f and a terminal penalty V_f satisfying* Assumption 1:

(i) *the origin is an asymptotically stable equilibrium point for the closed-loop system formed by (1) and (7) with maximal output admissible set* $X^0(t, N_p)$;
(ii) $X_s^0(N_p+1) \supseteq X_s^0(N_p)$, $\forall N_p$;
(iii) $X_s^0(N_p) \supseteq X_f$, $\forall N_p$;
(iv) *there exist a finite* $\bar{N}_p$ *such that* $X_s^0(\bar{N}_p) \supseteq \bar{X}_f$, *where* $\bar{X}_f$ *is the maximal output admissible set for (8).*

Proof of Theorem 2. In view of Definitions 1–4, if $X_s^0(N_p)$ is the maximal sampled output admissible set of (4) with $\kappa(\cdot,\cdot) = \kappa^{RH}(\cdot,\cdot)$ then $X^0(t, N_p)$ is the maximal output admissible set of (4) with $\kappa(\cdot,\cdot) = \kappa^{RH}(\cdot,\cdot)$. Note in fact that, from Definition 4 it follows that $\forall \bar{x}_c(t) \notin X^0(t, N_p)$ the constraints are not satisfied for $\tau \in [t, t_k)$ and/or $\varphi_x^{RH}(t_k, t, x_c) \notin X_s^0(N_p)$. But in view of Definition 2 this means that $\bar{x}_c(t)$ cannot belong to any output admissible set of (4) with $\kappa(\cdot) = \kappa^{RH}(\cdot)$. □

Moreover by Assumption 1 it follows that X_f is nonempty, then also $X^0(t, N_p)$ is nonempty. Let now show that $X_s^0(N_p)$ is the maximal sampled output admissible set for systems (1) and (7). In fact letting $x(t_k) := x_{t_k} \in X_s^0(N_p)$ and the associated solution $u^o_{t_k,t_{k+N_p}}$ of the *FHOCP*[1] at time t_k, a feasible solution at time t_{k+1} for the *FHOCP*[1] is

$$\tilde{u}_{t_{k+1},t_{k+N_p+1}} := \begin{bmatrix} u^o_{t_{k+1},t_{k+N_p}} & t \in [t_{k+1}, t_{k+N_p}) \\ \kappa_f(\varphi_f(t, t_{k+N_p}, \xi^{N_p}) & t \in [t_{k+N_p}, t_{k+N_p+1}) \end{bmatrix} \quad (10)$$

with $\xi^{N_p} := \varphi(t_{k+N_p}, t_k, x_{t_k}, u^o_{t_k,t_{k+N_p}})$. Then, by definition, $\xi := \varphi_x^{RH}(t_{k+1}, t_k, [x'_{t_k} \;\; x'_{t_k}]') \in X_s^0(N_p)$ and, in view of constraints (ii) of the *FHOCP*[1], (2) are satisfied along the trajectory of (4) with $\kappa(\cdot,\cdot) = \kappa^{RH}(\cdot,\cdot)$. Finally, for $\forall x(t_k) \notin X_s^0(N_p)$ the *MPC* control law is not defined so that $X_s^0(N_p)$ is the maximal sampled output admissible set.

Let now show that the origin is an asymptotically stable equilibrium point for the closed-loop systems (1), (7). To this end define

$$V(x_c(t), t) := J_{FH}(x(t), u^o_{t_k,t_{k+N_p}}, N_p)$$

if $t = t_k$ and

$$\begin{aligned} V(x_c(t), t) :&= \int_t^{t_{k+1}} \{\|\varphi_x^{RH}(\tau, t_k, x_c(t_k))\|_Q^2 \\ &\quad + \|\kappa^{RH}(\tau, \varphi_{x_1}^{RH}(\tau, t_k, x_c(t_k)))\|_R^2\}\, \mathrm{d}\tau \\ &\quad + J_{FH}(\varphi_x^{RH}(t_{k+1}, t, x_c(t)), u^o_{t_{k+1},t_{k+N_p}}, N_p - 1) \end{aligned}$$

if $t \in (t_k, t_{k+1})$. Note that $V(x_c(t_k), t_k)$ is bounded $\forall x \in X_k^o(N_p)$. Moreover,

- $\forall t \in [t_k, t_{k+1})$

$$\begin{aligned} &V(\varphi^{RH}(t, t_k, x_c(t_k)), t) \\ &\quad = V(x_c(t_k), t_k) - \int_{t_k}^t \{\|\varphi_x^{RH}(\tau, t_k, x_c(t_k))\|_Q^2 \\ &\qquad + \|\kappa^{RH}(\tau, \varphi_{x_1}^{RH}(\tau, t_k, x_c(t_k)))\|_R^2\}\, \mathrm{d}\tau \end{aligned} \quad (11)$$

- At time $t = t_{k+1}$, $\tilde{u}_{t_{k+1},t_{k+N_p+1}}$ given by (10) is a (sub-optimal) feasible solution for the new *FHOCP*[1] so that

$$\begin{aligned} &V(\varphi^{RH}(t_{k+1}, t_k, x_c(t_k)), t_{k+1}) \\ &\leq J_{FH}(\varphi_x^{RH}(t_{k+1}, t_k, x_c(t_k)), \tilde{u}_{t_{k+1},t_{k+N_p+1}} N_p) \\ &= V(\varphi^{RH}(t_{k+1}^-, t_k, x_c(t_k)), t_{k+1}^-) \\ &\quad + \int_{t_{k+N_p}}^{t_{k+N_p+1}} \{\|\varphi_c^x(\tau, t_{k+N_p}, \xi^{Np})\|_Q^2 \\ &\quad + \|\kappa_f(\tau, \varphi_f(\tau, t_{k+N_p}, \xi^{Np}))\|_R^2\}\, \mathrm{d}\tau \\ &\quad + V_f(\varphi_f(t_{k+N_p+1}, t_{k+N_p}, \xi^{Np})) - V_f(\xi^{Np}) \\ &\leq V(\varphi^{RH}(t_{k+1}^-, t_k, x_c(t_k)), t_{k+1}^-) \end{aligned} \quad (12)$$

In conclusion using (11) for $t \in [t_{k+i}, t_{k+i+1})$, $i \geq 0$, and (12) for $t = t_{k+i+1}$, $i \geq 0$, $\forall t \geq t_k$

$$\begin{aligned} &V(\varphi^{RH}(t, t_k, x_c(t_k)), t) \\ &\quad + \int_{t_k}^t \{\|\varphi_x^{RH}(\tau, t_k, x_c(t_k))\|_Q^2 \\ &\quad + \|\kappa^{RH}(\tau, \varphi_{x_1}^{RH}(\tau, t_k, x_c(t_k)))\|_R^2\}\, \mathrm{d}\tau \leq V(x_c(t_k), t_k) \end{aligned}$$

and, since Q and R are positive definite matrices, if $x_c(t_k) \in X^0(t_k, N_p)$, both $V(\varphi^{RH}(t, t_k, x_c(t_k)), t)$ and

$$\int_{t_k}^t \{\|\varphi_x^{RH}(\tau, t_k, x_c(t_k))\|_Q^2 + \|\kappa^{RH}(\tau, \varphi_{x_1}^{RH}(\tau, t_k, x_c(t_k)))\|_R^2\}\, \mathrm{d}\tau$$

are bounded. These facts prove that

$$\lim_{t\to\infty} \varphi_x^{RH}(t, t_k, x_c(t_k))) = 0$$

(Michalska & Vinter, 1994).

The proof of (ii)–(iv) can be derived as in Theorem 6 in Magni et al. (2001).

4. Stabilizing *MPC* control algorithm

Many stabilizing *MPC* algorithms can be obtained depending on the choices made to satisfy Assumption 1. Here, the main algorithms proposed in the literature are briefly described.

4.1. *Terminal equality constraint (EC)*

The first algorithm presented in literature is characterized by a terminal quality constraint $x(t_{k+N_p}) = 0$ so that $X_f =$

{0} (Keerthi & Gilbert, 1988; Mayne & Michalska, 1990). The auxiliary control law and the terminal penalty are defined only in the origin so that the following trivial functions can be chosen: $V_f(x) \equiv 0$ and $\kappa_f(x) \equiv 0$. The fulfillment of Assumption 1 is easily checked. In fact $X_f = \{0\} \in X$, $\kappa_f(0) = 0 \in U$, $f(0, \kappa_f(0)) = 0$ so that X_f is positively invariant and $V_f(0) - V_f(0) \leq 0$.

4.1.1. Performance/complexity trade-off

If a short optimization horizon N_p is used, the terminal constraint forces an excessive control effort, while increasing the optimization horizon increases the computational load.

4.1.2. Enlargement property

The maximal output admissible set coincides with the (constrained) controllability region $X^{\text{con}}(N_p)$ (Gilbert & Tan, 1991). Note that $X^{\text{con}}(N_p)$ may be "small". In particular, there is no guarantee that $X^c(N_p)$ is larger than the maximal output admissible set guaranteed by the trivial control law $\kappa(x) \equiv 0$.

4.2. Quadratic terminal penalty (QP)

A second well known method was presented in Chen and Allgöwer (1998) where a quadratic terminal penalty and a linear auxiliary control law are considered. More precisely, assume that system (1) is linearizable and denote the linearized matrices with

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0,u=0}, \qquad B = \left.\frac{\partial f}{\partial u}\right|_{x=0,u=0}$$

The auxiliary control law is given by $\kappa_f(x) = Kx$ where K is such that $A_{\text{cl}} = A + BK$ is Hurwitz. The terminal penalty is a quadratic function $V_f(x) = x'Px$ where P is the solution of the following Lyapunov function:

$$(A_{\text{cl}} + \kappa_\varepsilon I)'P + P(A_{\text{cl}} + \kappa_\varepsilon I) = \bar{Q} \tag{13}$$

where $\bar{Q} = Q + K'RK$ and the scalar $\kappa_\varepsilon \in [0, \infty)$ satisfies

$$\kappa_\varepsilon < -\lambda_{\max}(A_{\text{cl}})$$

The terminal region is defined as a level set of the terminal penalty.

$$X_f := \{x \in R^n | x'Px \leq \alpha\} \subset X \tag{14}$$

such that

(i) $Kx \in U$, for all $x \in X_f$;
(ii) X_f is a positively invariant for the closed-loop systems with $u = Kx$;
(iii) $\forall x \in X_f$

$$\frac{\mathrm{d}}{\mathrm{d}t} x'Px \leq -x'(Q + K'RK)x$$

subject to the closed-loop dynamic with $u = Kx$.

Satisfaction of Assumption 1 is easily checked. In fact $X_f \subset X$, X_f closed, $0 \in X_f$ in view of (14); $\kappa_f(x) \in U$, $\forall x \in X_f$ from (i); X_f is positively invariant for the closed-loop system (8) from (ii); finally $\forall x(t_k) \in X_f$

$$\begin{aligned} V_f(\varphi_f(t_{k+1}, t_k, x(t_k))) - V_f(x(t_k)) &\leq -\int_{t_k}^{t_{k+1}} \|\varphi_f(\tau, t_k, x(t_k))\|^2_{Q+K'RK}\, \mathrm{d}\tau \\ &= -\int_{t_k}^{t_{k+1}} \{\|\varphi_f(\tau, t_k, x(t_k))\|^2_Q \\ &\quad + \|\kappa_f(\varphi_f(\tau, t_k, x(t_k)))\|^2_R\}\, \mathrm{d}\tau \end{aligned}$$

4.2.1. Performance/complexity trade-off

In view of the constant κ_ε introduced in the Lyapunov equation (13), the infinite horizon optimal performance are not recovered even if the auxiliary control law is locally optimal and $\|x\| \to 0$. The infinite horizon optimal control law can be reached only at the cost of a "long" prediction (optimization) horizon N_p.

4.2.2. Enlargement property

For any horizon N_p the maximal output admissible set includes the terminal set X_f, but a sufficient long optimization horizon N_p occurs in order to enlarge the maximum output admissible set of the auxiliary control law.

4.3. Infinite-Horizon closed-loop costing (CL)

A third method was presented in De Nicolao et al. (1998) where the infinite horizon cost associated with a generally nonlinear auxiliary control law is used as the terminal penalty. More precisely, assume that an auxiliary locally stabilizing control law is given by $\kappa_f(x)$. The terminal penalty is given by

$$\begin{aligned} V_f(x(t_k)) = \int_{t_k}^{\infty} &\{\|\varphi_f(\tau, t_k, x(t_k))\|^2_Q \\ &+ \|\kappa_f(\varphi_f(\tau, t_k, x(t_k)))\|^2_R\}\, \mathrm{d}\tau \end{aligned}$$

The terminal region is implicitly defined as

$$\begin{aligned} X_f := \{&\bar{x} \in R^n | \varphi_f(t, t_k, x(t_k)) \in X, \\ &\kappa_f(\varphi_f(t, t_k, x(t_k))) \in U, t > \bar{t}, \\ &V_f(\bar{x}) \text{ bounded}\} \subset X \end{aligned} \tag{15}$$

The fulfillment of Assumption 1 is easily checked. In fact $X_f \subset X$, X_f closed, $0 \in X_f$, $\kappa_f(x) \in U$, $\forall x \in X_f$ in view of (15); X_f is positively invariant for (8) because, from the definition of X_f, it follows that if $\bar{x} \in X_f$, $\varphi_f(t, t_k, x(t_k) \in X_f$, $t > \bar{t}$; finally $\forall x(t_k) \in X_f$

$$\begin{aligned} V_f(x(t_{k+1})) - V_f(x(t_k)) = -\int_{t_k}^{t_{k+1}} &\{\|\varphi_f(\tau, t_k, x(t_k))\|^2_Q \\ &+ \|\kappa_f(\varphi_f(\tau, t_k, x(t_k)))\|^2_R\}\, \mathrm{d}\tau \end{aligned}$$

4.3.1. Performance/complexity trade-off

If the auxiliary control law is locally optimal, then also the *MPC* control law is locally optimal. If the auxiliary control law is the solution of the unconstrained nonlinear infinite horizon control problem, then the *MPC* control law is the solution of the constrained infinite horizon optimal control problem.

4.3.2. Enlargement property

The maximal output admissible set guaranteed by this control scheme is larger than the maximal output admissible set of the auxiliary control law for any optimization horizon N_p. In fact X_f is equal to the maximal output admissible set of the auxiliary control law. Note that in the *QP* algorithm the terminal region X_f must be explicitly computed off-line, while in this one the explicit computation of X_f is not needed. This difference is crucial in view of the difficulty to compute the maximal output admissible set for a nonlinear system with constraints, so that only very conservative approximations can be obtained.

4.4. Infinite-Horizon closed-loop costing with control and prediction horizons (CL-2H)

The *CL* algorithm achieves better properties with respect to *EC* and *QP* both for performance and for the enlargement property with a lower computational burden. Its drawback is that the terminal penalty cannot be computed exactly because the integration of the closed-loop system (8) should be performed for an infinite time. However, since the auxiliary control law is stabilizing, in practice it is possible to compute the cost function until the state is sufficiently close to the origin. In Magni et al. (2001), this problem is analyzed and a rule to compute a finite integration horizon that preserves closed-loop stability is given. Otherwise, it is necessary to guarantee that the state is in an output admissible set of the auxiliary control law. Consequently, an output admissible set for the auxiliary control law must be computed off line as in the *QP* algorithm. In order to recover the properties of the *CL* algorithm without increasing the computational burden, the use of a control (optimization) horizon shorter than the prediction horizon was proposed in Magni et al. (2001). In particular the terminal penalty and the terminal inequality constraint are imposed at the end of the prediction horizon, while the optimization is performed only with respect to a shorter horizon, the so called control horizon. The control signal from the end of the control horizon to the end of the prediction horizon is given by the auxiliary control law. Accordingly, the optimization problem is changed in the following way.

4.4.1. Finite Horizon Optimal Control Problem (FHOCP[2])

Given X_f, a compact subset of R^n containing the origin, the sampling time T_s, the control horizon N_c, the prediction horizon N_p, an auxiliary control law $u = \kappa(x)$, two positive definite matrices Q and R, a penalty function $V_f(\cdot) : R^n \to R$, at every sampling time instant t_k, minimize, with respect to $u_{t_k,t_{k+N_c}}$, the performance index

$$J_{FH}(x_{t_k}, u_{t_k,t_{k+N_c}}, N_c, N_p) = \int_{t_k}^{t_{k+N_p}} \{\|x(\tau)\|_Q^2 + \|u(\tau)\|_R^2\}\,\mathrm{d}\tau + V_f(\varphi(t_{k+N_p}, t_k, x(t_k), u_{t_k,t_{k+N_p}}))$$

The minimization of (6) must be performed under the following constraints:

(i) the state dynamics (1) with $x(t_k) = x_{t_k}$;

(ii) $$u(t) := \begin{cases} u_{t_k,t_{k+N_c}}, & t \in [t_k, t_{k+N_c})] \\ \kappa_f(\varphi_f(t, t_{k+N_c}, \xi^{N_c})) \\ t \in [t_{k+N_c}, t_{k+N_p})] \end{cases}$$

where $\xi^{N_c} := \varphi(t_{k+N_c}, t_k, x_{t_k}, u_{t_k,t_{k+N_c}})$.

(iii) the constraints (2), with $u(t) = \kappa(x)$, $t \in [t_k, t_{k+N_p})$;
(iv) the terminal state constraint $x(t_{k+N_p}) \in X_f$.

4.4.2. Performance/complexity trade-off

If the auxiliary control law is locally optimal, then also the *MPC* control law obtained with $N_p \to \infty$, is locally optimal. If the auxiliary control law is the solution of the unconstrained nonlinear infinite horizon control problem, then the *MPC* control law obtained with $N_p \to \infty$, is the solution of the constrained infinite horizon optimal control problem.

4.4.3. Enlargement property

The maximal output admissible set guaranteed by this control scheme is larger than the maximal output admissible set of the auxiliary control law for any optimization horizon N_c and for a sufficiently large prediction horizon N_p. Note that the computational burden is mostly related the length of the control horizon.

5. Implementation and nominal stability

In view of Theorem 1, the *EC*, *QP*, *CL* and *CL–2H MPC* algorithms surveyed in the previous section guarantee nominal stability provided that the underlying optimization problem is efficiently solved on line. As a matter of fact, this is not possible in practice for a couple of reasons, namely the requirement to fulfill the state and control constraints at any continuous time instant and the necessity to use a suitable parametrization of the input signal. As for the first issue, it is apparent that the numerical integration of the system over the future prediction horizon is such that the state (and control) constraints can be verified only at the integration time instants. However, in the original problem formulation they are required to be fulfilled at any continuous time t. The second problem is even more intriguing and can be explained as follows: the on line optimization problem underlying the *MPC* algorithms can be completed in the sampling period only with respect to a finite number of parameters, instead of functions, so that the input profile must be parametrized. For example, a typical procedure consists in assuming that the input signal is held constant between two successive sampling instants. In turn, this means that also the adopted auxiliary control law must produce a congruent signal, that is a signal that satisfies the adopted parametrization, otherwise the signal $\tilde{u}_{t_{k+1},t_{k+N_p+1}}$ computed at t_k and given in (10) would not be a feasible solution for the (parametrized) *FHOCP* at time t_{k+1}. With respect to this problem, note that the *EC* method (where the auxiliary control law is obtained by setting equal to zero the control variable) cannot be solved in a finite number of iterations in view of the zero terminal equality constraints, while in all the other *MPC* algorithms the stability proof relies on an auxiliary control law providing a general continuous time control signal, a priori not congruent with the adopted parametrization.

In the following, the two implementation issues above discussed are analyzed in a reverse order. Specifically, two algorithms are first presented guaranteeing closed-loop stability when a piece-wise constant control parametrization is used. Then, the problem of the fulfillment of (2) is solved by specifying the (more restrictive) constraints to be imposed at any integration time during the on line optimization.

5.1. The piece-wise constant MPC control law

In Magni and Scattolini (2004), a piece-wise constant signal parametrization has been considered in the numerical solution of the optimization problem. Specifically, it has been suggested to use a time invariant auxiliary control law

$$u(t) \equiv \kappa(x(t_k)), \quad t \in [t_k, t_{k+1}), \tag{16}$$

with $\kappa(0) = 0$, satisfying the following assumption:

Assumption 2. The feasible control law (16) is a C^1 function with Lipschitz constant L_κ.

For control law (16), an associated sampled output admissible set can be computed as follows. First, define the linearization of system (1) at the origin

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{17}$$

Then introduce the discretization of (17) given by

$$x(t_{k+1}) = A_D x(t_k) + B_D u(t_k), \quad x(0) = x_0 \tag{18}$$

with

$$A_D := \mathrm{e}^{AT_s}, \qquad B_D := \int_0^{T_s} \mathrm{e}^{A\eta} B \mathrm{d}\eta$$

Finally, let

$$K = \left. \frac{\partial \kappa(x)}{\partial x} \right|_0$$

In view of the feasibility of (16), it is then easy to show that the closed-loop matrix $A_D^{\mathrm{cl}} := A_D + B_D K$ of the linearized discrete-time system (18) is Hurwitz and the following result holds.

Lemma 1. *Let $\kappa(x)$ be a feasible control law, suppose that Assumption 2 is satisfied and consider a positive definite matrix $\tilde{Q}$ and two real positive scalars γ and γ_2 such that $\gamma < \lambda_{\min}(\tilde{Q})$. Define by Π the unique symmetric positive definite solution of the following Lyapunov equation*:

$$A_D^{\prime\mathrm{cl}} \Pi A_D^{\mathrm{cl}} - \Pi + \bar{Q} = 0 \tag{19}$$

where

$$\bar{Q} = \int_0^{T_s} A_c^{ZOH}(\eta)' \tilde{Q} A_c^{ZOH}(\eta)\, \mathrm{d}\eta + \gamma_2 I_n$$

and

$$A_c^{ZOH}(t) := \mathrm{e}^{At} + \left(\int_0^t \mathrm{e}^{A(t-\tau)}\, \mathrm{d}\tau \right) BK$$

Then, there exist two constants $T_s \in (0, \infty)$ and $c \in (0, \infty)$ specifying a neighborhood $\Omega_c(\kappa, T_s)$ of the origin of the form

$$\Omega_c(\kappa, T_s) = \{x \in \Re^n \,|\, \|x\|_\Pi^2 \le c\} \tag{20}$$

such that $\forall x \in \Omega_c(\kappa, T_s)$:

(i) $\varphi_c^x(t, t_k, [x'\ x']') \in X,\ t \in [t_k, t_{k+1}),\ \kappa(x) \in U$;

(ii) $$\|\varphi_c^x(t_{k+1}, t_k, [x'x']')\|_\Pi^2 - \|x\|_\Pi^2 \le -\gamma \int_{t_k}^{t_{k+1}} \|\varphi_c^x(\eta, t_k, [x'x']')\|^2 \,\mathrm{d}\eta - \gamma_2 \|x\|^2 \tag{21}$$

The Lemma states that, in view of (i) and (ii), $\Omega_c(\kappa, T_s)$ is a sampled output admissible set for (4); moreover, from (ii) $V_L(x) = x'\Pi x$ is a positive definite function decreasing in the sampling times along the trajectory of (4).

Remark 1. An obvious way to determine a feasible sampled control law is to choose a suitable T_s, to consider the linearization of (1) around the origin and the sampled linear model described by (18) and to synthesize with any standard linear control synthesis technique, a linear control law

$$u(t) = Kx(t_k), \quad t \in [t_k, t_{k+1}) \tag{22}$$

such that $A_D + B_D K$ is Hurwitz. □

Let us suppose that the auxiliary control law $u = \kappa(x)$ is known, together with an associated sampled output admissible set and the Lyapunov function both given in Lemma 1. It is now shown how *MPC* allows one to stabilize the closed-loop system, to extend the maximal output admissible set of κ and to improve the control performance by minimizing a cost function suitably chosen by the designer.

To this end, given a control sequence

$$\bar{u}_{1,N_c}(t_k) := [u_{1_{t_k}}, u_{2_{t_k}}, ..., u_{N_{ct_k}}]$$

with $N_c \geq 1$, define the Finite Horizon piece-wise constant control signal

$$u_{t_k}^{FH}(t) = \begin{cases} u_{j_{t_k}} \\ t \in [t_{k+j-1}, t_{k+j})\ j = 1, \ldots, N_c \\ \kappa(\varphi(t_{k+j-1}, t_k, x(t_k), u_{t_k}^{FH}(\cdot))) \\ t \in [t_{k+j-1}, t_{k+j})\ j = N_c + 1, \ldots, N_p \end{cases} \tag{23}$$

where $N_p \geq N_c$. Moreover, denote by $\bar{u}_{t_k}^{FH}(t_{fin}, t_{in})$ the signal $u_{t_k}^{FH}(t)$ in the interval $t \in [t_{in}, t_{fin})$.

For system (1) the *MPC* control problem here considered is based on the solution of the following

5.1.1. Finite Horizon Optimal Control Problem (FHOCP[3]*)*

Given the sampling time T_s, the control horizon N_c, the prediction horizon N_p, $N_c \leq N_p$, two positive definite matrices Q and R, a feasible auxiliary control law $\kappa(x)$, the matrix Π and the region $\Omega_c(\kappa, T_s)$ given in Lemma 1 with $\gamma > \lambda_{\max}(Q)$ and $\gamma_2 > T_s\lambda_{\max}(R)L_\kappa$, at every sampling time instant t_k, minimize, with respect to $\bar{u}_{1,N_c}(t_k)$, the performance index

$$\begin{aligned} &J_{FH}(x_{t_k}, \bar{u}_{1,N_c}(t_k), N_c, N_p) \\ &\quad = \int_{t_k}^{t_{k+N_p}} \{\|x(\tau)\|_Q^2 + \|u(\tau)\|_R^2\}\, d\tau \\ &\qquad + V_f(\varphi(t_{k+N_p}, t_k, x(t_k), \bar{u}_{t_k}^{FH}(t_{k+N_p}, t_k))) \end{aligned} \tag{24}$$

where the terminal penalty V_f is selected as

$$V_f(x) = \|x\|_\Pi^2$$

The minimization of (24) must be performed under the following constraints:

(i) the state dynamics (1) with $x(t_k) = x_{t_k}$;
(ii) the constraints (2), $t \in [t_k, t_{k+N_p})$ with u given by (22);
(iii) the terminal state constraint $x(t_{k+N_p}) \in \Omega_c(\kappa, T_s)$.

According to the Receding Horizon approach, the state-feedback *MPC* control law is derived by solving *FHOCP*[3] at every sampling time instant t_k, and applying the constant control signal $u(t) = u^o_{1_{t_k}}$, $t \in [t_k, t_{k+1})$ where $u^o_{1_{t_k}}$ is the first column of the optimal sequence $\bar{u}^o_{1,N_c}(t_k)$. In so doing, one implicitly defines the sampled state-feedback control law

$$u(t) = \kappa^{RH}(x(t_k)), \quad t \in [t_k, t_{k+1}) \tag{25}$$

Remarkably, the algorithm proposed here satisfies all the assumptions of Theorem 1, so that closed-loop stability can be guaranteed.

5.2. Prestabilized MPC control scheme

In many cases, a stabilizing continuous time control law $\kappa(x(t))$ is already known and applied to the plant. For this reason, the scheme already proposed in Magni et al. (2002) is presented here. It can be used to improve the performance provided by $\kappa(x(t))$ with a reduced computational effort.

Given the control law $\kappa(x(t))$, the problem is to determine with the *MPC* approach an additive feedback control signal $v(t)$, such that the overall resulting control law:

$$u(t) = \kappa(x(t)) + v(t) \tag{26}$$

enlarges the stability region of $\kappa(x(t))$ and enhances the overall control performance with the fulfillment of the constraints (2).

The closed-loop system (1), (26) is described for $t \geq \bar{t}$ by

$$\dot{x}(t) = f(x(t), \kappa(x(t)) + v(t)) \tag{27}$$

with $x(\bar{t}) = \bar{x}$. Hence, for system (27) the *MPC* problem can be formally stated as follows: consider the control sequence

$$\bar{v}_{1,N_c}(t_k) := [v_{1_{t_k}}, v_{2_{t_k}}, \ldots, v_{N_{ct_k}}]$$

with $N_c \geq 1$, for any $t \geq t_k$ define the associated piece-wise constant control signal

$$v(t) = \begin{cases} v_{j_{t_k}} & t \in [t_{k+j-1}, t_{k+j}),\ j = 1, ..., N_c \\ 0 & t \geq t_k + N_c T_s \end{cases} \tag{28}$$

and consider the following

5.2.1. Finite Horizon Optimal Control Problem (FHOCP[3]*)*

Given the positive integers N_c and N_p, $N_c \leq N_p$ at every "sampling time" instant t_k, minimize, with respect to $\bar{v}_{1,N_c}(t_k)$, the performance index

$$\begin{aligned} &J_{FH}(x_{t_k}, \bar{v}_{1,N_c}(t_k), N_c, N_p) \\ &\quad = \int_{t_k}^{t_k+N_pT_s} \{\|x(\tau)\|_Q + \|v(\tau)\|_R\}\, d\tau + V_f(x(t_k+N_pT_s)) \end{aligned} \tag{29}$$

As for the terminal penalty V_f, it is here selected such that Assumption 1 is satisfied with $\kappa_f(x) = 0$.

The minimization of (29) must be performed under the following constraints:

(i) the state dynamics (27) with $x(t_k) = x_{t_k}$;
(ii) the constraints (2), $t \in [t_k, t_k + N_pT_s)$ with u given by (26);
(iii) $v(t)$ given by (28);
(iv) the terminal state constraint $x(t_k + N_pT_s) \in X_f$, where X_f is a set satisfying Assumption 1 with $\kappa_f(x) \equiv 0$.

According to the well known Receding Horizon approach, the state-feedback *MPC* control law is derived by solving the *FHOCP*[3] at every sampling time instant t_k, and applying the constant control signal $u(t) = \kappa(x) + v^o_{1_{t_k}}$, $t \in [t_k, t_{k+1})$ where $v^o_{1_{t_k}}$ is the first column of the optimal sequence $\bar{v}^o_{1,N_c}(t_k)$. In so doing, one implicitly defines the discontinuous (with respect to time) state-feedback control law

$$v(t) = \kappa(x) + \kappa^{RH}(x_k), \quad t \in [t_k, t_{k+1}) \tag{30}$$

Again, all the assumptions of Theorem 1 are satisfied by the algorithm, so that closed-loop stability is achieved.

5.3. *Continuous time state space constraint fulfillment*

Recall that the second issue related to the practical implementation of *MPC* algorithms for continuous time systems was related to the fulfillment of the continuous time constraints (2) at any time instant t. This problem can be easily solved by forcing in the on line optimization a finite number of suitable constraints on the state variable at any integration time step. In particular, letting δ be the (maximum) integration step used in the optimization phase to simulate the plant (1) with the control signal (23) and defining by $B_r^\nu = \{x \in R^n : \|x\|_\nu \le r\}, \nu > 0$, the following result holds (Magni & Scattolini, 2004).

Theorem 2. *Let*

$$M := \|\max_{x\in B_g^\nu, u\in U} f(x,u)\|_\nu$$

if (a) $0 < \delta < g/M$, $g > 0$, *(b)* $x(\bar t) \in B_{\bar g}^\nu$, $\bar g = g - \delta M$, *then* $\varphi(\bar t + t, \bar t, x(\bar t), \bar u) \in B_g^\nu$, $\forall t \in [0, \delta)$, $\bar u \in U$.

From this result it is clear that one can choose the maximum integration step δ and a more conservative discrete-time state constraint (defined by $\bar g$) so as to guarantee continuous-time state constraint satisfaction. More precisely, given ν and g such that $B_g^\nu \subseteq X$, a nonnegative integer n_s, a constant integration step $\delta = T_s/n_s$, condition (*ii*) in the *FHOCP* can be replaced by

$$\|\varphi(t_k + n_\delta\delta, t_k, x(t_k), u_{t_k}^{FH}(\cdot))\|_\nu \le \bar g,$$
$$n_\delta = 1, 2, \dots, < N_p n_s - 1, \quad u(t_{k+j}) \in U, j = 0, \dots, N_p$$

The use of a more conservative constraints set has already been proposed for linear systems in Berardi, De Santis, Di Benedetto and Pola (2001). Notably the conservatism introduced is substantially less than the one of the discrete-time *MPC*, because the maximum integration time δ can be chosen much smaller than the sampling time T_s.

6. Global stabilization of a pendulum

In this section the global stabilization of a pendulum is solved using as pre-stabilizing control law the nonlinear energy control proposed in Åström and Furuta (2000). The *MPC* control law, according to the scheme described in Section 5.2, is used to improve performance and to achieve global stability. The equation of motion of a pendulum, written in normalized variables (Åström & Furuta, 2000), is

$$\ddot\theta(t) - \sin\theta(t) + u(t)\cos\theta(t) = 0, \qquad (31)$$

where θ is the angle between the vertical and the pendulum, assumed to be positive in the clockwise direction, and u is the normalized acceleration, positive if directed as the positive real axis. The system has two state variables, the angle θ and its rate of change $\dot\theta$ (i.e. $x = [\theta\ \dot\theta]'$), defined taking θ modulo 2π, with two equilibria, i.e. $u = 0$, $\theta = 0$, $\dot\theta = 0$, and $u = 0$, $\theta = \pi$, $\dot\theta = 0$. Moreover, it is assumed that $|u| \le n$.

The normalized total energy of the uncontrolled system ($u = 0$) is

$$E_n(t) = \tfrac{1}{2}\dot\theta^2(t) + \cos\theta(t) - 1$$

Consider now the energy control law

$$u(t) = \mathrm{sat}_n(k_u E_n(t)\mathrm{sign}(\dot\theta(t)\cos\theta(t))) \qquad (32)$$

where sat_n is a linear function which saturates at n. In Åström and Furuta (2000) it is shown that the control law (32) is able to bring the pendulum at the upright position provided that its initial condition does not coincide with the download stationary position (in fact, with $\theta = \pi$, $\dot\theta = 0$, (32) gives $u = 0$ so that the pendulum remains in the download equilibrium). However, the upright equilibrium is an unstable saddle point. For this reason, when the system approaches the origin of the state space, a different strategy is used to locally stabilize the system. In the reported simulations, a linear control law computed with the *LQ* method applied to the linearized system has been used. This switching strategy, synthetically called in the sequel again "energy control", is described by the control law

$$\kappa(x) = \begin{cases} \mathrm{sat}_n(k_u E_n \mathrm{sign}(\dot\theta\cos\theta)), & \text{if } x_{2\pi} \notin \Omega(K) \\ -Kx'_{2\pi} & \text{if } x_{2\pi} \in \Omega(K), \end{cases} \qquad (33)$$

where $x_{2\pi} := [\mathrm{mod}_{2\pi}(\theta)\dot\theta]$, K is the gain of the locally stabilizing *LQ* control law and $\Omega(K)$ is an associated output admissible set.

The *NMPC* control algorithm described in Section 5.2 has been applied to the closed-loop systems (31) and (33), with

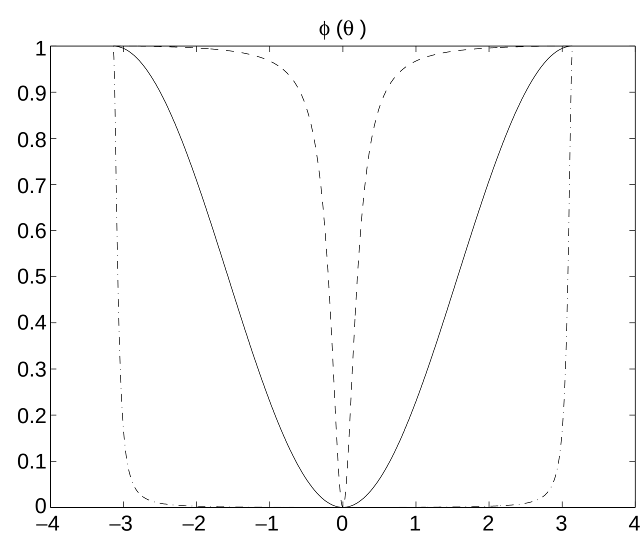

Fig. 1. $\phi(\theta)$ with $\beta = 0.01$ (dash-dot line), $\beta = 1$ (continuous line), $\beta = 100$ (dashed line).

Table 1

N_c	0	2	4	8
$\beta = 0$				
J_{1H}	73.9	70.0	68.7	67.0
%	0	−5.3	−7.0	−9.4
$\beta = 0.01$				
J_{1H}	64.7	60.6	61.0	58.7
%	0	−6.3	−5.7	−9.2
$\beta = 1$				
J_{1H}	37.0	35.9	35.5	35.5
%	0	−3.2	−4.2	−4.2
$\beta = 100$				
J_{1H}	26.4	26.2	25.9	25.7
%	0	−0.6	−1.8	−2.3

the aim of enhancing the performance provided by (33) in terms of the energy required to swing up the pendulum and of the time required to reach the upright position.

For this reason, the stage-cost of the *FHOCP* is given by

$$\psi(x, u) = \phi(\theta)E_n^2 + (1 - \phi(\theta))V_n \tag{34}$$

where

$$V_n = k_{v1}\frac{1}{2}\sin^2\left(\frac{1}{2}\theta\right) + \frac{1}{2}\dot{\theta}^2 \tag{35}$$

and

$$\phi(\theta) = \frac{\beta\tan^2(\theta/2)}{1 + \beta\tan^2(\theta/2)} \tag{36}$$

The function V_n given by (35) penalizes the state deviation from the origin, while $\phi(\theta)$ allows to balance the need to reduce the total energy applied and to bring the state to zero. The dependence of $\phi(\theta)$ from the parameter β is shown in Fig. 1.

In the following simulation examples the saturation limit is $n = 0.29$, the *FHOCP* is solved every $T_s = 0.1$ sec and the following parameters are used to synthesize the *NMPC* control law.

- Auxiliary control law (33): $k_u = 100$, K is the *LQ* control gain with state penalty matrix $Q = \text{diag}\{2.5, 1\}$, and control penalty matrix $R = 1$; $\Omega(K)$ is given by

  $$\Omega(K) = \{x \in \Re^n | x'_{2\pi} P x_{2\pi} \leq c\}$$

  where P is the solution of the Riccati equation for the solution of the *LQ* control problem and $c = 0.001$.
- *FHOCP*: $N_p = 2500$, $k_{v1} = 10$, $\Omega_c(\kappa) = \Omega(K)$, $\tilde{Q} = Q + K'RK$, $\gamma = \lambda_{\min}(\tilde{Q})/20$.

For different choices of the tuning parameters, and starting from the initial condition $[\pi, 0]$, the results summarized in the Table 1 have been obtained. In the table, J_{IH} is the

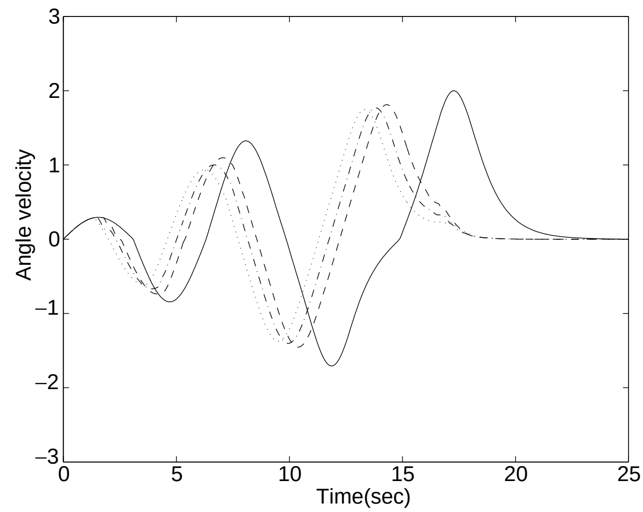

Fig. 3. Angle velocity movement with "energy control" (continuous line), *MPC* with $\beta = 0$ and $N_c = 2$ (dash-dot line), $N_c = 4$ (dashed line) and $N_c = 8$ (dotted line).

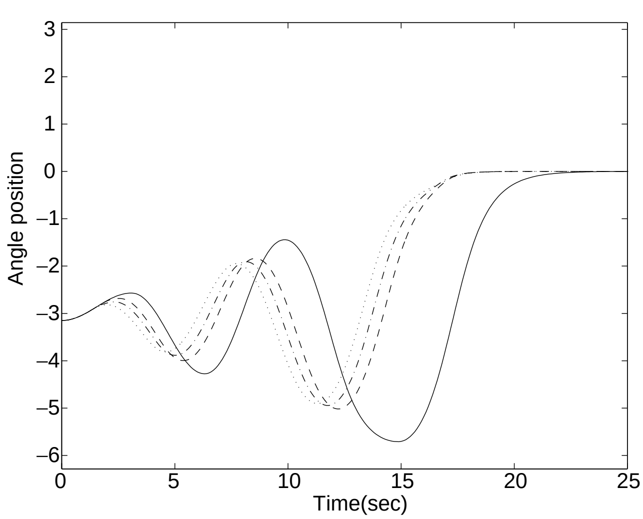

Fig. 2. Angle position movement with "energy control" (continuous line), *MPC* with $\beta = 0$ and $N_c = 2$ (dash-dot line), $N_c = 4$ (dashed line) and $N_c = 8$ (dotted line).

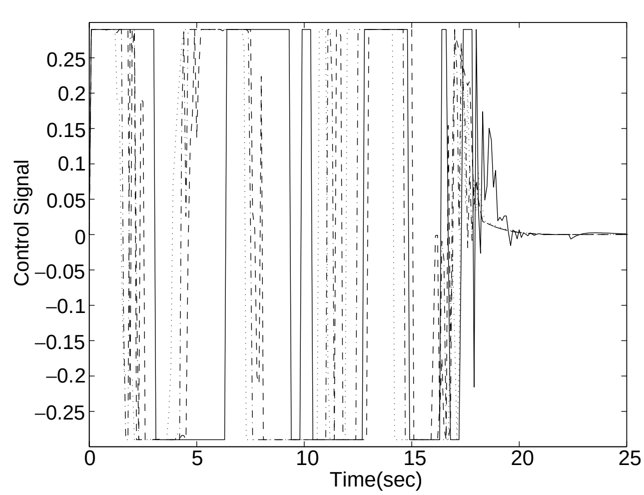

Fig. 4. Control signal with "energy control" (continuous line), *MPC* with $\beta = 0$ and $N_c = 2$ (dash-dot line), $N_c = 4$ (dashed line) and $N_c = 8$ (dotted line).

infinite horizon performance index with stage cost (34) and % is the variation with respect to the performance provided by the "energy control" strategy. Note that for $N_c = 0$, when the "energy control' strategy is used, a numerical error is sufficient to move the pendulum in the maximal output admissible set guaranteed by the energy control strategy. On the contrary, with $N_c \geq 1$, the *MPC* control law guarantees the global stabilization of the inverted pendulum. Moreover note that the best improvement is obtained with a low value of β because in this case the energy is not considered in the cost function. In Figs. 2–4 the movement of the angle position, of the velocity and of the control signal are reported for the control strategies with $\beta = 0$ and for different control horizons N_c.

7. Conclusions

In this paper, the main algorithms for the stabilization of continuous time nonlinear systems with the *MPC* approach have been critically analyzed. In particular, emphasis has been given on the necessity to resort to a "sampled" implementation of the methods and on all the theoretical issues related to the implementation phase. It has been shown how the need to use a suitable parametrization of the control signal forces the selection of a coherent auxiliary control law, usually required to compute the terminal penalty and the terminal state constraint providing closed-loop stability. Also the problem to guarantee the satisfaction of the state constraints has been addressed and solved by reformulating these constraints in the integration times. Many problems are still open and widely studied in *MPC* for nonlinear systems, among them we recall the development of stabilizing output feedback solutions, for which some significant results have already been suggested in Magni, De Nicolao, and Scattolini (1998) and Findeisen, Imsland, Allgöwer, and Foss (2003), or the analysis of the algorithms when disturbances act on the system.

Acknowledgements

The authors acknowledge the partial support by *MURST* Project "New techniques for the identification and adaptive control of industrial systems".

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