On the solution of the tracking problem for non-linear systems with MPC

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This paper presents a Model Predictive Control (MPC) algorithm for non-linear systems which solves the tracking problem for asymptotically constant references. Closed-loop stability of the equilibrium and asymptotic zero-error regulation are guaranteed. The performance of the method is discussed with the classical Continuous Stirred Tank Reactor (CSTR) control application.

Keywords: Non-linear systems; Nonlinear Model Predictive Control; Discrete-time systems; Tracking

1. Introduction

Nowadays, many Model Predictive Control (MPC) algorithms are available to solve the regulation problem with guaranteed stability for non-linear systems; see Mayne et al. (2000). However, for a practical use of non-linear MPC, some issues are still to be faced. Among them, the fundamental problem of tracking constant exogenous reference signals using output feedback stabilizing regulators is still largely open. In this context, an input–output approach was used in De Nicolao et al. (1997), while in Magni et al. (2001) a state-space algorithm for tracking exogenous signals (not necessarily constant) and the asymptotic rejection of disturbances generated by a properly defined exosystem was developed. Although this method solves the tracking and disturbance rejection problems, its applicability can be limited by the computational load. For example, in the case of piecewise constant references, it needs to compute the equilibrium corresponding to the required constant output. In Findeisen et al. (2000), the pseudolinearization method described in Reboulet and Champetier (1984) is adopted to derive a model predictive control strategy. In so doing, an invertible state variable change and a state feedback law are introduced such that the transformed and controlled closed-loop system has the same (constant) linearization independent of the constant reference signal. Considering systems for which a pseudolinearization can be obtained, the MPC control law stabilizes all the fixed set points in the considered set point family. However, the computation of the pseudolinearization transformation and of the feedback law may be rather cumbersome.

This paper presents a new MPC algorithm for non-linear models which guarantees local stability and asymptotic tracking of constant references. The proposed method builds on the state-space regulation approach described in De Nicolao et al. (1998), and its main peculiarities with respect to other approaches e.g. Magni et al. (2001), are:

1. It does not require the knowledge of the steady-state value of the input and state variables corresponding to the desired equilibrium. In fact, the control parameters, like the terminal penalty and the terminal constraint, are directly implicitly related to the reference signal, and their explicit computation is not required.
2. It allows for the use of pre-programmed references, while in Magni et al. (2001), the reference is viewed as the output of a given exosystem.
3. It takes advantage of the a priori knowledge of a locally stabilizing linear regulator, which can be the one already used to control the plant. This property allows one to easily integrate the MPC algorithm.

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in an already existing structure, made for example by standard PID regulators, with the aim to enhance performance. Alternatively, the linear regulator can be derived by means of empirical design procedures, for example the classical Ziegler–Nichols tuning rules. In this case, the synthesis of the linear regulator does not even need the computation of the linearized plant model.

4. It can be applied also in the output feedback case by resorting to the ‘separation principle’ for non-linear systems presented in Magni et al. (2004).

The paper is structured as follows. In section 2, the control problem is stated, while the new MPC method is presented in section 3, together with its stabilizing properties. Finally, in section 4, the performances of the proposed algorithm are tested on a simulated Continuous Stirred Tank Reactor (CSTR) control application.

2. Problem formulation

Consider a non-linear dynamic system

\[ \xi(k + 1) = f(\xi(k), u(k)) \]
\[ y(k) = h(\xi(k)), \]  

where \( k \) is the discrete time index, \( \xi \in \mathbb{R}^n \), \( u \in \mathbb{R}^m \), \( y \in \mathbb{R}^m \) are the state, input and output vectors respectively, \( f(\cdot, \cdot) \) and \( h(\cdot) \) are \( C^1 \) functions of their arguments. Assume now that, for a given vector \( \tilde{y}^0 \in \mathbb{R}^m \), there exists an equilibrium \( (\tilde{\xi}(\tilde{y}^0), \tilde{u}(\tilde{y}^0)) \) such that

\[ \tilde{\xi}(\tilde{y}^0) = f(\tilde{\xi}(\tilde{y}^0), \tilde{u}(\tilde{y}^0)) \]
\[ \tilde{y}^0 = h(\tilde{\xi}(\tilde{y}^0)). \]

The problem is to design a control algorithm such that for any constant reference signal \( y^0 \in \Gamma \), where \( \Gamma \) is an open neighbourhood of \( \tilde{y}^0 \), the resulting closed-loop system reaches an asymptotically stable equilibrium point \( (\tilde{\xi}(y^0), \tilde{u}(y^0)) \) such that \( y^0 = h(\tilde{\xi}(y^0)) \). Moreover, the state and control variables are required to fulfil the following constraints:

\[ \xi(k) \in \Xi, \quad u(k) \in U, \quad k \geq t, \]  

where \( \Xi \) and \( U \) are compact subsets of \( \mathbb{R}^n \) and \( \mathbb{R}^m \), containing \( \tilde{\xi}(\tilde{y}^0) \) and \( \tilde{u}(\tilde{y}^0) \), respectively, as an interior point. Although this appears as a reasonable assumption, the reader should be warned that it rules out the case of constraints that are active at a steady state, which may be a case of industrial interest. Assuming that the equilibrium \( (\tilde{\xi}(y^0), \tilde{u}(y^0)) \) exists, let

\[ \tilde{A}(\tilde{y}^0) = \frac{\partial \tilde{f}_k}{\partial \xi} \bigg|_{\xi(\tilde{y}^0), u(\tilde{y}^0)}, \quad \tilde{B}(\tilde{y}^0) = \frac{\partial \tilde{f}_k}{\partial u} \bigg|_{\xi(\tilde{y}^0), u(\tilde{y}^0)} \]
\[ \tilde{C}(\tilde{y}^0) = \frac{\partial \tilde{h}}{\partial \xi} \bigg|_{\xi(\tilde{y}^0), u(\tilde{y}^0)}. \]

Assumption A1: The linear system \( (\tilde{A}(\tilde{y}^0), \tilde{B}(\tilde{y}^0), \tilde{C}(\tilde{y}^0)) \) is reachable and observable, and does not have transmission zeros equal to one.

Theorem 2.1: Consider \( \tilde{y}^0 \in \mathbb{R}^m \) and the associated equilibrium \( (\tilde{\xi}(\tilde{y}^0), \tilde{u}(\tilde{y}^0)) \) of (1) with \( h(\tilde{\xi}(\tilde{y}^0)) = \tilde{y}^0 \). If A1 holds, there exists one and only one pair of functions \( \tilde{\xi}(y^0), \tilde{u}(y^0) \), continuous in an open neighbourhood \( \Gamma \subseteq \mathbb{R}^m \) of \( \tilde{y}^0 \), such that, \( \forall y^0 \in \Gamma, \) \( \tilde{y}^0 = f(\tilde{\xi}(y^0), \tilde{u}(y^0)), \) \( y^0 = h(\tilde{\xi}(y^0)), \) \( \tilde{y}^0 \in \Xi, \tilde{u}(y^0) \in U. \)

According to the Internal Model principle, an integrator is preliminarily plugged in front of the system to guarantee the solution of the asymptotic tracking problem. Then, the control variable \( u(k) \) is given by

\[ z(k + 1) = z(k) + \delta u(k) \]
\[ u(k) = z(k) \]

so that the overall system (1), (3) is described by

\[ x(k + 1) = f(x(k), \delta u(k)), \quad x(t) = \hat{x}, \quad k \geq t \]
\[ y(k) = h(x(k)) \]
\[ \epsilon(k) = y^0 - y(k), \]

where \( x = [x' \; z'], \) \( \hat{x}(y^0) = [\tilde{\xi}(y^0)' \; m(y^0)'] \) and

\[ f(x, \delta u) = \begin{bmatrix} \tilde{f}(\xi, z) \\ z + \delta u \end{bmatrix}. \]

For the augmented system (4)–(5), the following well-known result holds (see De Nicolao et al. 1997, for example, for a complete proof).

Lemma 2.2: Consider \( \tilde{y}^0 \in \mathbb{R}^m \) and the associated equilibrium \( (\tilde{\xi}(\tilde{y}^0), \tilde{u}(\tilde{y}^0)) \) of (1) with \( h(\tilde{\xi}(\tilde{y}^0)) = \tilde{y}^0 \). Then, under A1, for system (4)–(6) there exist an open neighbourhood \( \Gamma \subseteq \Gamma \) of \( y^0 \) and a Linear Dynamic Regulator (LDR):

\[ w(k + 1) = A_w w(k) + B_w \epsilon(k), \quad w(t) = \hat{w}, \quad k \geq t \]
\[ \delta u(k) = C_w w(k) + D_w \epsilon(k), \]
with \( w \in \mathbb{R}^r \) such that \( \forall y^0 \in \Gamma', \ \hat{x}_c(y^0) = [\hat{x}(y^0), 0] \) is a stable equilibrium point of the closed-loop system (4)–(8).

Assumption A2: With reference to system (4)–(5), with \( \delta u(k) = 0 \), if \( y(k) = y^0 \), \( \forall k \geq t \), then \( x(t) = \hat{x}(y^0) \).

Remark: Under A1, from standard continuity arguments it follows that there exist finite neighbourhoods of \( y^0 \) and of the equilibrium point \( \hat{x}(y^0) \) where A2 is satisfied. In fact, A2 is implied by the observability of the linearization of (4)–(5) around \( (\hat{x}(y^0), 0) \).

3. Non-linear predictive control algorithm

In order to enlarge the stability region and improve the performance provided by the LDR (7)–(8), a non-linear MPC algorithm is now derived by suitably extending the method proposed in De Nicolao et al. (1998) for the regulation problem. To this end, for any \( y^0 \in \Gamma' \), denote by \( X(LDR, y^0) \) the set of all states \( \hat{x}_c = [\hat{x}, \hat{\omega}] \in \mathbb{R}^{n+m+r} \) of (4)–(8) which are steered to \( \hat{x}_c(y^0) \) without violating the state and control constraints (2), that is (under A2)

\[
X(LDR, y^0) = \left\{ \hat{x}_c \in \mathbb{R}^{n+m+r} : \lim_{k \to \infty} \| e(k) \| = 0 \right\}
\]

subject to (4)–(8) and (2) with \( x(t) = \hat{x}_c \).

3.1. Finite Horizon Optimal Control Problem (FHOCP)

Given the prediction horizon \( N_p \), four positive definite matrices \( Q_e, R_u, Q_w \) and \( R_w \) of suitable dimensions, a feasible auxiliary regulator (7), (8), the control sequence \( \delta u_{t, t+N-1} = [\delta u(t), \delta u(t+1), \ldots, \delta u(t+N-1)] \), minimize with respect to \( \delta u_{t, t+N-1}, w(t+N) \), the performance index

\[
J(\hat{x}, [\delta u_{t, t+N-1}, w(t+N)], y^0)
\]

\[
= \sum_{i=t}^{t+N-1} \{ e(i)'Q_e e(i) + \delta u(i)'R_u \delta u(i) \}
\]

\[
+ V_f(x(t+N), w(t+N), LDR)
\]

subject to (4)–(6), (2) and

\[
\begin{bmatrix} x(t+N) \\ w(t+N) \end{bmatrix} \in X(LDR, y^0).
\]

In (9), the terminal penalty is

\[
V_f(\hat{x}, \hat{w}, LDR) = w(t)'R_u w(t) + \sum_{i=t}^{\infty} \{ e(i)'Q_e e(i) + 2e(i)'Q_{ew} w(i) \} + w(i)'Q_{ew} w(i) \]

subject to (4)–(8) with

\[
\hat{Q}_e = [Q_e + D_u^T R_u D_u + B_u^T R_u B_u],
\]

\[
\hat{Q}_w = [Q_w + C_u R_u C_u + A_u^T R_u A_u]
\]

\[
\hat{Q}_{ew} = [D_u^T R_u C_u + B_u^T R_u A_u].
\]

Note that

\[
\begin{bmatrix} \hat{Q}_e & \hat{Q}_{ew} \\ \hat{Q}_{ew} & \hat{Q}_w \end{bmatrix} > 0.
\]

In the sequel, a sequence \( [\delta u_{t, t+N-1}, w(t+N)] \) is termed admissible with respect to \( \hat{x} \) if, when applied to (4) with \( x(t) = \hat{x} \), it satisfies (10) and (2).

Assuming that the plant state \( x \) is known, the MPC control law is obtained by solving at any time instant the FHOCP and by applying only the first control move \( \delta u(t) = k^{MPC}(\hat{x}) \), where \( k^{MPC}(\hat{x}) \) is the first column of the optimal sequence \( \delta u_{t, t+N-1} \). Correspondingly, the closed-loop system is

\[
\hat{x}^{MPC}(k+1) = f(\hat{x}^{MPC}(k), k^{MPC}(\hat{x}^{MPC}(k)));
\]

Denoting by \( \hat{x}_0(N, y^0) \) the set of states \( \hat{x} \) for which an admissible solution for the FHOCP exists, the following stability result holds.

Theorem 3.1: Consider a constant reference signal \( y^0 \in \Gamma' \), and suppose that A1–A2 hold. Then \( \hat{x}_0(N, y^0) \) is non-empty and (13) admits the state \( \hat{x}(y^0) \) as an asymptotically stable equilibrium point with region of attraction \( \hat{x}_0(N, y^0) \).

Remark 1: In the proposed approach, (9) must be minimized with respect both to \( \delta u_{t, t+N-1} \) and to the state \( w(t+N) \) of the LDR at the end of the control horizon. In so doing, the algorithm does not require computation of steady state value of the state at the desired equilibrium. The optimization is performed subject to the constraint that at the end of the control horizon, the LDR is applied, while the constraint (10) is added to guarantee that at the end of the control horizon, this linear regulator is stabilizing for the considered operating condition. However, the LDR is never applied in practice, and the terminal set \( X(LDR, y^0) \) needs not be computed explicitly off-line; in fact, the terminal
constraint (10) is satisfied if \( V_f \) is finite and the state constraint (2) is satisfied for all the times. In this way an implicit direct relation between \( y^0 \) and the controller parameters like the terminal region and terminal penalty is obtained.

**Remark 2:** In principle, the terminal penalty (11) should be computed over an infinite horizon. However, in a practical implementation of the method, its computation can be approximated as suggested in De Nicolao et al. (1998). Remark 3. In fact, after a sufficiently long horizon, it can be assumed that the system has been driven to a sufficiently small neighbourhood of the equilibrium, where the cost to go is negligible.

**Remark 3:** The output admissible set \( X_0(N, y^0) \) is not smaller than \( X(LDR, y^0) \) and the performance in terms of the cost function (9) improved; otherwise, the set of constant references that can be tracked with the NMPC is no larger than that associated with the LDR.

**Remark 4:** In the statement of Theorem 3.1, only constant references \( y^0 \) have been considered. However, it is easy to extend the problem definition and the proof of Theorem 3.1 to the case of asymptotically constant reference signals. This possibility allows one to use pre-programmed set-points with an asymptotically constant behaviour, as shown in section 4.

**Remark 5:** When the system state is not measurable, one can combine the stabilizing control law with an asymptotic state observer, such as the popular Extended Kalman Filter (EKF). By resorting to the results in Magni et al. (2004) it is possible to prove that the corresponding output feedback control law enjoys regionally stabilizing properties similar to those stated in Theorem 3.1.

### 4. Illustrative example

The Continuous Stirred Tank Reactor (CSTR) for an exothermic, irreversible reaction, \( A \rightarrow B \), is described by the following dynamic model based on a component balance for reactant \( A \) and an energy balance (see Seborg et al. 1989):

\[
\dot{C}_A = \frac{q}{V} (C_{Af} - C_A) - k_0 \exp\left(-\frac{E}{RT}\right) C_A
\]

\[
T = \frac{q}{V} (T_f - T) - \frac{\Delta H}{\rho C_p} k_0 \exp\left(-\frac{E}{RT}\right) C_A + \frac{UA}{V \rho C_p} (T_c - T),
\]

where \( C_A \) is the concentration of \( A \), \( T \) is the reactor temperature, and \( T_c \) is the temperature of the coolant stream. The constraints are \( 280 \text{ K} \leq T_c \leq 380 \text{ K}, \) \( 280 \text{ K} \leq T \leq 380 \text{ K}, \) \( 0 \leq C_A \leq 1 \). The objective is to control \( T \) by manipulating \( T_c \).

Letting \( q = 1001 \text{ min}^{-1}, \) \( C_{Af} = 1 \text{ mol l}^{-1}, \) \( T_f = 350 \text{ K}, \) \( V = 1001, \) \( \rho = 1000 \text{ g l}^{-1}, \) \( C_p = 0.239 \text{ J g}^{-1} \text{ K}^{-1}, \) \( \Delta H = -5 \times 10^4 \text{ J mol}^{-1}, \) \( E/R = 8750 \text{ K}, \) \( k_0 = 7.2 \times 10^{10} \text{ min}^{-1}, \) \( UA = 5 \times 10^4 \text{ J min} \text{ K}^{-1}, \) the constant input \( T_c = 300 \text{ K} \) corresponds to the unstable equilibrium state \( C_A = 0.5 \text{ mol l}^{-1}, T = 350 \text{ K}. \) The open-loop responses for \( \pm 5 \text{ K} \) changes in \( T_c \), reported in figure 1, demonstrate

![Figure 1](image-url)
that the reactor exhibits highly non-linear behaviour in this operating regime. The non-linear discrete-time state-space model (1) of system (14) can be obtained by defining the state vector $\xi = [C_A \ T]'$, the manipulated input $u = T_c$, and the controlled output $y = T$, and by discretizing equations (14) with sampling period $\Delta t = 0.05 \text{ min}$ (numerical integrations were performed using the function ode23 of Matlab).

The MPC algorithm, together with a standard EKF, was used to compute the output feedback non-linear control law. Specifically, an optimization horizon $N = 4$ was used together with $Q_e = 10$, $Q_w = R_w = R_u = 1$, while the EKF was designed assuming that two mutually independent white noises $\omega \sim WGN(0, Q^*)$, $Q^* = \text{diag}(1, 1000, 1000)$ and $\psi \sim WGN(0, R^*)$, $R^* = 10$, act on the enlarged state $([C_A \ T \ z]')$ and output vector $(T)$, respectively. The LDR used in the implementation of the MPC algorithm is given by

$$A_r = -0.98, \ B_r = 4, \ C_r = -5.734, \ D_r = 12.2,$$

and it is computed with the root-locus technique applied to the model obtained by linearization of the plant around the nominal operating conditions.

### 4.1. Simulation results

#### 4.1.1. Simulation 1. Starting from the nominal operating conditions, a pre-programmed step reference signal and a reference signal with a truncated ramp-type behaviour were used in order to reach a steady-state value of the temperature set-point $T$ equal to 365 K and 375 K, respectively. In figures 2 and 3, the non-linear (continuous line) closed-loop responses are reported together with the considered reference signals. For comparison, these transients are compared with those obtained with the CRHPC algorithm (Clarke and Scattolini 1991) (dashed line), an MPC method for linear systems which guarantees closed-loop stability. As the figures clearly show, even with a small control horizon ($N = 4$), the

![Figure 2](image1.png)  **Figure 2.** Pre-programmed step reference for $T$ from 350 to 365 K: linear CRHPC regulator (dashed), non-linear MPC regulator (continuous), constraints (dotted) and set-point (dashed-dotted).

![Figure 3](image2.png)  **Figure 3.** Pre-programmed truncated ramp-type behaviour reference for $T$ from 350 to 375 K: linear CRHPC regulator (dashed), nonlinear MPC regulator (continuous), constraints (dotted), and set-point (dashed-dotted).
control performance is significantly improved by the proposed non-linear \textit{MPC} method. Notably, the Root Mean Square Error (\textit{RMSE}), computed in the intervals \([0, 3]\) for the step response are 37 for the linear \textit{CRHPC} and 16 for the non-linear \textit{MPC}. Note also that the linear controller violates substantially the input constraint on \(T_c\) in the interval \([0.5, 1]\) min. For the truncated-ramp response experiment, the \textit{RMSE} computed in the intervals \([0, 4]\) is 103 for the linear \textit{CRHPC} and 2.3 for the non-linear \textit{MPC}, respectively. This time, the linear controller heavily violates the state constraint on \(T\) as well as the input constraint \(T_c\).

4.1.2. Simulation 2. The robustness of the proposed \textit{MPC} method was investigated by forcing a ramp-type change on the parameter \(E/R\) starting from nominal operating conditions, as shown in figure 4(a). Correspondingly, the transients of the state and control variables reported in figure 4(b)–(d) were computed. These results clearly show that the \textit{MPC} algorithm can reject plant-parameter perturbations without the need for an explicit estimation of the plant parameters.

5. Conclusions

The novel output-feedback \textit{MPC} algorithm for non-linear systems proposed in this paper ensures local closed-loop stability and zero-error regulation in the face of asymptotically constant reference signals. The stability results rely on very mild assumptions on the linearized plant model. The method requires the availability of a (dynamic) output feedback linear regulator stabilizing the ensemble plant + integrator in the considered operating conditions.

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Appendix

Proof of Theorem 2.1: Define \(\varphi(\xi, u, y^0) = \left[(f(\xi, u) - \xi)'(h(\xi) - y^0)\right]'\). By assumption, it follows that \(\varphi(c(j^0), \hat{u}(y^0), y^0) = 0\), and, in view of A1,

\[
\begin{align*}
\text{rank} & \left[ \frac{\partial \varphi(\xi, u, y^0)}{\partial(\xi, u)} I_{y^0} \right]_{y^0, \tilde{u}(y^0), \hat{y}(y^0)} = \text{rank} \begin{bmatrix} \hat{A}(\hat{y}^0) - I & \hat{B}(\hat{y}^0) \\ \tilde{C}(\tilde{y}^0) & 0 \end{bmatrix} \\
& = n + m.
\end{align*}
\]

Then, the proof follows from the implicit function theorem.

Proof of Theorem 3.1: Given a constant reference signal \(y^0 \in \Gamma^c\), we show that \(\forall x \in X_0(N, y^0), V(\hat{x}) \coloneqq J(\hat{x}, [\delta y^0(t, t+1, N-1, w(t + N)^0], y^0)\) is a Lyapunov function
for (13). First, note that in view of Theorem 2.1, A2 and (9), \( V(x) > 0 \ \forall x \neq \hat{x}(y^0) \in X_0(N, y^0) \) and \( V(\hat{x}(y^0)) = 0 \).

Then, the closed-loop system (4)–(8) formed by the non-linear system and the linear dynamic regulator can be rewritten in the following way

\[
\sigma(k+1) = \hat{f}(\sigma(k), \eta(k))
\]

\[
\varepsilon(k) = \hat{h}(\sigma(k), \eta(k), y^0(k))
\]

\[
\eta(k) = K \varepsilon(k)
\]

with

\[
\sigma(k) = \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}, \quad \eta(k) = \begin{bmatrix} \delta u(k) \\ \eta_{2}(k) \end{bmatrix}, \quad \varepsilon(k) = \begin{bmatrix} \varepsilon(k) \\ w(k) \end{bmatrix}
\]

\[
\hat{f}(\sigma(k), \eta(k)) = \begin{bmatrix} f(x(k), \delta u(k)) \\ \eta_2(k) \end{bmatrix},
\]

\[
\hat{h}(\sigma(k), \eta(k)) = \begin{bmatrix} y^0(k) - h(x(k)) \\ w(k) \end{bmatrix}
\]

\[
K = \begin{bmatrix} D_r & C_r \\ B_r & A_r \end{bmatrix}
\]

Then, the optimization problem can be rewritten as the minimization, with respect to \( \eta_{t, t+N-1} = [\eta(t), \eta(t+1), \ldots, \eta(t+N-1)] \), of

\[
J_\sigma(\hat{x}, \eta_{t, t+N-1}, y^0) = \sum_{i=t}^{t+N-1} \varepsilon(i)^T Q \varepsilon(i) + \eta(i)^T R \eta(i) + \eta(i)^T R \eta(i)
+ \sum_{i=t+N}^\infty \varepsilon(i)^T Q \varepsilon(i) + \eta(i)^T R \eta(i),
\]

with \( Q = \text{diag}(Q_r, Q_w), \ R = \text{diag}(R_u, R_w) \), subject to the open-loop dynamics (15) and

\[
\sigma(t) = [\hat{x}' \ 0']
\]

\[
\eta(i) = K \varepsilon(i), \quad \text{for } i \geq t + N.
\]

To see this, first note that for \( t \leq k \leq t+N-1 \), the variables \( \varepsilon(k), \delta u(k), \) and \( x(k) \) are not affected by \( \eta_2(i) \). Hence, the choice \( \eta_2(k) = 0 \), for \( t \leq k \leq t+N-2 \) is the one that minimizes the cost function, and only \( \eta_2(t+N-1) = w(t+N) \) is a real optimization parameter. Then, the minimization of \( J_\sigma \) is equivalent to the minimization (subject to (4)) of

\[
\sum_{i=t}^{t+N-1} \{ \varepsilon(i)^T Q \varepsilon(i) + \delta u(i)^T R_u \delta u(i) \}
+ \eta_2(t + N - 1)^T R_u \eta_2(t + N - 1)
+ \sum_{i=t+N}^\infty \{ \vee(i)^T \hat{Q}_r \vee(i) + 2 \vee(i)^T \hat{Q}_w w(i) + w(i)^T \hat{Q}_w w(i) \},
\]

with \( \hat{Q}_r, \hat{Q}_w, \hat{Q}_w \) given by (12) and, for \( i \geq t+N \), with \( \delta u(i) \) given by (7)–(8) where \( w(t+N) = \eta_2(t+N-1) \). In this way, we have proven the equivalence of the minimization of \( J \) and \( J_\sigma \).

We can now show that

\[
V(\hat{x}) = J(\hat{x}, [\delta u^0_{t,t+N-1}, w(t+N)^0, y^0])
= J_\sigma(\hat{x}, \eta_{t, t+N-1}^0, y^0)
\]

is a Lyapunov function. Letting \( \eta_{t, t+N-1}^0 \) be the optimal solution at time \( t \), the sequence \( \eta_{t, t+N-1}^0 = [\eta(t), \eta(t+1), \ldots, \eta(t+N-1)] \) is a feasible (but not necessarily optimal) solution at time \( t + 1 \), in view of the invariance of \( X(LDR, y^0) \) with respect to the LDR control. Then,

\[
V(f(\hat{x}, k_{MPC}(\hat{x}))) \leq J_\sigma(f(\hat{x}, k_{MPC}(\hat{x})), \eta_{t, t+N-1}^0, y^0)
\leq V(\hat{x}) - \varepsilon(t)^T Q \varepsilon(t) - \eta(t)^T R \eta(t)
\]

which implies \( V(f(\hat{x}, k_{MPC}(\hat{x}))) - V(\hat{x}) < 0 \), \( \forall \varepsilon(t), \eta(t) \neq 0 \).

Therefore, we are left to show that there are no perturbed closed-loop trajectories of \( x_{MPC}(\cdot) \) such that \( V(x_{MPC}(\cdot)) \) remains constant. By contradiction this would entail \( \varepsilon(t) = \eta(t) = 0 \), \( \forall t \), that is \( \varepsilon(t) = \delta u(t) = 0 \), \( \forall t \), or also \( y^0 - h(x_{MPC}(t)) = 0 \), \( \forall t \). Then, in view of A2, \( x_{MPC} \) coincides with the equilibrium \( \hat{x}(y^0) \) and the thesis is proven. Finally, note that \( X_0(N, y^0) \) is a positive invariant set. In fact, if \( \hat{x} \in X_0(N, y^0) \), letting \( \eta_{t, t+N-1}^0 \) be the optimal solution at time \( t \), the sequence \( \eta_{t, t+N-1}^0 = [\eta(t), \eta(t+1), \ldots, \eta(t+N-1)] \) is a feasible sequence at time \( t + 1 \), so that \( f(\hat{x}, k_{MPC}(\hat{x})) \in X_0(N, y^0) \).

References


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