Decentralized MPC of nonlinear systems: An input-to-state stability approach

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SUMMARY
This paper presents stabilizing decentralized model predictive control (MPC) algorithms for discrete-time nonlinear systems. The overall system under control is composed by a number of subsystems, each one locally controlled with an MPC algorithm guaranteeing the input-to-state stability (ISS) property. Then, the main stability result is derived by considering the effect of interconnections as perturbation terms and by showing that also the overall system is ISS. Both open-loop and closed-loop min–max formulations of robust MPC are considered. Copyright © 2007 John Wiley & Sons, Ltd.

1. INTRODUCTION
Decentralized model predictive control (MPC) techniques are of paramount interest in the process industry; in fact a decentralized control structure is often the most appropriate one due to topological constraints and limited exchange of information between subsystems, while the MPC approach allows one to include in the problem formulation both performance requirements and state and control constraints. Moreover, a decentralized implementation of MPC often has the advantage to reduce an original, large size, optimization problem into a number of smaller and easily tractable ones. For these reasons, decentralized MPC has already been studied for discrete-time linear systems in, e.g. [1, 2] and in a number of papers quoted there. Recently, in [3] a decentralized MPC algorithm for nonlinear systems has been proposed,

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where closed-loop stability of the origin is achieved through the inclusion of a contractive constraint (see also [4]).

Distributed MPC algorithms can be developed either by assuming that there is a partial exchange of information between the subsystems, as in [1, 5], or by considering a fully decentralized control structure, as in [3]. This second possibility is obviously more critical than the former and requires a more conservative solution, since the amount of information available to any local controller is less. However, this setting more closely resembles most of the real-world cases, where complex control structures are built according to fully decentralized schemes. In this paper, stabilizing fully decentralized MPC algorithms for nonlinear, discrete-time systems are derived under the assumption that no information is exchanged between subsystems. They rely on the concept of input-to-state stability (ISS), which is now well recognized as a powerful tool to investigate the stability properties of nonlinear continuous and discrete-time perturbed systems, see, e.g. [6–8]. ISS has already been used in the analysis and synthesis of MPC algorithms with enhanced robustness properties, see e.g. [9–12]. However, the main limitation in the application of standard ISS concept for MPC with state and control constraints is that global results are not useful, while local results do not allow to study the properties of a predictive control law in terms of its region of attraction. For this reason, the concept of regional ISS has been introduced in [13], where this tool has been used to study the properties of robust MPC algorithms based on open-loop and closed-loop formulations. Relying on these results, the approach taken in the following to derive decentralized MPC implementations consists in considering the overall system as composed by a number of interconnected subsystems, each one of them controlled by a robust (open-loop or closed-loop) MPC algorithm guaranteeing ISS, and by considering the effect of interconnections as a perturbation term. A similar approach has also been taken in [14, 15] where global results are given for interconnected systems. Then, by suitably combining and extending the results reported in [13–15], it is shown that under suitable assumptions the ISS property of the controlled subsystems guarantees the ISS of the overall (controlled) system.

The paper is organized as follows. In Section 2 the notation and the basic definitions used in the sequel are reported. Section 3 deals with analysis of the regional ISS property of an overall system formed by a number of ISS subsystems. In Section 4, these results are extended to the ISS property of a system controlled with MPC implemented according to a decentralized control structure, and the main results of the paper are reported for open-loop and closed-loop MPC formulations. All the proofs of lemmas and theorems are given in the Appendix A.

## 2. NOTATIONS AND BASIC DEFINITIONS

Let $\mathbb{R}, \mathbb{R}_{\geq 0}, \mathbb{Z}$ and $\mathbb{Z}_{\geq 0}$ denote the real, the non-negative real, the integer and the non-negative integer numbers, respectively. Euclidean norm is denoted as $|\cdot|$. For any $\phi: \mathbb{Z}_{\geq 0} \to \mathbb{R}^{m}$, $\|\phi\|_1 = \sup_{k\geq 0} \{ |\phi(k)| \}$ and $\|\phi\|_\infty = \sup_{0 \leq k \leq 1} \{ |\phi(k)| \}$. The set of signals $\psi$ taking values in some subset $\Psi \subseteq \mathbb{R}^n$ is denoted by $\mathcal{A}_\Psi$, while $\hat{\psi} = \max_{\phi \in \Psi} |\psi|$. The symbol id represents the identity function from $\mathbb{R}$ to $\mathbb{R}$, while $\gamma_1 \circ \gamma_2$ is the composition of two functions $\gamma_1$ and $\gamma_2$ from $\mathbb{R}$ to $\mathbb{R}$. Given a set $A \subseteq \mathbb{R}^n$, $d(\zeta, A) = \inf \{ |\eta - \zeta|, \eta \in A \}$ is the point-to-set distance from $\zeta \in \mathbb{R}^n$ to $A$. The difference between two given sets $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^n$ with $B \subseteq A$, is denoted by $A \setminus B = \{ x : x \in A, x \notin B \}$. Given a closed set $A \subseteq \mathbb{R}^n$, $\partial A$ denotes the border of $A$. For $x, y \in \mathbb{R}^n$, $x \geq y \Leftrightarrow x_i \geq y_i$, $i = 1, \ldots, n$ and $x \not\leq y$ means the negation of $x \geq y$. 

Definition 1 ($\mathcal{K}$-function)
A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{K}$ (or a ‘$\mathcal{K}$-function’) if it is continuous, positive definite and strictly increasing.

Definition 2 ($\mathcal{K}_\infty$-function)
A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{K}_\infty$ if it is a $\mathcal{K}$-function and $\gamma(s) \to +\infty$ as $s \to +\infty$.

Definition 3 ($\mathcal{KL}$-function)
A function $\beta : \mathbb{R}_+ \times \mathcal{L}_+ \to \mathbb{R}_+$ is of class $\mathcal{KL}$ if, for each fixed $t \geq 0$, $\beta(\cdot, t)$ is of class $\mathcal{K}$, for each fixed $s \geq 0$, $\beta(s, \cdot)$ is decreasing and $\beta(s, t) \to 0$ as $t \to \infty$.

Definition 4 (Upper limit)
Given a bounded function $s : \mathcal{L}_+ \to \mathbb{R}_+$, the upper limit is defined
$$\limsup_{t \to \infty} s(t) = \inf_{t \geq 0} \sup_{\tau \geq t} s(\tau)$$

Consider the following nonlinear discrete-time dynamic system
$$x(k + 1) = F(x(k), w(k)), \quad k \geq t, \quad x(t) = \bar{x}$$
where $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz continuous, $F(0, 0) = 0$, $x(k) \in \mathbb{R}^n$ is the state, $w(k) \in \mathbb{R}^m$ is the input (disturbance), limited in a compact set $\mathcal{W}$ containing the origin $w(k) \in \mathcal{W}$

The transient of system (1) with initial state $\bar{x}$ and input $w$ is denoted by $x(k, \bar{x}, w), \ k \geq t$.

Definition 5 (Robust positively invariant set)
A set $\Xi \subseteq \mathbb{R}^n$ is a robust positively invariant set for system (1) if $F(x, w) \in \Xi, \ \forall x \in \Xi$ and $\forall w \in \mathcal{W}$.

Definition 6 (0-AS in $\Xi$)
Given a compact set $\Xi \subset \mathbb{R}^n$ including the origin as an interior point, system (1) with $w = 0$ is said to be 0-AS in $\Xi$, if there exists a $\mathcal{KL}$-function $\beta$ such that
$$|x(k, \bar{x}, 0)| \leq \beta(|\bar{x}|, k) \quad \forall k \geq t \ \forall \bar{x} \in \Xi$$

Definition 7 (ISS in $\Xi$)
Given a compact set $\Xi \subset \mathbb{R}^n$ including the origin as an interior point, system (1) with $w \in \mathcal{W}$ is said to be ISS in $\Xi$, if $\Xi$ is robust positively invariant for (1) and if there exist a $\mathcal{KL}$-function $\beta$ and a $\mathcal{K}$-function $\gamma$ such that
$$|x(k, \bar{x}, w)| \leq \beta(|\bar{x}|, k) + \gamma(\|w\|) \quad \forall k \geq t \ \forall \bar{x} \in \Xi$$

Definition 8 (AG in $\Xi$)
Given a compact set $\Xi \subset \mathbb{R}^n$ including the origin as an interior point, system (1) satisfies the asymptotic gain (AG) property in $\Xi$, if $\Xi$ is robust positively invariant for (1) and if there exists a function $\gamma_{AG} \in \mathcal{K}_\infty$ such that for all initial values $\bar{x} \in \Xi$ and all inputs $w \in \mathcal{W}$
$$\lim_{k \to \infty} |x(k, \bar{x}, w)| \leq \gamma_{AG}(\|w\|)$$
Definition 9 (LISS)

System (1) is said to be locally input-to-state stable (LISS) if there exist a $\rho > 0$, a $\mathcal{K}$-function $\gamma$ and a $\mathcal{K} L$-function $\beta$ such that

$$|x(k, \bar{x}, w)| \leq \beta(|\bar{x}|, k) + \gamma(|w|) \quad \forall k \geq t$$

for all $|\bar{x}| \leq \rho$ and all $||w|| \leq \rho$.

Definition 10 (ISS-Lyapunov function in $\Xi$)

A function $V : \mathbb{R}^n \to \mathbb{R} \geq 0$ is called an ISS-Lyapunov function in $\Xi$ for system (1), if $\Xi$ is a compact robust positively invariant set including the origin as an interior point and if there exist compact sets $\Omega$ and $D$, including the origin as an interior point with $D \subset \Omega \subset \Xi$, some $\mathcal{K}_{\infty}$-functions $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_3$, $\rho$, some $\mathcal{K}$-functions $\sigma$, $b = \varepsilon_4^{-1} \rho^{-1} \sigma$, with $\varepsilon_4 = \varepsilon_3 \varepsilon_2^{-1}$, such that $(id - \rho)$ is a $\mathcal{K}_{\infty}$-function and the following relations hold:

$$V(x) \geq \varepsilon_1(|x|) \quad \forall x \in \Xi$$

$$V(x) \leq \varepsilon_2(|x|) \quad \forall x \in \Omega$$

$$\Delta V(x) \overset{\triangle}{=} V(F(x, w)) - V(x) \leq - \varepsilon_3(|x|) + \sigma(|w|) \quad \forall x \in \Xi \forall w \in \mathbb{W}$$

$$D \overset{\triangle}{=} \{ x : d(x, \partial \Omega) > c, \quad V(x) \leq b(\bar{w}) \} \subset \Omega, \quad c > 0$$

Then, the following sufficient condition for regional ISS of system (1) can be stated.

Theorem 1 (Magni et al. [13])

If system (1) admits an ISS-Lyapunov function in $\Xi$, then it is ISS in $\Xi$ and $\lim_{k \to \infty} d(x(k, \bar{x}, w), D) = 0$.

3. NONLINEAR INTERCONNECTED LOCAL SUBSYSTEM

Let the system under control be composed by the interconnection of $N$ local subsystems described by the following nonlinear, discrete-time model:

$$x_i(k+1) = f_i(x_i(k)) + g_i(x(k)) + \psi_i(k), \quad k \geq t, \quad x_i(t) = \bar{x}_i$$

where $x_i(k) \in \mathbb{R}^n$ is the state of the $i$th subsystem, $\psi_i(k) \in \mathbb{R}^v$ is the disturbance, $f_i(0) = 0$, while $g_i$, which depends on the overall state

$$x(k) \overset{\triangle}{=} [x_1(k)^t x_2(k)^t \ldots x_N(k)^t]^t \in \mathbb{R}^v, \quad v = \sum_{i=1}^N v_i$$

describes the influence of $N$ subsystems on the $i$th subsystem and is such that $g_i(0) = 0$. The state variables and the disturbances fulfil the following constraints:

$$x_i \in X_i$$

$$\psi_i \in \Psi_i$$

where $X_i$ and $\Psi_i$ are compact sets of $\mathbb{R}^v$ containing the origin as an interior point.
There exist positive constants $L_{ij}$, $i, j \in [1, 2, \ldots, N]$ such that

$$|g_i(x)| \leq \sum_{j=1}^{N} L_{ij}|x_j|, \quad i = 1, \ldots, N$$

**Assumption 1**
There exist positive constants $L_{ij}$, $i, j \in [1, 2, \ldots, N]$ such that

Introduce the following assumptions.

**Assumption 2**
For every $k \geq t$, the state trajectories of subsystems (8) are continuous in $x = 0$ and $\psi_j = 0$ with respect to the initial condition $\vec{x}$ and the input sequence $\psi_i$.

**Assumption 3**
Given a compact robust positively invariant set $\Xi_i \subseteq X_i$, assume that each subsystem (8) admits an ISS-Lyapunov function $V_i$ in $\Xi_i$, with $g_i(x) + \psi_i$ as disturbance terms.

By Assumptions 1 and 3, there exist some $\mathcal{K}_\infty$-functions $x_{i1}, x_{i2}, x_{i3}, \rho_i$, some $\mathcal{K}$-functions $\sigma_{ij}, \sigma_{ij}^\psi$ such that $(id - \rho_i)$ is a $\mathcal{K}_\infty$-function and the following relations hold:

$$V_i(x_i) \geq x_{i1}(|x_i|) \quad \forall x_i \in \Xi_i$$

$$V_i(x_i) \leq x_{i2}(|x_i|) \quad \forall x_i \in \Omega_i$$

$$\Delta V_i(x_i) \leq -x_{i3}(|x_i|) + \sum_{j=1}^{N} \sigma_{ij}^\psi(|x_j|) + \sigma_{ij}^\psi(|\psi_j|) \quad \forall x_i \in \Xi_i \quad \forall x_j \in \Xi_j \quad \forall \psi_i \in \Psi_i$$

$$D_i := \left\{ x_i : d(x_i, \partial \Omega_i) > c_i, V_i(x_i) \leq b_i = x_{i4}^{-1} \cdot \rho_i^{-1} \left( \sum_{j=1}^{N} \sigma_{ij}^\psi(x_j) + \sigma_{ij}^\psi(\psi_j) \right) \right\} \subseteq \Omega_i, \quad c_i > 0$$

where $x_{i4} = x_{i3} \cdot x_{i2}^{-1}$ and with $\Omega_i \subseteq \Xi_i$.

Introduce now an intermediate result useful in the following.

**Lemma 1**
Given any $\mathcal{K}_\infty$-function $\rho_i$ such that $(id - \rho_i) \in \mathcal{K}_\infty$, there exist some $\mathcal{K}_\infty$-functions $a_{i1}, a_{i2}, \ldots, a_{i9}$ such that

$$\rho_i^{-1}(\theta_1 + \cdots + \theta_N + \theta_\psi) \leq \max\{(id + a_{i1})(\theta_1), (id + a_{i2})(\theta_2), \ldots, (id + a_{i9})(\theta_N), (id + a_{i9})(\theta_\psi)\}$$

In order to introduce the following theorem first define $\Delta : \mathbb{R}_{\geq 0}^{n} \rightarrow \mathbb{R}_{\geq 0}$ as:

$$\Delta(s_1, \ldots, s_n) \triangleq ((id + z_1)(s_1), \ldots, (id + z_N)(s_N))$$
with $x_i \in \mathcal{H}_\infty$, $i = 1, \ldots, N$ and $\Gamma : \mathcal{P}_1^N \to \mathcal{P}_1^{\geq 0}$ as

$$\Gamma(s_1, \ldots, s_N) = \sum_{j=1}^{N} x_{1j}^{-1}(\text{id} + a_{ij}) \eta_j^*(s_j), \ldots, \sum_{j=1}^{N} x_{Nj}^{-1}(\text{id} + a_{ij}) \eta_j^*(s_j)$$

where $x_{ij} = x_{i2}^{-1}$, $\eta_j^* = \sigma_j^* x_{i1}^{-1}$, while $(\text{id} + a_{ij})$ are obtained starting from $\rho_i$ in (15) using Lemma 1. Moreover, define by $\Xi = \Xi_1 \times \cdots \times \Xi_N$ the compositions of regions where the subsystems are robust positively invariant.

**Theorem 2**
Consider systems (8) and suppose that Assumptions 1–3 are satisfied. Let $\Gamma$ be given by (17). If there exists a mapping $\Delta$ as in (16), such that

$$(\Gamma \circ \Delta)(s) \in \Xi$$

then the overall system (11) is ISS in $\Xi$ from $\psi$ to $x$.

**Remark 1**
As discussed in [15], condition (18) is the generalization to nonlinear interconnected systems of the well-known small gain theorem, which, in the case of only two interconnected systems was previously given in [16]. Many interesting interpretations of this conditions are given in [15]. Note also that condition (18) together with Assumption 1 is necessary to guarantee that the interconnections between the subsystems do not cause instability.

**4. NONLINEAR MODEL PREDICTIVE CONTROL**

In this section, the results derived in Theorems 1 and 2 are used to analyse the ISS property of open-loop and closed-loop min–max formulations of stabilizing MPC for nonlinear systems. Notably, in the following it is not necessary to assume the regularity of the value function and of the resulting control law.

Consider the nonlinear perturbed discrete-time dynamic subsystem

$$x_i(k + 1) = f_i(x_i(k), u_i(k)) + g_i(x(k)) + \psi_i(k), \quad k \geq t, \quad x_i(t) = x_i$$

where $x_i(k) \in \mathcal{R}^m$ is the state of the $i$th subsystem, $u_i(k) \in \mathcal{R}^m$ is the control variable, $f_i(0, 0) = 0$, $g_i(0) = 0$ and $\psi_i(k) \in \mathcal{R}^m$ is the additive uncertainty. The state and the disturbance are required to fulfill constraints (9) and (10), respectively, while the control variable is required to fulfill the following constraint:

$$u_i(k) \in U_i$$

where $U_i$ is a compact set of $\mathcal{R}^m$ containing the origin as an interior point.

Defining $f(x, u) = [f'_1(x_1, u_1), \ldots, f'_N(x_N, u_N)]', g(x) = [g'_1(x), \ldots, g'_N(x)]'$ and $\psi = [\psi'_1, \ldots, \psi'_N]'$, the whole system can be written as

$$x(k + 1) = f(x(k), u(k)) + g(x(k)) + \psi(k), \quad k \geq t, \quad x(t) = \bar{x}$$

Assumption 4
There exists positive Lipschitz constant $L_{ij}$ such that
\[ |f_i(a, u_i) - f_i(b, u_i)| \leq L_{ij} |a - b| \quad \forall a, b \in X_i \quad \forall u_i \in U_i \]

Definition 11 (Robust output admissible set)
Given a control law $u_i = \kappa_i(x_i)$, $\hat{X}_i \subseteq X_i$ is a robust output admissible set for the closed-loop system (19) with $u_i(k) = \kappa_i(x_i(k))$, if it is ISS in $\hat{X}_i$ and $\bar{x}_i \in \hat{X}_i$ implies $\kappa_i(x_i(k)) \in U_i$, $\forall w_i \times (k) \in W_i$, $k \geq t$.

4.1. Open-loop MPC formulation
In order to introduce the MPC algorithm formulated according to an open-loop strategy, first let $u_{[t,t_1]} \triangleq [u_i(t_1) \ u_i(t_1+1) \ldots u_i(t_2)]$, $t_2 \geq t_1$. Then, the following problem can be stated.

Definition 12 (FHOCP)
Given the positive integer $H_i$, the stage cost $l_i$, the terminal penalty $V_i^f$ and the terminal set $X_i^f$, the finite horizon optimal control problem (FHOCP) for the $i$th subsystem consists in minimizing, with respect to $u_{[t,t+H_i-1]}$, the performance index
\[
J_i(\bar{x}_i, u_{[t,t+H_i-1]}; H_i) = \sum_{k=t}^{t+H_i-1} l_i(x_i(k), u_i(k)) + V_i^f(x_i(t + H_i))
\]
subject to
(i) the nominal state dynamics (19) with $g_i = 0$, $\psi_i = 0$ and $x_i(t) = \bar{x}_i$;
(ii) the constraints (9), (20), $k \in [t, t + H_i - 1]$;
(iii) the terminal state constraints $x_i(t + H_i) \in X_i^f$.

It is now possible to define a ‘prototype’ of a nonlinear MPC algorithm: at every time instant $t$, define $\bar{x}_i = x_i(t)$ and find the optimal control sequence $u_{[t,t+H_i-1]}^0$ by solving the FHOCP. Then, according to the receding horizon (RH) strategy, define
\[
\kappa_i^{\text{MPC}}(\bar{x}_i) = u_{[t,t]}^0(\bar{x}_i)
\]
where $u_{[t,t]}^0(\bar{x}_i)$ is the first column of $u_{[t,t+H_i-1]}^0$, and apply the control law
\[
u_i = \kappa_i^{\text{MPC}}(x_i)
\]
Define the overall control law
\[
u = [\kappa_1^{\text{MPC}}(x_1)', \kappa_2^{\text{MPC}}(x_2)', \ldots, \kappa_N^{\text{MPC}}(x_N)']' \quad (23)
\]

Assumption 5
For every $k \geq t$, the state trajectories of subsystems (19) and (22) are continuous at $\bar{x} = 0$ and $\psi_i = 0$ with respect to the initial condition $\bar{x}$ and the input sequence $\psi_i$.

Although the FHOCP has been stated for nominal conditions, under suitable assumptions and by choosing accurately the terminal cost function $V_i^f$ and the terminal constraint $X_i^f$, it is
possible to guarantee the ISS property of the closed-loop system formed by (19) and (22), subject to constraints (9), (10), (20).

**Assumption 6**
The function \( l(x_i, u_i) \) is such that \( l(0, 0) = 0, l(x_i, u_i) \geq \alpha_l(|x_i|) \) where \( \alpha_l \) is a \( \mathcal{K}_\infty \)-function. Moreover, \( l(x_i, u_i) \) is Lipschitz with respect to \( x_i \), in \( X_i \times U_i \), with constant \( L_l \), i.e.

\[
|l(a, u_i) - l(b, u_i)| \leq L_l |a - b| \quad \forall a, b \in X_i \quad \forall u_i \in U_i
\]

**Assumption 7**
The terminal penalty \( V_{if} \) and the terminal set \( X_{if} \) are such that, given an auxiliary control law \( \kappa_{if} \)

1. \( X_{if} \subseteq X_i, X_{if} \) closed, \( 0 \in X_{if} \);
2. \( \kappa_{if}(x_i) \in U_i, \quad \forall x_i \in X_{if} \);
3. \( f(x_i, \kappa_{if}(x_i)) \in X_{if}, \quad \forall x_i \in X_{if} \);
4. \( \alpha_{V_{if}}(x_i) \leq V_{if}(x_i) < \beta_{V_{if}}(x_i), \quad \forall x_i \in X_{if} \), where \( \alpha_{V_{if}} \) and \( \beta_{V_{if}} \) are \( \mathcal{K}_\infty \)-functions;
5. \( V_{if}(f(x_i, \kappa_{if}(x_i))) - V_{if}(x_i) \leq -l(x_i, \kappa_{if}(x_i)), \quad \forall x_i \in X_{if} \);
6. \( V_{if} \) is Lipschitz in \( X_{if} \) with a Lipschitz constant \( L_{V_{if}} \).

**Assumption 8**
The set \( X_i^{\text{MPC}}(H_i) \) of states such that a solution of the FHOCO exists is a robust positively invariant set for the closed-loop system (19), (22).

**Assumption 9**
The values \( \tilde{\psi}_i, \tilde{x}_j, j = 1, \ldots, N \), are such that (15) is satisfied, with \( V_i(x_i) := J_i(x_i, x_i^0, u_i^{0,[0,t\_1+H_i-1]}), H_i) \), \( \alpha_3 = \alpha_l, \beta_2 = \beta_{V_{if}}, \Xi = X_i^{\text{MPC}}(H_i), \Omega_i = X_{if}, \sigma_i^\gamma = L_l - J_i, \sigma_i^\delta = L_l \), where \( L_l = L_{V_{if}}L_{if}^{-1} + L_l \times (L_{if}^{-1} - 1)/(L_{if} - 1) \).

**Remark 2**
The assumptions above can appear quite difficult to be satisfied, but they are standard in the development of nonlinear stabilizing MPC algorithms. Moreover, many methods have been proposed in the literature to compute \( V_{if}, X_{if} \) satisfying Assumptions 5 and 7 (see, e.g. [17–21]). However, with the MPC based on FHOCO defined above, Assumption 8 is not \textit{a priori} satisfied. A way to fulfil it is shown in [9] by properly restricting the state constraints (ii) and (iii) in the formulation of FHOCO.

The following preliminary result must be introduced first.

**Lemma 2**
Under Assumptions 6–7, \( (\text{id} - \alpha_{id}) \) with \( \alpha_{id} = \alpha_{i3}z_i^1, \alpha_{i2} = \beta_{V_{if}}, \alpha_{i3} = \alpha_l \), is a \( \mathcal{K}_\infty \)-function.

**Assumption 10**
Given systems (19), (22), \( i = 1, \ldots, N \), there exists a mapping \( \Delta \) as in (16), such that condition (18) is satisfied.

Define \( X_i^{\text{MPC}}(H) \equiv X_1^{\text{MPC}}(H_1) \times \cdots \times X_N^{\text{MPC}}(H_N) \) as the vector of regions where the subsystems are robust positively invariant. The main result can now be stated.
Theorem 3
Under Assumptions 1, 4–10 the overall system (21), (23) is ISS in $X^{\text{MPC}}(H)$ from $\psi$ to $x$.

4.2. Closed-loop min–max optimization

As underlined in Remark 2, the positive invariance of the feasible set $X^{\text{MPC}}(H_i)$ in a standard open-loop MPC formulation can be achieved through a wise choice of constraints (ii) and (iii) in the FHCGP. However, this solution can be extremely conservative and can provide a small robust output admissible set, so that a less stringent approach explicitly accounting for the intrinsic feedback nature of any RH implementation has been proposed, see e.g. [11, 22–28]. In the following, it is shown that the ISS result of the previous section is also useful to derive the ISS property of min–max MPC. In this framework, at any time instant the controller for the $i$th subsystem chooses the input $u_i$ as a function of the current state $x_i$, so as to guarantee that the influence of the disturbance of the $N$ subsystems are compensated. Hence, instead of optimizing with respect to a control sequence, at any time $t$ the controller has to choose a vector of feedback control policies $\kappa_{i[t,t+H_i-1]} = [\kappa_{i0}(x_i(t))\kappa_{i1}(x_i(t + 1)) \ldots \kappa_{iH_i-1}(x_i(t + H_i - 1))]$ minimizing the cost in the worst case.

Assumption 11
For each subsystem, the sum of the interaction with the other subsystems and the disturbance is restricted to fulfil the following constraint:

$$\forall x \in X \quad \forall \psi_i \in \Psi_i$$

where $\Psi_i$ is a compact set of $\mathbb{R}^n$, containing the origin as an interior point while $X = X_1 \times \cdots \times X_N$.

Note that, in view of Assumption 1, the sets $W_i$ can be derived in view of the knowledge of $X$ and $\Psi_i$.

The following optimal min–max problem can be stated for the $i$th subsystem.

Definition 13 (FHCG)
Given the positive integer $H_i$, the stage cost $l_i - l_{iw}$, the terminal penalty $V_{if}$ and the terminal set $X_{if}$, the finite horizon closed-loop game (FHCG) problem consists in minimizing, with respect to $\kappa_{i[t,t+H_i-1]}$ and maximizing with respect to $w_{i[t,t+H_i-1]}$ the cost function

$$J_i(\bar{x}_i, \kappa_{i[t,t+H_i-1]}, w_{i[t,t+H_i-1]}, H_i) \triangleq \sum_{k=t}^{t+H_i-1} \{l_i(x_i(k), u_i(k)) - l_{iw}(w_i(k))\} + V_{if}(x_i(t + H_i))$$

subject to

(i) the state dynamics (19) with $x_i(t) = \bar{x}_i$;
(ii) the constraints (9), (20), (24) $k \in [t, t + H_i - 1]$;
(iii) the terminal state constraint $x_i(t + H_i) \in X_{if}$.

Letting $\kappa_{i0}^{\text{FHCG}}(x_i)$, $w_{i0}^{\text{FHCG}}(x_i)$ be the solution of FHCG, according to the RH paradigm, the feedback control law

$$u_i = \kappa_{i}^{\text{MPC}}(x_i)$$

is obtained by setting $\kappa_{i}^{\text{MPC}}(x_i) = \kappa_{i0}^{\text{FHCG}}(x_i)$ where $\kappa_{i0}^{\text{FHCG}}(x_i)$ is the first element of $\kappa_{i0}^{\text{FHCG}}(x_i)$.
Define the overall control law

\[ u \triangleq [\kappa_1^{\text{MPC}}(x_1)^	op, \ldots, \kappa_N^{\text{MPC}}(x_N)^	op] \]

(27)

**Assumption 12**

For every \( k \geq t \), the state trajectories of subsystems (19), (26) are continuous at \( \bar{x} = 0 \) and \( \psi_i = 0 \) with respect to the initial condition \( \bar{x} \) and the input sequence \( \psi_i \).

In order to derive the main stability and performance properties associated with the solution of FHCG, the following assumptions are introduced.

**Assumption 13**

\( l_{iw}(w_i) \) is such that \( \alpha_{iw}(|w_i|) \leq l_{iw}(w_i) \leq \beta_{iw}(|w_i|) \), where \( \alpha_{iw} \) and \( \beta_{iw} \) are \( \mathcal{K}_\infty \)-functions.

**Assumption 14**

The terminal penalty \( V_{if} \) and the terminal set \( X_{if} \) are such that, given an auxiliary law \( \kappa_{if} \)

1. \( X_{if} \subseteq X_i, \ X_{if} \) closed, \( 0 \in X_{if} \);
2. \( \kappa_{if}(x_i) \in U_i, \ \forall x_i \in X_{if} \);
3. \( f_i(x_i, \kappa_{if}(x_i)) + w_i \in X_{if}, \ \forall x_i \in X_{if}, \forall w_i \in U_i \);
4. \( \alpha_{V_{if}}(|x_i|) \leq V_{if}(x_i) \leq \beta_{V_{if}}(|x_i|), \ \forall x_i \in X_{if}, \) where \( \alpha_{V_{if}} \) and \( \beta_{V_{if}} \) are \( \mathcal{K}_\infty \)-functions;
5. \( V_{if}(f_i(x_i, \kappa_{if}(x_i)) + w_i) \leq V_{if}(x_i) \leq l_i(x_i, \kappa_{if}(x_i)) + l_{iw}(w_i), \forall x_i \in X_{if}, \forall w_i \in U_i \);
6. \( V_{if} \) is Lipschitz in \( X_{if} \) with Lipschitz constant \( L_{V_{if}} \).

Observe that, in view of Assumptions 1 and 11, given any \( \mathcal{K}_\infty \)-function \( \beta_{iw} \)

\[ \beta_{iw}(|w_i|) \leq \beta_{iw}(|g_i(x) + \psi_i|) \leq \beta_{iw} \left( \sum_{j=1}^{N} L_{ij} |x_j| + |\psi_j| \right) \]

and in view of Lemma 1 there exist some \( \mathcal{K}_\infty \)-functions \( \tau_{i1}, \tau_{i2}, \ldots, \tau_{i\psi} \) such that

\[ \beta_{iw} \left( \sum_{j=1}^{N} L_{ij} |x_j| + |\psi_j| \right) \leq \beta_{iw} \cdot (id + \tau_{i1}) \cdot L_{i1}(|x_1|) + \beta_{iw} \cdot (id + \tau_{i2}) \cdot L_{i2}(|x_2|) + \ldots + \beta_{iw} \cdot (id + \tau_{iN}) \cdot L_{iN}(|x_N|) + \beta_{iw} \cdot (id + \tau_{i\psi})(|\psi_i|) \]

(28)

**Assumption 15**

The set \( \mathcal{W}_i \) is such that (15) is satisfied, with \( \alpha_{i2} = \beta_{V_{if}}, \ \alpha_{i3} = \alpha_{il}, \ \Omega_i = X_{if}, \ \sigma_{ij} = \beta_{iw} \cdot (id + \tau_j) \cdot L_{ij}, \ \sigma_{i}^{\mathcal{W}} = \beta_{iw} \cdot (id + \tau_{i\psi}), \ V_i(x_i, H_i) = J_i(x_i, \kappa_{i2}, H_{i-1}, H_{i-1}), \)

\[ w_{ii}^{\mathcal{W}}(H_{i-1}, H_{i-1}, H_i). \]

The following preliminary result must be introduced first (for the proof see the last part of Lemma 2).

**Lemma 3**

If \( \beta_{V_{if}} \) is chosen such that \( \alpha_{il} \leq \beta_{V_{if}} \), then \( (id - \alpha_{i4}) \) with \( \alpha_{i4} = \alpha_{i3}^{-1} \), \( \alpha_{i2} = \beta_{V_{if}}, \ \alpha_{i3} = \alpha_{il} \), is a \( \mathcal{K}_\infty \)-function.
**Assumption 16**

Given systems (19), (26), \( i = 1, \ldots, N \), there exists a mapping \( \Delta \) as in (16), such that condition (18) is satisfied.

The main result can now be stated.

**Theorem 4**

Let \( X^\text{MPC}_i(H_i) \) be the set of states of system (19) where there exists a solution of FHCG. Under Assumptions 1, 6, 11–16, the overall system (21), (27) is ISS from \( \psi \) to \( x \) with robust output admissible set \( X^\text{MPC}(H) \).

**Remark 3**

Note that the term \( l_{w_i}(w_i(k)) \) is included in the performance index (25) in order to obtain the ISS. In fact, without this term only practical ISS can be proved, see [11].

**Remark 4**

The computation of the auxiliary control law, of the terminal penalty and of the terminal inequality constraint satisfying Assumption 14, is not trivial at all. In this regard, a solution has been proposed for affine system in [25], where it is shown how to compute a non-linear auxiliary control law based on the solution of a suitable \( H_\infty \) problem for the linearized system under control.

**Remark 5**

The usual way to derive the upper bound for the value function in \( X^\text{MPC}_i(H_i) \) requires the assumption that the solution \( u^o_{[t,t+H_i-1]} \) of the FHOC and \( u^o_{[t,t+H_i-1]} \) of the FHCG are Lipschitz in \( X^\text{MPC}_i(H_i) \). On the contrary, Theorem 1 gives the possibility to find the upper bound for the ISS Lyapunov function only in a subset of the robust output admissible set. This can be derived in \( X_{if} \), without assuming any regularity of the control strategies, by using the monotonicity properties (A9) and (A13), respectively. However, in order to enlarge the set \( W_i \) that satisfies Assumptions 9 and 15 for the FHOC and FHCG, respectively, it could be useful to find an upper bound \( \tilde{z}_i \) of \( V_i \) in a region \( \Omega_i \supseteq \Omega_i \). In this regard, define

\[
\tilde{z}_i = \max \left( \frac{V_i}{z_{2}(r_i)} \right) z_{2}
\]

where \( V_i = \max_{x \in \Omega_i} (V_i(x_i)) \) and \( r_i \) is such that \( B_{r_i} = \{ x_i \in \mathbb{R}^n : |x_i| \leq r_i \} \subseteq \Omega_i \), as suggested in [11]. This idea can either enlarge or restrict the set \( \Omega_i \) since \( \Omega_i \supseteq \Omega_i \) but \( \tilde{z}_i \geq z_i \).

5. CONCLUSIONS

Recent results on regional ISS have been used in this paper to study the properties of two classes of decentralized MPC algorithms applied for the control of nonlinear discrete time systems. Specifically, the stability analysis has been performed by considering the interconnections between the subsystems composing the overall system under control like perturbation terms and by using local MPC control laws with robustness properties. Both open-loop and closed-loop MPC formulations have been studied. Further research is required to establish the effect of partial exchange of information between subsystems on the stability conditions to be fulfilled.
APPENDIX A

Proof of Lemma 1
As it is proved in [6], observe that given \( \rho_i \), there exists a \( \mathcal{K}_\infty \)-function \( \zeta_i \) such that \( \rho_i^{-1} = (\text{id} + \zeta_i) \) and for any \( \mathcal{K} \)-function \( \gamma \)

\[
\gamma \left( \sum_{i=1}^{N} r_i \right) \leq \max \{ \gamma(Nr_1), \gamma(Nr_2), \ldots, \gamma(Nr_N) \} \tag{A1}
\]

Using these observations, it is obtained that

\[
\rho_i^{-1}(\theta_1 + \cdots + \theta_N + \theta_\phi) = \text{id} + \zeta_i(\theta_1 + \cdots + \theta_N + \theta_\phi) \\
\leq \max\{ (\text{id} + \zeta_i)((N + 1)\theta_1), (\text{id} + \zeta_i)((N + 1)\theta_2) \times (\text{id} + \zeta_i)((N + 1)\theta_3), \ldots, (\text{id} + \zeta_i)((N + 1)\theta_\phi) \}
\]

Finally, it is obtained that there exist some \( \mathcal{K}_\infty \)-functions \( a_1, a_2, \ldots, a_\phi \) such that

\[
\rho_i^{-1}(\theta_1 + \cdots + \theta_N + \theta_\phi) \\
\leq \max\{ (\text{id} + a_1)(\theta_1), (\text{id} + a_2)(\theta_2), \ldots, (\text{id} + a_N)(\theta_N), (\text{id} + a_\phi)(\theta_\phi) \} \quad \square
\]

Proof of Theorem 2
From Equation (14) it follows:

\[
\Delta V_i(x_i) \leq -z_{id}(V_i(x_i)) + \sum_{j=1}^{N} \sigma_{ij}^y(|x_j|) + \sigma_{ij}^y(|\psi_j|) \quad \forall x_i \in \Omega_i \quad \forall x_j \in \Xi_j \quad \forall \psi_j \in \Psi_i \tag{A2}
\]

where \( z_{id} = z_{ij} \circ z_{ij}^{-1} \). Without loss of generality, assume that \( \text{id} - z_{id} \) is a \( \mathcal{K} \)-function. Using (12), Equation (A2) implies

\[
\Delta V_i(x_i) \leq -z_{id}(V_i(x_i)) + \sum_{j=1}^{N} \eta_{ij}^y(V_j(x_j)) + \sigma_{ij}^y(|\psi_j|) \quad \forall x_i \in \Omega_i \quad \forall x_j \in \Xi_j \quad \forall \psi_j \in \Psi_i \tag{A3}
\]

where \( \eta_{ij}^y = \sigma_{ij}^y \circ z_{ij}^{-1} \). Given \( e_i \in \mathcal{R}_{\geq 0} \), let \( R_i(e_i) := \{ x_j : V_j(x_j) \leq e_i \} \). Define \( \Theta_i := \{ x_j : V_j(x_j) \leq \bar{e}_i = \max_{R_i \subseteq \Omega_i} e_i \}. \) Note that \( \bar{e}_i > b_i \) and \( \bar{D}_i \subseteq \Theta_i \). By Theorem 1, the region \( \bar{D}_i \) is reached asymptotically. This means that the state will arrive in \( \Theta_i \) in a finite time, that is, there exists \( T_{\bar{D}} \) such that \( V_i(x_i(k)) \leq \bar{e}_i, \forall k \geq T_{\bar{D}} \). Hence, the region \( \Theta_i \) is a robust positively invariant set for subsystem (8). By Remark 3.6 [6], there exist \( \beta_i \in \mathcal{K} \mathcal{L} \) such that

\[
V_i(x_i(k)) \leq \max \left\{ \beta_i(V_i(x_i(t), k), z_{id}^{-1} \circ \rho_i^{-1} \left( \sum_{j=1}^{N} \eta_{ij}^y(||V_j(x_j)||_{[k-1]} + \sigma_{ij}^y(||\psi_j||)) \right) \right\} \\
\forall x_i \in \Theta_i \quad \forall x_j \in \Xi_j \quad \forall \psi_j \in \Psi_i
\]

Now, using Lemma 1, one has

\[
V_i(x_i(k)) \leq \max\{ \beta_i(V_i(x_i(t), k), z_{id}^{-1} \circ (\text{id} + a_1) \circ \eta_{id}^y(||V_i(x_i)||_{[k-1]})), \ldots, \}
\]
\[
z_{id}^{-1} \circ (\text{id} + a_N) \circ \eta_{id}^y(||V_N(x_N)||_{[k-1]}), z_{id}^{-1} \circ (\text{id} + a_\phi) \circ \sigma_{id}^y(||\psi_i||) \} \quad \tag{A4}
\]
where $a_{i1}, a_{i2}, \ldots, a_{iN}$ and $a_{iy}$ are $\mathcal{K}_\infty$-functions. Then

$$V_i(x_i(k)) \leq \beta_i(V_i(x_i(t), k) + \sum_{j=1}^{N} \sigma_i^{-1}(|a_{ij}| + \sigma_i^j(||V_j(x_j)|| - 1))$$

Moreover, under Assumption 2, using (17) and following the same steps of the proof of Theorem 4 in [14], using (A5) instead of (2.4) in [14], it can be shown that (A5) satisfies AG property and is 0-AS in $\Theta_i$ and hence there exist a $\mathcal{K}_\infty$-function $\tilde{\beta}_i$ and a $\mathcal{K}$-function $\tilde{\gamma}_i$ such that

$$V_i(x_i(k)) \leq \tilde{\beta}_i(V_i(x_i(t), k) + \tilde{\gamma}_i(||\psi_i||)) \quad \forall x_i \in \Theta_i, \quad \psi_i \in \Psi_i$$

In fact AG in $\Theta_i + 0$-AS in $\Theta_i$ is equivalent to ISS in $\Theta_i$ [6]. Now, using properties (12) and (13), considering that, from (A1), for any $\mathcal{K}_\infty$-function $\gamma$, $\gamma(r + s) \leq \gamma(2r) + \gamma(2s)$, one has

$$|x_i(t)| \leq \tilde{\beta}_i(|x_i(t)|, k) + \tilde{\gamma}_i(||\psi_i||) \quad \forall x_i \in \Theta_i, \quad \psi_i \in \Psi_i$$

where $\tilde{\beta}_i(|x_i(t)|, k) = \sigma_i^{-1} - 2\tilde{\beta}_i(x_i(t)))$, and $\tilde{\gamma}_i(||\psi_i||) = \sigma_i^{-1} - 2\tilde{\gamma}_i(||\psi_i||)$. Hence, system (8) is ISS in $\Theta_i$ from $\psi_i$ to $x_i$. Then, considering that starting from $\Xi_i$, the state will reach the region $\Theta_i$ in a finite time, AG in $\Theta_i$ implies AG in $\Xi_i$. Now under Assumptions 2, considering that ISS in $\Theta_i \Rightarrow$ LISS and using the property that LISS + AG in $\Xi_i \Rightarrow$ ISS in $\Xi_i$ (see [7] for the continuous time case), system (8) is ISS in $\Xi_i$ from $\psi_i$ to $x_i$ and hence the overall system (11) is ISS in $\Xi$ from $\psi$ to $x$.

Proof of Lemma 2
From Assumption 6 and point 5 of Assumption 7

$$V_{\delta}(f_i(x_i, \kappa_{\delta}(x_i))) - V_{\delta}(x_i) \leq -l_i(x_i, \kappa_{\delta}(x_i)) \leq -a_{i\delta}(|x_i|)$$

Then, from point 4 of Assumption 7

$$a_{i3}(|x_i|) \leq V_{\delta}(x_i) - V_{\delta}(f_i(x_i, \kappa_{\delta}(x_i))) \leq V_{\delta}(x_i) - a_{i\delta}(|x_i|)$$

and $a_{i3} < a_{i2}$, so that $a_{i4} = a_{i3} - a_{i2}^{-1}$ is a $\mathcal{K}_\infty$-function with $a_{i4} \leq \text{id}$. Hence (id $- a_{i4}$) is a $\mathcal{K}_\infty$-function.

Proof of Theorem 3
By Theorem 1, if the subsystems admit ISS-Lyapunov functions in $X_i^{\text{MPC}}(H_i)$, then they are ISS in $X_i^{\text{MPC}}(H_i)$.

In the following, it will be shown that $V_i(x_i, H_i) \geq J_i(x_i, u^{\text{MPC}}_{i,t+H_i-1} + H_i)$ is an ISS-Lyapunov function in $X_i^{\text{MPC}}(H_i)$ for the $i$th subsystem. To this end, first note that

$$V_i(x_i, H_i) \geq l_i(x_i, \kappa_{i}^{\text{MPC}}(x_i)) \geq a_{i\delta}(|x_i|) \quad \forall x_i \in X_i^{\text{MPC}}(H_i)$$
Moreover, in view of Assumption 7

\[ \bar{u}_{\{t,t+H_i\}} = [u_{\{t,t+H_i-1\}}^0, \kappa_I(x_i(t+H_i))] \]

is an admissible, possible suboptimal, control sequence for the FHOC with horizon \( H_i + 1 \) at time \( t \) with cost

\[
J_i(x_i, \bar{u}_{\{t,t+H_i\}}, H_i + 1) \\
= V_i(x_i, H_i) - V_{Iy}(x_i(t + H_i)) + V_{Iy}(x_i(t + H_i + 1)) + l_i(x_i(t + H_i), \kappa_I(x_i(t + H_i)))
\]

Since \( \bar{u}_{\{t,t+H_i\}} \) is a suboptimal sequence

\[
V_i(x_i, H_i + 1) \leq J_i(x_i, \bar{u}_{\{t,t+H_i\}}, H_i + 1)
\]

But using point 5 of Assumption 7, it follows that

\[
J_i(x_i, \bar{u}_{\{t,t+H_i\}}, H_i + 1) \\
= V_i(x_i, H_i) - V_{Iy}(x_i(t + H_i)) + V_{Iy}(x_i(t + H_i + 1)) + l_i(x_i(t + H_i), \kappa_I(x_i(t + H_i)))
\]

Then,

\[
V_i(x_i, H_i + 1) \leq V_i(x_i, H_i) \quad \forall x_i \in X_{\text{MPC}}(H_i)
\]

with \( V_i(x_i, 0) = V_{Iy}(x_i), \forall x_i \in X_{Iy} \). Therefore,

\[
V_i(x_i, H_i) \leq V_i(x_i, H_i - 1) \leq V_i(x_i, 0) = V_{Iy}(x_i) \gg V_{Iy}(|x_i|) \quad \forall x_i \in X_{Iy} \quad \text{(A9)}
\]

Finally, following the proof of Theorem 2 in [9], it is possible to show that

\[
\Delta V_i(x_i, H_i) \leq -z_{Iy}(|x_i|) + \sum_{j=1}^{N} L_{Iy} L_{Iy}(|x_j|) + L_{Iy}(|\psi_j|) \quad \forall x_i \in X_{\text{MPC}} \forall x_j \in X_{\text{MPC}} \forall \psi_j \in \Psi_{Iy}
\]

\[
\text{(A10)}
\]

where

\[
L_I = L_{Iy} L_{Iy}^H + L_{Iy} \frac{L_{Iy}^H - 1}{L_{Iy} - 1}
\]

Therefore, by (A8)–(A10) and under Assumption 8, the optimal cost \( J_i(x_i, u_{\{t,t+H_i-1\}}^0, H_i) \) is an ISS-Lyapunov function for the closed-loop system (19), (22) in \( X_{\text{MPC}}(H_i) \) and hence, under Assumption 5, the closed-loop system is ISS with robust output admissible set \( X_{\text{MPC}}(H_i) \). Moreover, in view of Assumption 10, the overall system (21), (23) is ISS in \( X_{\text{MPC}} \) from \( \psi \) to \( x \).
Proof of Theorem 4
The robust positively invariant of $X^\text{MPC}_i(H_i)$ and consequently of $X^\text{MPC}(H)$, is easily derived from Assumption 14 by taking
\[
\bar{\kappa}_{[t+1,t+H_i]} = \begin{cases} 
\kappa^o_{[t+1,t+H_i-1]} & t+1 \leq k \leq t + H_i - 1 \\
\kappa_f(x(t + H_i)) & k = t + H_i 
\end{cases}
\]
as admissible policy vector at time $t+1$ starting from the optimal sequence $\kappa^o_{[t,t+H_i-1]}$ at time $t$. Moreover, it is possible to show that $V_i(x_i, H_i) = J_i(\bar{x}_i, \kappa^o_{[t,t+H_i-1]}, w^o_{[t,t+H_i-1]}, H_i)$ is an ISS-Lyapunov function for the closed-loop system (19), (26). In fact
\[
V_i(x_i, H_i) = J_i(\bar{x}_i, \kappa^o_{[t,t+H_i-1]}, w^o_{[t,t+H_i-1]}, H_i) \geq \min_{\kappa^o_{[t,t+H_i-1]}} J_i(\bar{x}_i, \kappa^o_{[t,t+H_i-1]}, 0, H_i)
\]
\[
\geq l_i(x_i, \kappa^o_{[t,H_i]}) \geq \beta_{\text{ui}}(|x_i|) \quad \forall x_i \in X^\text{MPC}_i(H_i) \tag{A11}
\]
In order to derive the upper bound, consider the following policy vector for the FHCG with horizon $H_i + 1$
\[
\bar{\kappa}_{[t+1,t+H_i]} = \begin{cases} 
\kappa^o_{[t,t+H_i-1]} & t \leq k \leq t + H_i - 1 \\
\kappa_f(x(t + H_i)) & k = t + H_i 
\end{cases}
\]
Then
\[
J_i(\bar{x}_i, \bar{\kappa}_{[t,t+H_i]}, W^o_{[t,t+H_i]}, H_i + 1)
\]
\[
= V_f(x(t + H_i + 1)) - V_f(x(t + H_i)) + l_i(x(t + H_i), u(t + H_i)) - l_{\text{wi}}(w_i(t + H_i))
\]
\[
+ \sum_{k=t}^{t+H_i-1} [l_i(x_i(k), u_i(k)) - l_{\text{wi}}(w_i(k))] + V_f(x_i(t + H_i))
\]
so that in view of Assumption 14
\[
J_i(\bar{x}_i, \bar{\kappa}_{[t,t+H_i]}, W^o_{[t,t+H_i]}, H_i + 1) \leq \sum_{k=t}^{t+H_i-1} [l_i(x_i(k), u_i(k)) - l_{\text{wi}}(w_i(k))] + V_f(x_i(t + H_i))
\]
which implies
\[
V_i(x_i, H_i + 1) \leq \max_{w_i \in \mathcal{W}_i} J_i(\bar{x}_i, \bar{\kappa}_{[t,t+H_i-1]}, W^o_{[t,t+H_i-1]}, H_i + 1)
\]
\[
\leq \max_{w_i \in \mathcal{W}_i} \sum_{k=t}^{t+H_i-1} [l_i(x_i(k), u_i(k)) - l_{\text{wi}}(w_i(k))] + V_f(x_i(t + H_i)) = V_i(x_i, H_i) \tag{A12}
\]
which holds $\forall x_i \in X^\text{MPC}_i(H_i), \forall w_i \in \mathcal{W}_i$. Moreover,
\[
V_i(x_i, H_i) \leq V_i(x_i, H_i - 1) \leq \cdots \leq V_i(x_i, 0) = V_f(x_i) < \beta_{V_f}(|x_i|) \quad \forall x_i \in X_f
\tag{A13}
\]

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From monotonicity property (A12), Assumption 1, (24) and (28), it is derived that

\[ V_i(f(x_i, \kappa^{\text{MPC}}_i(x_i)) + w_i, H_i) - V_i(x_i, H_i) \]

\[ \leq V_i(f(x_i, \kappa^{\text{MPC}}_i(x_i)) + w_i, H_i - 1) - V_i(x_i, H_i) \]

\[ \leq -l_i(x_i, \kappa^{\text{MPC}}_i(x_i)) + l_i(w_i) \]

\[ \leq -\varphi_i(|x_i|) + \beta_i(|w_i|) \]

\[ \leq -\varphi_i(|x_i|) + \beta_{i\ell}(\text{id} + \tau_i\ell)L_1(|x_1|) + \beta_{i\ell}(\text{id} + \tau_i\ell)L_2(|x_2|) \]

\[ + \cdots + \beta_{i\ell}(\text{id} + \tau_i\ell)L_N(|x_N|) + \beta_{i\ell}(\text{id} + \tau_i\ell)(\psi_i) \]

\[ \forall x_i \in X_i^{\text{MPC}}(H_i) \forall w_i \in \psi_i \]

(A14)

Then, by (A11), (A13), (A14), the optimal cost \( J_i(x_i, \kappa^0_{ij;j+H_i-1}, w^0_H_{ij;j+H_i-1}, H_i) \) is an ISS-Lyapunov function for the closed-loop system (19), (26) in \( X_i^{\text{MPC}}(H_i) \) and hence, under Assumption 12, the closed-loop system is ISS with robust output admissible set \( X_i^{\text{MPC}}(H_i) \). Moreover, in view of Assumption 16, the overall system (21), (27) is ISS in \( X^{\text{MPC}} \) from \( \psi_i \) to \( x_i \). \( \square \)

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