Tracking of non-square nonlinear continuous time systems
with piecewise constant model predictive control

L. Magni a,*, R. Scattolini b

a Dipartimento di Informativa e Sistemistica, Università degli Studi di Pavia, 27100 Pavia, Italy
b Dipartimento di Elettronica e Informazione, Politecnico di Milano, 20133 Milano, Italy

Received 5 October 2005; received in revised form 26 January 2007; accepted 28 January 2007

Abstract

The paper presents a model predictive control (MPC) algorithm for continuous-time, possibly non-square nonlinear systems. The algorithm guarantees the tracking of asymptotically constant reference signals by means of a control scheme were the integral action is directly imposed on the error variables rather than on the control moves. The plant under control, the state and control constraints and the performance index to be minimized are described in continuous time, while the manipulated variables are allowed to change at fixed and uniformly distributed sampling times. The algorithm is used to control a continuous fermenter where the manipulated variables are the dilution rate and the feed substrate concentration while the controlled variable is the biomass concentration.

Keywords: Nonlinear systems; Nonlinear model predictive control; Tracking; Non-square systems; Sampled-data systems; Fermenter

1. Introduction

The extraordinary industrial success of model predictive control (MPC) based on linear plant models motivates the development of MPC algorithms for nonlinear systems, see e.g. [38,30,21,6]. Currently, many theoretical results and industrial applications witness the wide potential impact of nonlinear MPC in the process industry, see [37].

MPC for nonlinear systems is usually developed by assuming that the plant under control is either described by a continuous time model, see [29,33,2,28,15,8,7,32,31], or by a discrete time one, see [17,5,22]. A continuous time setting is more natural, since the plant model is usually derived by resorting to first principle equations, but it results in a more difficult development of the MPC control law. In fact, the performance index to be minimized is defined in continuous time and the overall optimization procedure is assumed to be continuously repeated after any vanishingly small sampling time, which often turns out to be a computationally intractable task. On the contrary, MPC algorithms based on a discrete time system representation are computationally simpler, but require the discretization of the model equations, so that they rely from the very beginning on an approximate system representation. Moreover, the performance index to be minimized as well as the state constraints only consider the system behavior in the sampling instants, so ignoring the intersample behavior, which in some cases could be significant in the evaluation of the control performance.

In this paper the nonlinear plant under control, the state and control constraints and the performance index to be minimized are described in continuous time, while the manipulated variables are allowed to change at fixed and uniformly distributed sampling times. In so doing, one has to deal with optimization with respect to sequences, as in discrete time nonlinear MPC, while taking into account the continuous time evolution of the system and avoiding the approximate discretization of the model, see [37].
forcing integral action on the error variables, according to a very well known scheme in multivariable control, see [4,18,19]. On the contrary, in linear and nonlinear MPC, the integral action is usually forced on the manipulated variables, although this choice has the significant drawback to require the use of a state observer even when the state itself is measurable, otherwise any plant-model mismatch could lead to closed loop instability, see [20]. A sampled implementation of MPC has been already used in [25], where however a pure regulation problem was considered, while the regulation structure adopted here is similar to the one proposed by the authors in [24] for discrete time systems.

The algorithm developed here is used to control a highly nonlinear model of a continuous fermenter. The control objective is to move the biomass concentration along a prescribed (asymptotically constant) trajectory by acting on the dilution rate and the feed substrate concentration. The joint use of the two control variables allows one to obtain results which could not be achieved with only one input. The robustness properties of the resulting control system with respect to noise and model uncertainty are examined through simulation experiments.

2. Problem statement and preliminary results

In the paper, for any vector \( x \in \mathbb{R}^n \), \( \| x \| \) denotes the Euclidean norm in \( \mathbb{R}^n \), \( \| x \|_p = x^T \Pi x \), where \( \Pi > 0 \). For any square matrix \( A \), \( \lambda_{\text{max}}(A) \) and \( \lambda_{\text{min}}(A) \) denote the largest and the smallest real part of the eigenvalues of the matrix \( A \), respectively, \( \| A \| \) stands for the induced 2-norm of \( A \) and \( B^T \Pi = \{ x \in \mathbb{R}^n : \| x \|_p \leq 1, \Pi > 0 \} \). To properly formulate the MPC problem, the plant and a number of state augmentations have to be defined first.

**Plant**

Consider a plant \( P \) described by the nonlinear continuous-time dynamic system

\[
\begin{align*}
\dot{x}_p(t) &= f_p(x_p(t), u(t)), \quad t \geq 0, \\
y(t) &= h_p(x_p(t))
\end{align*}
\]

where \( x_p \in \mathbb{R}^n \) is the state, \( u \in \mathbb{R}^m \) is the input, \( y \in \mathbb{R}^p \) is the output. In (1) it is assumed that: (i) the state \( x_p \) is measurable; (ii) \( m \geq p \); (iii) \( f_p(\cdot, \cdot) \) is a \( C^1 \) function of its arguments; (iv) \( h_p(\cdot) \) is a Lipschitz function with Lipschitz constant \( L_h \); (v) given a sampling period \( T_s \), and letting \( t_k = kT_s \), \( k \) nonnegative integer, be the sampling instants, the control variable \( u \) is restricted to be constant in \([t_k, t_{k+1})\); (vi) the following constraints must be fulfilled:

\[
\begin{align*}
x_p(t) &\in X_p, \quad u(t) \in U, \quad t \geq 0
\end{align*}
\]

where \( X_p \) and \( U \) are closed and compact subsets of \( \mathbb{R}^n \) and \( \mathbb{R}^m \) respectively. The movement of (1) from the initial time \( t \) and initial state \( x_p(t) \) for a control signal \( u(t) \) is denoted by \( \varphi_p(t, i, x_p(t), u(\cdot)) \).

The tracking problem considered here consists in finding a “sampled” feedback control law guaranteeing

\[
\lim_{t \to \infty} e(t) = 0
\]

with \( e(t) = y^0(t) - y(t) \), for reference signals \( y^0 \) such that \( y^0(t) = \bar{y} \) \( \forall t \geq t_{k+N} \), where \( N \) is a nonnegative integer.

Letting \( \tilde{x}_p(y^0) \) and \( \tilde{u}(y^0) \) be such that \( f_p(\tilde{x}_p(y^0), \tilde{u}(y^0)) = 0 \) and \( h_p(\tilde{x}_p(y^0)) = y^0 \), the following preliminary assumptions are required.

**Assumption 1.** For a given \( y^0 \), there exists at least one equilibrium \( \tilde{x}_p(y^0), \tilde{u}(y^0) \in U \) of system (1) such that, letting

\[
\begin{align*}
A_p(y^0) &= \left. \frac{\partial f_p}{\partial x_p(y^0) u(y^0)} \right|_{x_p(y^0) u(y^0)}, \\
B_p(y^0) &= \left. \frac{\partial f_p}{\partial u} \right|_{x_p(y^0) u(y^0)}, \\
C_p(y^0) &= \left. \frac{\partial h_p}{\partial x_p(y^0)} \right|_{x_p(y^0)}
\end{align*}
\]

(i) the pair \((A_p, B_p)\) is stabilizable;
(ii) there are no transmission zeros of \((A_p, B_p, C_p)\) equal to zero.

**Assumption 2.** For any equilibrium \( \tilde{x}_p(y^0), \tilde{u}(y^0) \), if \( \lim_{t \to \infty} u(t) = \tilde{u}(y^0) \) and \( \lim_{t \to \infty} y(t) = \tilde{y} \), then \( \lim_{t \to \infty} x_p(t) = \tilde{x}_p(y^0) \).

**Plant and integrators**

According to well known results on the Internal Model Principle, see [4], the tracking problem can be solved with the scheme of Fig. 1, where \( I \) is a set of \( p \) integral actions applied to the error variables and described by

\[
\dot{z}(t) = e(t), \quad z(0) = z_0, \quad t \geq 0
\]

In Fig. 1, \( R \) is a regulator implementing the “sampled” feedback control law which must stabilize the system formed by the cascade connection of (1) and (3) and described by

\[
\begin{align*}
\dot{x}_p(t) &= f_p(x_p(t), u(t), y^0(t)), \quad t \geq 0, \\
e(t) &= h_p(x_p(t), y^0(t))
\end{align*}
\]

where \( x_a = [x_p', z'] \in \mathbb{R}^{n+p}, x_a(0) = x_{a0} = [x_{a0}', z_{a0}'] \), while \( f_a(\cdot, \cdot) \), and \( h_a(\cdot, \cdot) \) are easily derived from (1) and (3).

![Fig. 1. Control scheme.](image-url)
Plant, integrators and past control variable (enlarged plant)

According to the MPC approach, at any sampling instant \( t_k \), a performance index penalizing the future error and control variations \( \bar{d} = d(t_{k+1}) - d(t_{k+1}) \), \( i \geq 0 \) is minimized with respect to the future control moves. To formulate the optimization problem, it is convenient to enlarge the state vector \( x_t \) of (4) with the previous value of the control variable, so obtaining the new augmented system

\[
\dot{x}(t) = \left[ f_{\mathcal{A}}(x(t), u(t), y^0(t)) \right], \quad t \in [t_k, t_{k+1}), \quad x(t_k) = \left[ x_a(t_k^+) \right]
\]

(5)

\[
e(t) = h(x(t), y^0(t)),
\]

(6)

where \( x = [x_a, x_c^\alpha, x_c^\beta] \in \mathbb{R}^{n+p+m} \), \( x(0) = x_0 = [x_{a0}, u(t_0)^+] \) and \( h(-) \) is easily derived from (4). In this way \( u(t_{k+1}) = \bar{D}x(t_k), \bar{D} = [0_{n+m}, I_m] \) where \( 0_{n+m} \) and \( I_m \) are a \( n \times m \) zero matrix and the identity matrix of dimension \( m \), respectively. For any control signal \( u(-) \), the movement of (5) from the initial time \( t \) and initial state \( x(t) \) is denoted by \( \phi_t, \bar{x}(t), u(t) \).

**Enlarged plant and sampled control law**

Given a generic sampled feedback control law

\[
u(t) \equiv k(x(t), y^0(t)), \quad t \in [t_k, t_{k+1})
\]

(7)

the description of the hold mechanism implicit in (7) calls for a further state augmentation. Then, letting \( x_c := [x_a, x_c^\alpha] \in \mathbb{R}^{n+p+m} \), the closed loop system (5)–(7) is

\[
\dot{x}_c(t) = \left[ f_{x_c}(x_c(t), u(t), y^0(t)) \right], \quad t \in [t_k, t_{k+1}), \quad x_c(t_k) = \left[ x_a(t_k^+), \kappa(x(t), y^0(t)) \right]
\]

(8)

and its movement from the initial time \( t \) and initial state \( x_c(t) \) is denoted by

\[
\phi_{x_c}(t, t, x_c(t), y^0(t)) = \begin{bmatrix}
\phi_{x_c}^1(t, t, x_c(t), y^0(t)) \\
\phi_{x_c}^2(t, t, x_c(t), y^0(t)) \\
\phi_{x_c}^3(t, t, x_c(t), y^0(t))
\end{bmatrix}
\]

(9)

\[
\phi_{x_c}^1 \in \mathbb{R}^n, \quad \phi_{x_c}^2 \in \mathbb{R}^{n+p+m}, \quad \phi_{x_c}^3 \in \mathbb{R}^{m+n}
\]

With reference to the closed-loop system (8) it is possible to define the following sets.

**Definition 1.** A sampled output admissible set associated to (8) is a set \( \Gamma_\varepsilon^u(k, y^0(-)) \in \mathbb{R}^{n+p+m} \) such that for all \( x \in \Gamma_\varepsilon^u(k, y^0(-)) \), \( \phi_{x_c}^1(t_k, t_{k+1}, \kappa(x(t), y^0(t))) \), \( y^0(-) \) \( \in \Gamma_\varepsilon^u(k, y^0(-)) \), \( \phi_{x_c}^2(t_{k+1}, t_k, \kappa(x(t), y^0(t))) \), \( \phi_{x_c}^3(t_{k+1}, t_k, \kappa(x(t), y^0(t))) \) \( \in X_p \), \( t \in [t_k, t_{k+1}) \), \( \kappa(x(t), y^0(t)) \) \( \in U \), \( \lim_{t \to -\infty} \| h_p(\phi_{x_c}^{-1}(t, t, x_c(t), y^0(t))) - y^0(t) \| = 0 \), \( \lim_{t \to -\infty} \| \kappa(\phi_{x_c}^1(t_k, t_{k+1}), \kappa(x(t), y^0(t))) \| \| y(t) \| = 0 \).

**Remark 1.** According to **Definition 1**, a sampled admissible set \( \Gamma_\varepsilon^u(k, y^0(-)) \) is a state invariant set, associated to the closed loop system (8), defined at the sampling instant \( t_k \) and such that (i) the state and control constraints (2) are satisfied in all the future continuous-time instants, (ii) the tracking problem is asymptotically solved. The (unique) maximal sampled output admissible set \( \Gamma_\varepsilon^u(k, y^0(-)) \) is defined as the union of all sampled output admissible sets.

**Definition 2.** An output admissible set associated to (8) is a set \( \Gamma^u(\varepsilon, t_k, y^0(-)) \in \mathbb{R}^{n+p+m} \) such that for all \( x \in \Gamma^u(\varepsilon, t_k, y^0(-)) \), \( \phi_{x_c}^1(t_k, t_{k+1}, x_c(t_k), y^0(t_k)) \), \( y^0(-) \) \( \in \Gamma^u(\varepsilon, t_k, y^0(-)) \), \( \phi_{x_c}^2(t_{k+1}, t_k, x_c(t_k), y^0(t_k)) \), \( \phi_{x_c}^3(t_{k+1}, t_k, x_c(t_k), y^0(t_k)) \) \( \in U \).

**Remark 2.** According to **Definition 2**, an output admissible set \( \Gamma^u(\varepsilon, t_k, y^0(-)) \) is a set, defined at any continuous-time instant \( t \), of states of the closed-loop system (8) such that (i) the state of (5) at the closest sampling time in the future belongs to \( \Gamma^u(\varepsilon, t_k, y^0(-)) \) and (ii) the state and control constraints (2) are satisfied in all the future continuous-time instants. The (unique) maximal output admissible set \( \Gamma^u(\varepsilon, t_k, y^0(-)) \) is defined as the union of all output admissible sets.

The tracking problem can now be formally stated as the problem of finding a feasible sampled control law (7) with the largest output admissible set \( \Gamma^u \) and which optimizes a given performance index.

Assume now to know a feasible control law (7) satisfying the following assumption.

**Assumption 3.** The feasible control law (7) is a \( C^1 \) function, with respect to its argument \( x \), with Lipschitz constant \( L_u \).

Let \( \bar{x}_p(y^0, \kappa) := [\bar{x}_p(y^0, \kappa)'] \bar{z}(y^0, \kappa)' \bar{u}(y^0, \kappa)' \bar{u}(y^0, \kappa)' \) be an equilibrium associated to the closed-loop system (8) such that \( \bar{\varepsilon} = h_p(\bar{x}_p(y^0, \kappa), y^0) = 0 \), with \( \bar{x}_p(y^0, \kappa) := [\bar{x}_p(y^0, \kappa)'] \bar{z}(y^0, \kappa)' \bar{u}(y^0, \kappa)' \). If **Assumption 1** is satisfied, in view of the Implicit Function Theorem, \((\bar{x}_p(y^0, \kappa), y^0)\) is an isolated equilibrium for system (8) (for simplicity the dependence of the equilibrium point on \( \kappa \) will be omitted whenever possible).

For this control law, an associated sampled output admissible set can be computed as follows. Define the linearization of system (5) around the equilibrium as

\[
\bar{\dot{x}}(t) = \begin{bmatrix}
A_{x}(y^0)\bar{\dot{x}}_x(t) + B_{x}(y^0)\bar{u}(t) \\
0_{m+p+m}
\end{bmatrix}
\]

(9)

\[
\bar{\dot{x}}_x(t_k) = \begin{bmatrix}
\bar{\dot{x}}_x(t_k^+) \\
\bar{\dot{u}}(t_k^+)
\end{bmatrix}
\]

\[
A_x(y^0) = \partial f_x / \partial x(t, x_c(t), y^0(t)) \quad B_x(y^0) = \partial f_x / \partial u(t, x_c(t), y^0(t))
\]

Then introduce the discretization of (9) given by

\[
\bar{\dot{x}}(t_{k+1}) = A_D(y^0)\bar{\dot{x}}(t_k) + B_D(y^0)\bar{u}(t_k)
\]
with

\[ A_D(\tilde{y}) := \begin{bmatrix} e^{A(t)p}T_e & 0_{n+p,m} \\ 0_{m,n+p} & 0_{m,m} \end{bmatrix} \]

\[ B_D(\tilde{y}) := \int_0^T e^{A(t)p}B(t)\,dt \]

Finally let

\[ K(\tilde{y}) = \frac{\partial \kappa(x,y)}{\partial x} \bigg|_{y=x} \]

In view of the feasibility of (7), it is then easy to show that the closed-loop matrix \( A_D(\tilde{y}) := A_D(\tilde{y}) + B_D(\tilde{y})K(\tilde{y}) \) of the linearized discrete-time system (10) is Hurwitz and the following result holds.

**Lemma 1.** Let \( \kappa(x,y) \) be a feasible control law. Consider an equilibrium \((\tilde{x}(\tilde{y}), \tilde{u}(\tilde{y}))\) of system (1) satisfying Assumptions 1 and 3, a positive definite matrix \( Q \) and two real positive scalars \( \gamma \) and \( \gamma_2 \) such that \( \gamma < \lambda_{\text{min}}(Q) \). Define by \( \Pi \) the unique symmetric positive definite solution of the following Lyapunov equation:

\[ A_D^T(\tilde{y})\Pi A_D(\tilde{y}) - \Pi + Q = 0 \]  

(11)

where

\[ \Pi = \int_0^T A_D^T(\tilde{y})(t)\tilde{A}_D(\tilde{y})(t)\,dt + \gamma_2I_{n+p+m} \]

and

\[ \tilde{A}_D(\tilde{y})(t) := \begin{bmatrix} \pi_1 & \pi_2 \\ 0_{m,n+p} & 0_{m,m} \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} T_e \\ t \end{bmatrix} \]

with \( \pi_1 = e^{A(t)p} + \left( \int_0^t e^{A(t)p}d\tau \right)B_D(\tilde{y})K_1 \), \( \pi_2 = \left( \int_0^t e^{A(t)p}d\tau \right)B_D(\tilde{y})K_2 \) and \( K := [K_1, K_2] \), \( K_1 \in \mathbb{R}^{n \times n+p} \).

Then, there exist two constants \( T_s \in (0, \infty) \) and \( \sigma \in (0, +\infty) \) specifying a neighborhood \( \Omega_s(\tilde{x}(\tilde{y}), \kappa, T_s) \) of \( \tilde{x}(\tilde{y}) \) of the form

\[ \Omega_s(\tilde{x}(\tilde{y}), \kappa, T_s) = \{ x \in \mathbb{R}^{n+p+m} \mid \| x - \tilde{x}(\tilde{y}) \|_T \leq \sigma \} \]  

(12)

such that \( \forall x \in \Omega_s(\tilde{x}(\tilde{y}), \kappa, T_s) \):

\begin{enumerate}
  \item \( \phi(x, t, \tilde{x}(\tilde{y}), \tilde{u}(\tilde{y}))) \in \Xi, t \in [t_k, t_{k+1}) \), \( \kappa(x, \tilde{y}) \in U \);
  \item \( \| \phi(x, t, \tilde{x}(\tilde{y}), \tilde{u}(\tilde{y}))) - \phi(x, \tilde{y})) \|_T^2 - \| x - \tilde{x}(\tilde{y}) \|_T^2 \leq -\gamma_2 \| x - \tilde{x}(\tilde{y}) \|_T^2 \)
\end{enumerate}

(13)

The proof of this Lemma is analogous to the one of Lemma 1 in [25] and is not reported for conciseness.

**Remark 3.** Given a sampling time \( T_s \), an equilibrium \((\tilde{x}, \tilde{u})\) of (1) such that \( h_\tilde{p}(\tilde{x}) = \tilde{y} \), the linearization of (5) around \((\tilde{x} := [\tilde{x}_p^T \ 0_{1,p} \ \tilde{u}^T \ \tilde{u}], \tilde{y})\) and the sampled linear model described by (10), a feasible sampled linear control law is

\[ u(t) = \tilde{u} + K[x(t) - \tilde{x} + K_1y(t)] \]  

(14)

where \( K \) is chosen to stabilize \( A_D + B_DK \). In (14), the term \( K_1y(t) \) is included to consider also a feedforward (possibly anticipative) action. A practical way to compute the pairs \( K \) and \( K_1 \) is to resort to a linear MPC stabilizing methods, see e.g. [3], so that also the adopted auxiliary control law has predictive capabilities.

### 2.1. The sampled MPC control law

Let \( u = \kappa(x, y) \) be a feasible auxiliary control law, assumed to be known together with its sampled output admissible set and Lyapunov function given by Lemma 1. Moreover, given a control sequence \( \tilde{u}_{i,N_2}(t_k) := [u_1(t_k), u_2(t_k), \ldots, u_{N_2}(t_k)] \), with \( N_2 \geq 1 \), define the Finite Horizon piecewise constant control signal

\[ u^{\text{FH}}(t) = \begin{cases} u_j(t) & t \in [t_k+j-1, t_{k+j}) \quad j = 1, \ldots, N_2 \\ \kappa(x, y) & t \in [t_{k+N_2}, t_k+N_2) \end{cases} \]  

(15)

where \( x_N = (x(t_k+N_2), x(t_k), \ldots, x(t_{k+N_2})) \) and \( N_2 \geq 1 \) and denote by \( u^{\text{FH}}(t) \) the signal \( u^{\text{FH}}(t) \) in the interval \( t \in [t_{k+N_2}, t_k+N_2) \).

For system (5) and (6) the MPC, based on the following optimization problem, is applied to enlarge the output admissible set of \( \kappa(x, y) \) and to improve the control performance.

**Finite horizon optimal control problem (FHOCP).**

Given the sampling time \( T_s \), the control horizon \( T_c \), the prediction horizon \( N_p, N_c \leq N_p \), two positive definite matrices \( Q \) and \( R \), the reference signal duration \( N_s \leq N_p \), a feasible auxiliary control law \( \kappa(x, y) \), the matrix \( \Pi \) and the region \( \Omega_s(\tilde{x}(\tilde{y}), \kappa, T_s) \) given in Lemma 1 with \( \gamma > c_{\max} \) \( \max(Q) \), \( x_N \) is defined by (16) \( \max(R) \), \( l_s \), at every sampling time instant \( t_k \), minimize, with respect to \( \tilde{u}_{i,N_2}(t_k) \),

\[ J_{\text{FH}}(x_N, \tilde{u}_{i,N_2}(t_k), N_c, N_p, y(t)) = \int_{t_k}^{t_{k+N_p}} \left\{ \| e(t) \|_Q^2 + \| u(t) - \mathbb{E}(e(t)) \|_R^2 \right\} \,dt + \int_{t_k}^{t_{k+N_p}} V_f(\phi(t, x(t), y(t), \tilde{u}^{\text{FH}}(t, x(t), y(t)))) \,dt \]

(16)

where the terminal penalty \( V_f \) is selected as

\[ V_f(x) = \| x - \tilde{x}(\tilde{y}) \|_R^2 \]

The minimization of (16) must be performed under the following constraints:

\begin{enumerate}
  \item \( (i) \) the state dynamics (5) and (6) with \( x(t_k) = x_{k} \);
  \item \( (ii) \) the constraints (2), \( t \in [t_k, t_{k+N_2}] \) with \( u \) given by (15);
  \item \( (iii) \) the terminal state constraint \( x(t_{k+N_2}) \in \Omega_s(\tilde{x}(\tilde{y}), \kappa, T_s) \).
\end{enumerate}
The state-feedback MPC control law
\[ u(t) = \kappa(t_x(t), y^0(\cdot)), \quad t \in [t_k, t_{k+1}) \] (17)
is then derived by solving the FHOCP at every sampling time instant \( t_k \), and applying the constant control signal \( u(t) = u^*_t(x(t_k)) \), \( t \in [t_k, t_{k+1}) \) where \( u^*_t(x(t_k)) \) is the first column of the optimal sequence \( \bar{u}^{t,*}_t(x(t_k)) \).

Let
\[
\varphi^{RH}(t, x_i(t), y^0(\cdot)) = \begin{bmatrix}
\varphi^{RH}_x(t, i, x_i(t), y^0(\cdot)) \\
\varphi^{RH}_u(t, i, x_i(t), y^0(\cdot)) \\
\varphi^{RH}_y(t, i, x_i(t), y^0(\cdot)) \\
\varphi^{RH}_x(t, i, x_i(t), y^0(\cdot)) \\
\varphi^{RH}_u(t, i, x_i(t), y^0(\cdot)) \\
\varphi^{RH}_y(t, i, x_i(t), y^0(\cdot))
\end{bmatrix},
\]
then
\[
\varphi^{RH} \in \mathbb{R}^n, \quad \varphi^{RH}_x \in \mathbb{R}^n, \quad \varphi^{RH}_u \in \mathbb{R}^m, \quad \varphi^{RH}_y \in \mathbb{R}^{m+p}, \quad \varphi^{RH}_x \in \mathbb{R}^m
\]
be the movement of (8) with \( \kappa(\cdot) = \kappa^{RH}(\cdot) \) and define the following sets.

**Definition 3.** Let \( X^0(N_c, N_p, y^0) \in \mathbb{R}^{n+p+2m} \) be the set of states \( x_i \) of system (5) and (6) at the sampling times \( t_k \) such that there exists a feasible control sequence \( \bar{u}_{1:N_c}(t_k) \) for the FHOCP.

**Definition 4.** Let \( X^0(t, N_c, N_p, y^0) \in \mathbb{R}^{n+p+2m} \) be the set of states \( x_i \) such that for all \( x_i(t) \in X^0(t, N_c, N_p, y^0) \), \( \varphi^{RH}(t, t, x_c, y^0) \in X^0(t, N_c, N_p, y^0) \), \( \varphi^{RH}(t, t, x_c, y^0(\cdot)) \in X_p, \quad \forall t \in [t_k, t_{k+1}) \), \( \varphi^{RH}(t, t, x_c, y^0(\cdot)) \in U \) where \( t_k \) is the closest sampling time in the future.

The main stability results of the proposed MPC algorithm can now be stated.

**Theorem 1.** Under Assumptions 1–3,

(i) \( (\bar{x}(y^0, \kappa)_t, \kappa^{RH}(x(y^0, \kappa), y^0(\cdot)), y^0) \) is an exponentially stable equilibrium point for the closed-loop system formed by (5), (6) and (17) with output admissible set \( X^0(t, N_c, N_p, y^0) \);

(ii) \( X^0(N_c, N_p + 1, y^0) \supset X^0(N_c, N_p, y^0) \), \( \forall N_c, N_p \);

(iii) \( \chi^{n}(N_c, N_p, y^0) \supset \Omega \{(x(y^0, \kappa), T_\delta), \forall N_c, N_p \};

(iv) there exist a finite \( N_p \) such that \( X^0(N_c, N_p, y^0) \supset X^0(\bar{x}(y^0, \kappa), T_\delta), \forall N_c \).

**Remark 4.** For a practical implementation of the algorithm, the knowledge of the sampled output admissible set \( X^0 \) is not required, but only the set \( \Omega \) must be computed. To this regard, numerical techniques based on Eqs. (11) and (12) can be easily implemented. Point (iii) of Theorem 1 states that inside the set \( \Omega \), the feasibility of MPC control law is guaranteed for any prediction \( N_p \) and control \( N_c \) horizon.

**Remark 5.** An usual way to solve the tracking problem is to compute the reference trajectories of the state and control variables corresponding to the reference signal and to resort to a proper coordinate transformation. However, it must be stressed that in so doing the result is not robust, that is asymptotic zero error regulation is not preserved for modelling errors or plant parameter variations. On the contrary, for asymptotically constant reference signals, the proposed method guarantees robust asymptotic zero error regulation, that is the error asymptotically vanishes even in the presence of a plant-model mismatch provided that stability is preserved, although the computation of the allowed uncertainty is usually difficult. This robustness property is due to the integrators directly applied on the error signal. In fact, if the feasibility of the FHOCP and the asymptotic stability are preserved, then the input to the integrators must go asymptotically to zero.

**Remark 6.** The robust stability property with respect to unmodelled dynamics and/or disturbances has not been considered in the problem formulation. As such, the algorithm guarantees the robustness properties of nominal MPC, see [27]. Possible ways to modify the problem statement in order to improve robustness are discussed again in [27].

**Remark 7.** In the FHOCP, continuous time state constraints are considered. It can appear that this approach is only conceptual, because any numerical implementation needs a time discretization and the constraints satisfaction can be checked only in the integration time instants. However this is not a significant limitation; in fact, following Theorem 3 in [25] one can choose the maximum integration step \( \delta \) and a more conservative discrete-time state constraint so as to guarantee the fulfillment of the original continuous-time state constraint.

**Remark 8.** If a dynamic input-output auxiliary regulator is available, one can avoid to compute the equilibrium point \( \bar{x}(y^0) \), \( u(y^0) \) by resorting to the algorithm presented in [26].

**Remark 9.** If the state is not available, the proposed algorithm must be combined with a suitable observer, such as a continuous-discrete Extended Kalman Filter. In a discrete time setting the “separation principle” of a closed-loop formed by an exponentially stabilizing state-feedback MPC and an asymptotically stable observer have been proven in [23]. The same problem has been addressed for continuous time systems in [10].

**Remark 10.** In order to develop an algorithm with guaranteed stability properties, it has been necessary to make the assumption that for any considered set-point signal there exists a suitable constant input within the feasibility region and that a feasible solution exists (see Assumption 1 and the existence of a non-empty \( X^0(t, N_c, N_p, y^0) \)). If these conditions are not satisfied, still the MPC problem can be solved with constraint relaxation techniques.
3. Control of a continuous fermenter

In this section, the MPC control law introduced in the paper is applied to the simulated model of a continuous fermenter. The volume of the fermenter is assumed constant, its contents well-mixed, and the feed sterile. The manipulated inputs are the dilution rate $D$ and the feed substrate concentration $S_f$ while the state variables are the effluent cell-mass or biomass concentration $X_b$, the substrate concentration $S$ and the product concentration $P$. In the sequel, $X_b$, $S$ and $P$ are assumed available for control design. Although this assumption is rarely satisfied in practice, their value can often be estimated from secondary variables, see e.g. [35,16].

Assuming that the fermenter culture consists of a single, homogeneously growing organism, a simple and widely used model is [1,12]

$$\dot{X}_b = -DX_b + \mu X_b$$

$$\dot{S} = D(S_0-S) - \frac{1}{Y_{X_b/S}} \mu X_b$$

$$\dot{P} = -DP + (\alpha + \beta)X_b$$

where

$$\mu = \frac{\mu_m(1-\frac{S}{K_m})}{K_m + S + \frac{S}{K_l}}$$

is the specific growth rate, $Y_{X_b/S}$ is the cell-mass yield, and $\alpha$ and $\beta$ are yield parameters for the product. In (21) the maximum specific growth rate $\mu_m$, the product saturation constant $P_m$, the substrate saturation constant $K_m$, and the substrate inhibition constant $K_l$ must be chosen to fit experimental data [36,9]. The nominal operating conditions and model parameters used throughout the paper are taken from [12,13] and listed in Table 1.

The control objective is to move the biomass concentration $X_b$ along a pre-prescribed trajectory within a large range. The open-loop behavior of the biomass concentration $X_b$ for step changes in the dilution rate and feed substrate concentration is shown in Fig. 2. Note that the responses to step changes of the dilution rate are not symmetrical. The model exhibits more severe nonlinear behavior for changes in the feed substrate concentration. In particular it is apparent that the gain from $S_f$ to $X_b$ can even change sign. In fact, for $1\%$ variation of $S_f$, the concentration $X_b$ increases, while it decreases for $2\%$ step change and an "inverse response" behavior occurs. Different single-input/single-output (SISO) control strategies have been proposed in order to control this class of systems [12,13]. However, the open loop behavior shows that with a single input control strategy is not possible to move the biomass concentration $X$ to a value greater than 7.4 g/L. This motivates the interest for a multi-input/single-output control law. In the following the MPC algorithm proposed in this paper will be used to control the plant along a pre-prescribed reference bringing the biomass concentration $X_b$ from the initial equilibrium value of Table 1 to the final steady state value of 7.5 g/L. The following input and state constraints are considered

0.05 h$^{-1} \leq D \leq 0.3$ h$^{-1}$

18 g/L $\leq S_f \leq 25$ g/L

3 g/L $\leq X_b \leq 10$ g/L

1 g/L $\leq S \leq 10$ g/L

10 g/L $\leq P \leq 35$ g/L

The nonlinear continuous-time state space model (1) of system (18)-(21) is obtained by defining the normalized state vector $x_0 = [\frac{X_b-X_b^0}{X_b^0}, \frac{S-S^0}{S^0}, \frac{P-P^0}{P^0}]$, manipulated input $u = [\frac{D-D^0}{D^0}, \frac{S_f-S_f^0}{S_f^0}, \frac{P_m-P_m^0}{P_m^0}]$ and output $y = \frac{X_b}{X_b^0}$. The initial equilibrium point is defined by $\bar{x} = 0$, $\bar{u} = 0$ and $\bar{y} = 0$, while the linear auxiliary stabilizing control law is given by

$$u(t) = K_r(t_k) + K_y \Delta Y(t_k)$$

where $K_r$ and $K_y$ are obtained with an MPC control law synthesized on the linearization of (5), (6) around $(x,u,y^0) = (0,0,0)$ discretized with a sampling period $T_s = 1$ h, see [3]. Finally, $\Delta Y^0(t_k) := [y^0(t_k),y^0(t_{k+1}),\ldots, y^0(t_{k+N_c^0})]$ and $N_c^0 = 30$ is the prediction and control horizon of the linear MPC. The cost function minimized to synthesized the linear MPC has the same stage cost of (16)

Table 1
Nominal operating conditions

<table>
<thead>
<tr>
<th>Variable</th>
<th>Value</th>
<th>Variable</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y_{X_b/S}$</td>
<td>0.4 g/L</td>
<td>$\alpha$</td>
<td>2.2 g/L</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.2 h$^{-1}$</td>
<td>$\mu_m$</td>
<td>0.48 h$^{-1}$</td>
</tr>
<tr>
<td>$P_m$</td>
<td>50 g/L</td>
<td>$K_m$</td>
<td>1.2 g/L</td>
</tr>
<tr>
<td>$K_l$</td>
<td>22 g/L</td>
<td>$S_f$</td>
<td>20 g/L</td>
</tr>
<tr>
<td>$D$</td>
<td>0.202 h$^{-1}$</td>
<td>$X_b$</td>
<td>6.0 g/L</td>
</tr>
<tr>
<td>$S$</td>
<td>5.0 g/L</td>
<td>$P_m$</td>
<td>19.14 g/L</td>
</tr>
</tbody>
</table>
with an additional terminal equality constraint and a penalty on the state variable with matrix $Q_x$ to guarantee the stability of the linearized closed-loop system. Letting $Q = 1$, $R = \text{diag}(1,1)$, $Q_x = \text{diag}(1,1,1,3,1,1) \times 10^{-3}$, $\gamma = 2$, $\gamma_2 = 0.99$, $Q = 2.02 + L_{\text{aux}}$, and $y^d = 0.25$ (which corresponds to $X_0 = 7.5 \text{ g/L}$) a region $Q_x(x)^{()}$, $k$, $T_v$ satisfying Lemma 1 is computed with the $\Pi$ solution of (11) and $\sigma = 0.027$. In order to guarantee continuous time state-constraints satisfaction, following Theorem 3 in [25], the constraints $\|x_p - \bar{x}_p\| \leq \varrho$ with $\bar{x}_p = [0.0833, 0.1000, 0.1755]$, $v = \text{diag}(2.9388, 1.2346, 2.3446)$, $g = 0.9$ is introduced so that, with a constant integration step $\delta = 0.05h$, $x_p(t) \in \{x_p : \|x_p - \bar{x}_p\| \leq 1\} \subseteq X$, $t > 0$.

Then, the sampled MPC control law described in Section 2.1 has been synthesized letting $Q = 1$, $R = \text{diag}(1,1)$, $N_c = 6$ and $N_D = 200$. The optimization problem has been iteratively solved using the MatLab Optimization Toolbox.

**Experiment 1.** The experiment is aimed at emphasize the property of the proposed method to weight in the cost function the output tracking error, while the state of the system is used only in the definition of the terminal cost used for stability. To this end the reference signal for the biomass concentration $X_0$ has first been varied according to a saw-tooth profile, while after time $t = 100$ h it has been maintained at the constant value 7.5 g/L.

The results obtained by applying the auxiliary linear MPC control law and the nonlinear MPC algorithm presented in this paper are reported in Fig. 3. From Fig. 3a, it is apparent that both the methods allow to reach the required final steady-state value. However, the nonlinear MPC law achieves a significant performance improvement. In fact the infinite horizon cost

$$J_{\text{lin}}(x_d, u(\cdot), y^d(\cdot)) = \int_0^\infty \{\|e(t)\|_Q^2 + \|u(t) - \mathbb{E}(x(t))\|_R^2\} \, dt$$

obtained with the linear MPC law is 0.2447 while with the nonlinear one is 1.390.

Note also that during the transient with the proposed nonlinear MPC algorithm, the feed substrate concentration $S_t$ reaches its lower limit (18 g/L) so that the corresponding constraint is active. Finally, as stated in Theorem 1, the state and control variables with nonlinear MPC converge to the same steady-state values of the auxiliary linear control law.

**Experiment 2.** The disturbance rejection properties of the proposed algorithm have been tested by adding to the process state Eqs. (18)–(20) three white noise terms with zero mean value and variance 0.1. By choosing the reference value for $X_0$ as in Experiment 1, the simulation results reported in Fig. 4 have been obtained. The transients of the state and control variables clearly show the reduced sensitivity to noise of nonlinear MPC algorithm with respect to the linear one. Note also that in this experiment the control variables exhibit a substantially different behavior in the two cases.

![Fig. 3. Responses of closed-loop system with linear MPC (dashed line) and nonlinear MPC (solid line): (a)–(c) state variables, (d) and (e) input variables.](image1)

![Fig. 4. Responses of noisy closed-loop system with linear MPC (dashed line) and nonlinear MPC (solid line): (a)–(c) state variables, (d) and (e) input variables.](image2)
the piecewise nature of the control signal and to the use of different control and prediction horizons. These two features lead to a tractable optimization problem, where the cost minimization is performed with respect to sequences and the number of future control moves to be selected can be small.

Notably, once a stabilizing auxiliary control law is known, the algorithm proposed here can only improve its performance with respect to the adopted cost function. This renders more attractive this approach also when a stabilizing control law is already available.

Appendix

Proof of Theorem 1. As in the proof of Theorem 2 in [25], it can be concluded that $X_0(t, N_c, N_p, y^0)$ is non-empty, $X_0(N_c, N_p, y^0)$ is a sampled output admissible set for system (5), (6) and (17) and $X(t, N_c, N_p, y^0)$ is an output admissible set of (8) with $\kappa(t) = \kappa^{RH}(t)$.

Let now show that $[\hat{x}(t), \hat{y}(t)]^{RH}(\hat{x}(t), y^0)$ is an asymptotically stable equilibrium point for the closed-loop system (5), (6), (17). First, following a derivation similar to the one in the proof of Theorem 2 in [25], define

$$V(x(t), t, y^0) := \int_{t}^{t+k} + \|h_0^{RH}(t, x(t), y^0)\|^2_{Q} \, \text{d}t$$

for $t \in (t_k, t_{k+1})$.

To this end, first note that $V(x(t_k), t_k, y^0)$ is bounded $\forall t \in X_2^0(N_c, N_p, y^0)$. Moreover

- $\forall t \in (t_k, t_{k+1})$

$$V(\phi^{RH}(t, t_k, x(t_k)), t, y^0) = V(x(t_k), t_k, y^0) - \int_{t_k}^{t} \|h_0^{RH}(t, t_k, x(t_k), y^0))\|^2_{Q} \, \text{d}t + \|\phi^{RH}_0(t, t_k, x(t_k), y^0)\|^2_{R}$$

(23)

- At time $t = t_k$ and $\widetilde{u}_{t+k+1}$ given by

$$\widetilde{u}_{t+k+1} := [\hat{x}_k^{RH}(t_k), \kappa(\phi(t_{k+1}, t_k, t_{k+1}, t_{k+1}, y^0))]$$

(24)

is a (sub-optimal) feasible solution for the new FHOCP so that

Experiment 3. A final simulation experiment has been performed in order to emphasize the robustness property of the proposed control scheme with respect to model uncertainty. In particular the parameter $\mu_m$ has been changed from the nominal value 0.48 h$^{-1}$ to the perturbed value 0.45 h$^{-1}$ at time 10 h, while the set point is constant at 6 g/L. The results obtained with the control algorithm based on the solution of the regulation problem through a standard change of coordinates (see Remark 5) and with the tracking control algorithm based on the solution of the FHOCP are reported in Fig. 5. It is apparent that the introduction of the integral action guarantees robust asymptotic zero error regulation. Moreover, the tracking control scheme is superior to the pure regulation one when a time-varying set-point is used (see Experiment 1) because it is not necessary to compute the state and input time-varying trajectory corresponding to the set-point. This is due to the fact that the cost function of the FHOCP penalizes only the output error and the variation of the control variable.

4. Conclusions

In this paper an MPC algorithm for nonlinear systems has been developed. Its main characteristics are related to

![Graphs and Figures]

Fig. 5. Closed-loop system responses with nonlinear MPC based on a pure regulation algorithm (dashed line) and on the tracking algorithm (solid line) when a change of $\mu_m$ from 0.48 h$^{-1}$ to 0.45 h$^{-1}$ occurs at time 10 h: (a)–(c) state variables, (d) and (e) input variables.
\[ V(q_{RH}^{(i)}(t_{k+i}, x_i(t_i)), t_{k+i}, y_i^0) \leq J_{\text{FF}}(q_{RH}^{(i)}(t_{k+i}, x_i(t_i), y_i^0), \tilde{u}_{\text{FF}}(t_{k+i}), N_c, N_p, y_i^0) = V(q_{RH}^{(i)}(t_{k+i}, x_i(t_i)), t_{k+i}, y_i^0) \]
\[ + \int_{t_{k+i}}^{t_{k+i+1}} \left\{ \| h(q_{RH}^{(i)}(t, x_i(t), y_i^0)) \|_Q^2 + \| \phi_{q_{RH}}^{(i)}(t, x_i(t), y_i^0) \|_{R}^2 + \| \phi_{x_{RH}}^{(i)}(t, x_i(t), y_i^0) \|_{R}^2 \right\} dt + V(h^0(x_{RH}^{(i)}(t, x_i(t), y_i^0)), t_{k+i+1}, y_i^0) \]
\[ \leq V(q_{RH}^{(i)}(t_{k+i}, x_i(t_i)), t_{k+i}, y_i^0) + c_{\text{max}} \int_{t_{k+i}}^{t_{k+i+1}} \left\{ \| \phi_{q_{RH}}^{(i)}(t, x_i(t), y_i^0) \|_{R}^2 + \| \phi_{x_{RH}}^{(i)}(t, x_i(t), y_i^0) \|_{R}^2 \right\} dt + T_{\text{max}}(R) \| \psi_{RH}^{(i)}(x_{RH}^{(i)}(t, x_i(t), y_i^0)) \| \leq V(q_{RH}^{(i)}(t_{k+i}, x_i(t_i)), t_{k+i}, y_i^0) \]
\[ + c_{\text{max}} \int_{t_{k+i}}^{t_{k+i+1}} \left\{ \| \phi_{q_{RH}}^{(i)}(t, x_i(t), y_i^0) \|_{R}^2 + \| \phi_{x_{RH}}^{(i)}(t, x_i(t), y_i^0) \|_{R}^2 \right\} dt + T_{\text{max}}(R) \| \psi_{RH}^{(i)}(x_{RH}^{(i)}(t, x_i(t), y_i^0)) \| \leq V(q_{RH}^{(i)}(t_{k+i}, x_i(t_i)), t_{k+i}, y_i^0) \]

where \( c_{\text{max}} \) is defined in the form of the FHOCP and
\[ \xi_{RH}^{(i)} := \phi(t_{k+i}, x_i(t_i), \tilde{u}_{\text{FF}}(t_{k+i}), t_{k+i}, y_i^0). \]

From Lemma 1
\[ V(q_{RH}^{(i)}(t_{k+i}, x_i(t_i), t_{k+i}, y_i^0) \leq V(q_{RH}^{(i)}(t_{k+i}, x_i(t_i), t_{k+i}, y_i^0) \]
\[ + \left\{ \| \phi_{q_{RH}}^{(i)}(t_{k+i}, x_i(t_i), y_i^0) \|_{R}^2 + \| \phi_{x_{RH}}^{(i)}(t_{k+i}, x_i(t_i), y_i^0) \|_{R}^2 \right\} dt + \left\{ \| \phi_{q_{RH}}^{(i)}(t_{k+i}, x_i(t_i), y_i^0) \|_{R}^2 + \| \phi_{x_{RH}}^{(i)}(t_{k+i}, x_i(t_i), y_i^0) \|_{R}^2 \right\} dt \]
\[ \leq V(q_{RH}^{(i)}(t_{k+i}, x_i(t_i), t_{k+i}, y_i^0) \]

and, since \( Q \) and \( R \) are positive definite matrices, if \( x_i(t_i) \in X^p(t, N_c, N_p, y_i^0), V(q_{RH}^{(i)}(t_{k+i}, x_i(t_i), t, y_i^0) \) and
\[ \int_{t_{k+i}}^{t_{k+i+1}} \left\{ \| h(q_{RH}^{(i)}(t_{k+i}, x_i(t_i), y_i^0)) \|_Q^2 + \| \phi_{q_{RH}}^{(i)}(t_{k+i}, x_i(t_i), y_i^0) \|_{R}^2 + \| \phi_{x_{RH}}^{(i)}(t_{k+i}, x_i(t_i), y_i^0) \|_{R}^2 \right\} dt \]
are bounded. Moreover \( h(q_{RH}^{(i)}(t_{k+i}, x_i(t_i), y_i^0)) \) is bounded and, from the properties of \( f \) and \( h \),
\[ \lim_{t \to \infty} \phi_{q_{RH}}^{(i)}(t_{k+i}, x_i(t_i), y_i^0) = 0 \]
\[ \lim_{t \to \infty} \phi_{x_{RH}}^{(i)}(t_{k+i}, x_i(t_i), y_i^0) = 0 \]
\[ \lim_{t \to \infty} \phi_{RH}^{(i)}(t_{k+i}, x_i(t_i), y_i^0) = 0 \]

References