Nonlinear systems

Course introduction

G. Ferrari Trecate

Dipartimento di Ingegneria Industriale e dell'Informazione
Università degli Studi di Pavia

Advanced automation and control

Course schedule

Advanced automation and control

Industrial automation + Nonlinear systems

Lectures

- Monday 14-16 room EF3, Thursday 16-18, room E1 (Industrial automation)
- Wednesday 14-16, room E1 (Nonlinear systems)

Office hours

- By appointment (`giancarlo.ferrari@unipv.it`). Office: Dipartimento di Ingegneria Industriale e dell'Informazione, floor F

Course website

http://sisdin.unipv.it/labsisdin/teaching/courses/ails/files/ails.php

- a copy of the slides can be downloaded after authentication with login/password

Textbooks

For a review of basic systems theory and automatic control

- G. F. Franklin, J. D. Powell, A. Emami-Naeini. *Feedback Control of Dynamic Systems* 6th ed., 2009 Prentice Hall
- P. Bolzern, R. Scattolini, N. Schiavoni. *Fondamenti di Controlli Automatici*, 2nd ed., 2004, McGraw-Hill, Italia

For the topics in nonlinear systems covered in the course

- J.-J. E. Slotine e W. Li. *Applied nonlinear control*. Prentice-Hall (1991)
- H.K. Khalil. *Nonlinear systems - third edition*. Prentice-Hall (2002)
- S. Sastry. *Nonlinear systems - Analysis, Stability and Control*. Springer-Verlag (1999) (and C. Tomlin - slides of the course "Advanced Nonlinear Control", Stanford University)

- All above books cover several topics that will be not discussed in the course. Khalil an Sastry's books are the most advanced (and difficult) ones
- The exam will focus only on topics covered in the course

Exams

Closed-books closed-notes written exam split in two parts

- First part: industrial automation
- Second part: nonlinear systems

Total duration: $\sim$ 3h. No graphic or programmable calculators are allowed

Registration to exams

Through the university website

Usually, registrations end 7 days before the exam date

Nonlinear (NL) systems

Analysis vs. simulation

- Steadily increasing computing power allows one to simulate complex NL systems
- Simulation and intuition allow one to understand several aspects of NL systems

However,

- Impossible to use only simulation to prove interesting properties (e.g. stability)
- Analysis procedures allow properties of NL systems to be rigorously assessed
  - Sometimes, results are surprising and highlight behaviors one had not tought to simulate !

Nonlinear (NL) systems

NL systems vs. linear systems

Several results on the analysis and control of linear systems
HOWEVER

- Most real systems are NL
- Linear systems do not capture behaviors such as
  - isolated multiple equilibria
  - limit cycles
  - subharmonics
  - complex dynamics, e.g. chaos

Next ...

- Review of systems theory !
- Examples of nonlinear behaviors

Review

NL system

$$\dot{x}(t) = f(x(t), u(t), t) \qquad (1) \qquad\qquad x(t) \in \mathbb{R}^n \text{ state}$$

$$y(t) = g(x(t), u(t), t) \qquad (2) \qquad\qquad u(t) \in \mathbb{R}^m \text{ input}$$

$$x(t_0) = x_0 \qquad (3) \qquad\qquad y(t) \in \mathbb{R}^p \text{ output}$$

- (1): state equation
- (2): output equation
- n: system order

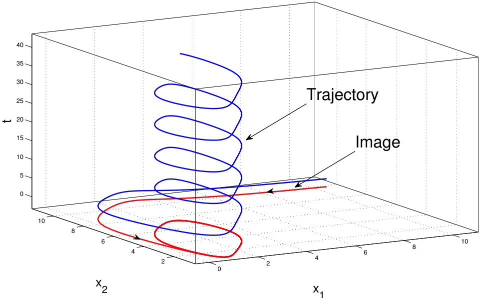
Definition

A state trajectory is a function $x(t)$ verifying (1) and (3). For highlighting the dependence on the input, initial time and initial states, we write $x(t) = \phi(t, t_0, x_0, u)$ and ϕ is called *transition map*

Drawing state trajectories

Often one draws the image of the trajectory $\phi(t, t_0, x_0, u)$, i.e. the set of points

$$\{\phi(t, t_0, x_0, u), t \geq t_0\} \subset \mathbb{R}^n$$



Review

NL system

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t), t) \\ y(t) &= g(x(t), u(t), t) \\ x(t_0) &= x_0 \end{aligned} \qquad \begin{aligned} x(t) &\in \mathbb{R}^n \\ u(t) &\in \mathbb{R}^m \\ y(t) &\in \mathbb{R}^p \end{aligned}$$

An NL system is:

- Invariant if f and g do not depend upon time
  - Without loss of generality, one can set $t_0 = 0$ and $\phi(t, t_0, x_0, u) = \phi(t, x_0, u)$
- Autonomous if the system does not depend upon the input $u(t)$
  - $\phi(t, t_0, x_0, u) = \phi(t, t_0, x_0)$
- Invariant and autonomous: $\dot{x}(t) = f(x(t)),\ y(t) = g(x(t))$
  - $\phi(t, t_0, x_0, u) = \phi(t, x_0)$
- Static if $n = 0$
  - described only through the output equation $y(t) = g(u(t), t)$

Review of linear systems

A system is linear if f and g are linear functions of x and u

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t) + D(t)u(t)$$

$A(t),\ B(t),\ C(t),\ D(t)$ matrices

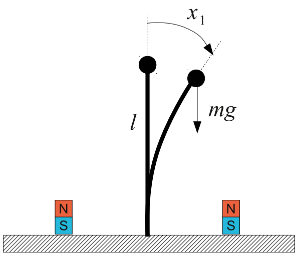
Linear Time-Invariant (LTI) system

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

$A,\ B,\ C,\ D$ matrices

Multiple isolated equilibria

NL vs. linear systems: Duffing oscillator



Model

$$ml^2\ddot{x}_1 = mgl\sin(x_1) - \alpha x_1 - k\dot{x}_1 + \tau$$

- $-\alpha x_1$: restoring torque ($\alpha > 0$)
- $-k\dot{x}_1$: damping torque ($k > 0$)
- τ: electromagnetic torque (input)

NL system

Defining $x_2 = \dot{x}_1$, $u = \dfrac{\tau}{ml^2}$,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{g}{l}\sin(x_1) - \frac{\alpha}{ml^2}x_1 - \frac{k}{ml^2}x_2 + u$$

Review

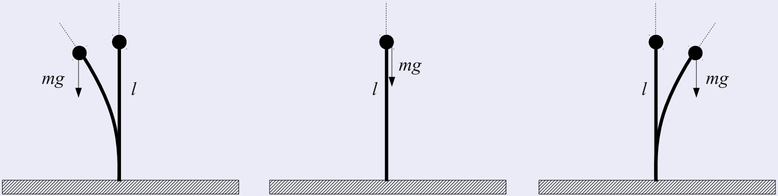
Equilibrium

Given $u(t) = \bar{u}$, $\forall t \geq 0$, the state $\bar{x} \in \mathbb{R}^n$ is an equilibrium state for the nonlinear time-invariant system $\dot{x} = f(x, u)$ if it verifies $f(\bar{x}, \bar{u}) = 0$[a]. The pair $(\bar{x}, \bar{u})$ is called an equilibrium.

[a] $\dot{x} = x^2 + 1$, $x(t) \in \mathbb{R}$ has no equilibrium state

Duffing oscillator: equilibra for $\bar{u} = 0$

Physical intuition: 3 equilibra



Duffing oscillator: equilibria of approximate models

NL system

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{g}{l}\sin(x_1) - \frac{\alpha}{ml^2}x_1 - \frac{k}{ml^2}x_2 + u$$

Linear approximation: $\sin(x_1) \simeq x_1$

LTI system $(u = 0)$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \left(\frac{g}{l} - \frac{\alpha}{ml^2}\right)x_1 - \frac{k}{ml^2}x_2$$

Equilibrium states:

$$\bar{x}_2 = 0$$

$$\left(\frac{g}{l} - \frac{\alpha}{ml^2}\right) \neq 0 \Rightarrow \bar{x}_1 = 0$$

Either one or infinite equilibrium states

Duffing oscillator: equilibria of approximate models

NL system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{g}{l}\sin(x_1) - \frac{\alpha}{ml^2}x_1 - \frac{k}{ml^2}x_2 + u$$

Approximation: $\sin(x_1) \simeq x_1 - x_1^3/6$

Approximated NL system ($u = 0$)

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \left(\frac{g}{l} - \frac{\alpha}{ml^2}\right)x_1 - \frac{g}{6l}x_1^3 - \frac{k}{ml^2}x_2$$

Equilibrium states:

$$\bar{x}_2 = 0$$
$$\left(\frac{g}{l} - \frac{\alpha}{ml^2}\right)\bar{x}_1 - \frac{g}{6l}\bar{x}_1^3 = 0$$

One can have 3 equilibrium states

Duffing oscillator: equilibria of the approximate NL model

If we set $\frac{g}{l} - \frac{\alpha}{ml^2} = 1$, $\frac{g}{6l} = 1$, $\frac{k}{ml^2} = \eta$, we get

Automonous Duffing model:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_1 - x_1^3 - \eta x_2$$

Equilibrium states:

$$p_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix},\ p_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ p_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

Linear approximations around an equilibrium

Review: linearization around an equilibrium

Let $(\bar{x}, \bar{u})$ be an equilibrium for the NL invariant system

$$\dot{x} = f(x, u)$$
$$y = g(x, u)$$

Deviations: $\delta x(t) = x(t) - \bar{x}, \; \delta u(t) = u(t) - \bar{u}, \; \delta y(t) = y(t) - \bar{y}$

First order Taylor expansion about the equilibrium:

$$f(x,u) \simeq f(\bar{x},\bar{u}) + D_x f(x,u)\Big|_{\substack{x=\bar{x}\\ u=\bar{u}}}(x-\bar{x}) + D_u f(x,u)\Big|_{\substack{x=\bar{x}\\ u=\bar{u}}}(u-\bar{u})$$
$$g(x,u) \simeq g(\bar{x},\bar{u}) + D_x g(x,u)\Big|_{\substack{x=\bar{x}\\ u=\bar{u}}}(x-\bar{x}) + D_u g(x,u)\Big|_{\substack{x=\bar{x}\\ u=\bar{u}}}(u-\bar{u})$$

$$D_x f(x,u) = \begin{bmatrix} \frac{\partial f_1(x,u)}{\partial x_1} & \cdots & \frac{\partial f_1(x,u)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n(x,u)}{\partial x_1} & \cdots & \frac{\partial f_n(x,u)}{\partial x_n} \end{bmatrix}$$ Jacobian with respect to the variables x

Review: linearization around an equilibrium

One gets:

$$\dot{\delta x} = \dot{x} - \dot{\bar{x}} = f(x,u) \simeq \underbrace{f(\bar{x},\bar{u})}_{=0} + D_x f(x,u)\Big|_{\substack{x=\bar{x}\\u=\bar{u}}} \delta x + D_u f(x,u)\Big|_{\substack{x=\bar{x}\\u=\bar{u}}} \delta u$$

$$\delta y = -\bar{y} + y \simeq \underbrace{-g(\bar{x},\bar{u}) + g(\bar{x},\bar{u})}_{=0} + D_x g(x,u)\Big|_{\substack{x=\bar{x}\\u=\bar{u}}} \delta x + D_u g(x,u)\Big|_{\substack{x=\bar{x}\\u=\bar{u}}} \delta u$$

Linearized system

Defining

$$A = D_x f(x,u)\Big|_{\substack{x=\bar{x}\\u=\bar{u}}},\ B = D_u f(x,u)\Big|_{\substack{x=\bar{x}\\u=\bar{u}}},\ C = D_x g(x,u)\Big|_{\substack{x=\bar{x}\\u=\bar{u}}},\ D = D_u g(x,u)\Big|_{\substack{x=\bar{x}\\u=\bar{u}}}$$

the linearized system around the equilibrium $(\bar{x},\bar{u})$ is

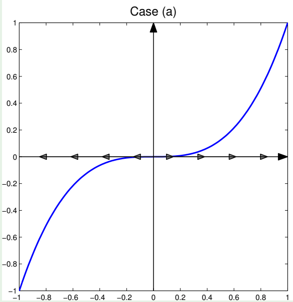
$$\dot{\delta x} = A\delta x + B\delta u$$
$$\delta y = C\delta x + D\delta u$$

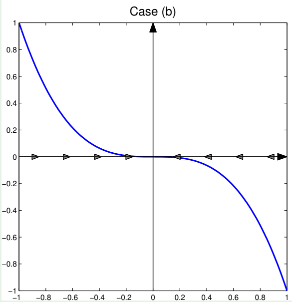
Review: linearization around an equilibrium

We hope state trajectories of the linearized system are good approximations of $x(t) - \bar{x}$ ... but this does not always happen

Example: (a): $\dot{x} = x^3$, (b): $\dot{x} = -x^3$

Linearized systems around $\bar{x} = 0$ are the same: $\dot{\delta x} = 0 \Rightarrow \delta x(t) = x_0$ but NL systems have different behaviors



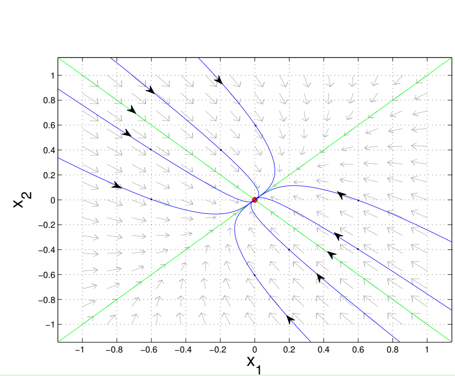


Phase plane

For second-order systems it is possible to graphically study the projection of trajectories in the plane (x_1, x_2) that is called phase plane

Example

$$\dot{x}_1 = -x_1 + 0.5x_2$$
$$\dot{x}_2 = 0.5x_1 - x_2$$



Some types of equilibria for second-order LTI systems

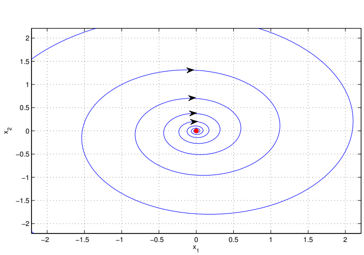
Autonomous LTI system

$$\dot{x} = Ax$$

- The origin $\bar{x} = 0$ is always an equilibrium state
- When $x(t) \in \mathbb{R}^2$ one can classify the behavior of state trajectories using the eigenvalues λ_1, λ_2 of the matrix A.

Complex eigenvalues with real part < 0

The origin is a stable focus



Some types of equilibria for second-order LTI systems

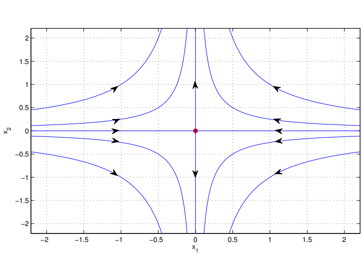
Autonomous LTI system

$$\dot{x} = Ax$$

- The origin $\bar{x} = 0$ is always an equilibrium state
- When $x(t) \in \mathbb{R}^2$ one can classify the behavior of state trajectories using the eigenvalues λ_1, λ_2 of the matrix A.

Real eigenvalues and $\lambda_1 < 0 < \lambda_2$

The origin is a saddle



Equilibria of second-order NL systems

Idea: analyze the behavior of state trajectories around an equilibrium state using the linearized system

Example: Duffing model for $\eta = 1$

NL system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_1 - x_1^3 - \eta x_2, \;\; \eta = 1$$

Linearized system

$$\dot{\delta x}_1 = \delta x_2$$
$$\dot{\delta x}_2 = \delta x_1 - 3\bar{x}_1^2 \delta x_1 - \delta x_2$$

Around $p_1 = \begin{bmatrix} -1 & 0 \end{bmatrix}^{\mathrm{T}}$ and $p_3 = \begin{bmatrix} 1 & 0 \end{bmatrix}^{\mathrm{T}}$

$$D_x f = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix} \Rightarrow \text{ Eigenvalues: } -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$$

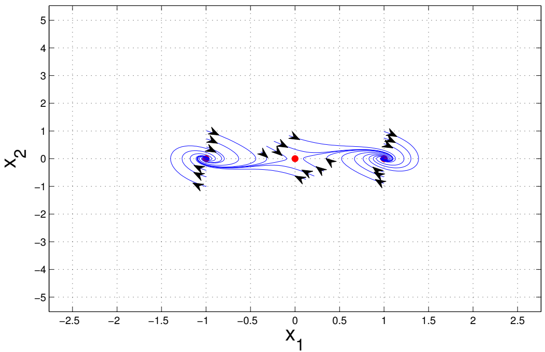
Can we conlude that p_1 and p_3 are stable foci ?

Equilibria of second-order NL systems

Around $p_2 = \begin{bmatrix} 0 & 0 \end{bmatrix}^{\mathrm{T}}$

$$D_x f = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow \text{ Eigenvalues: } -1 \pm \frac{\sqrt{5}}{2}$$

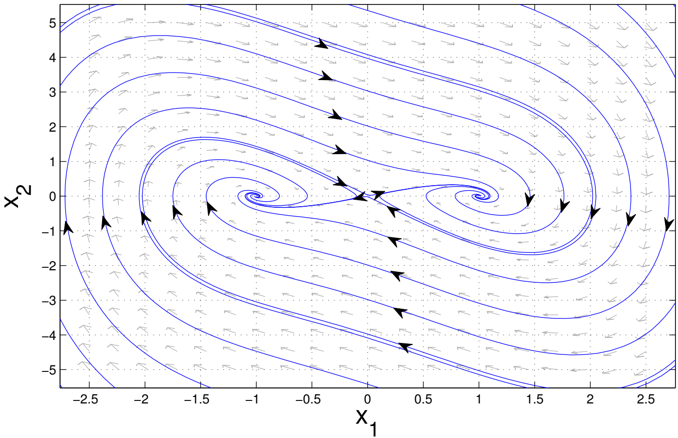
Can we conlude that p_2 is a saddle ?



From the state trajectories it seems the answer is yes...
The analysis is local.

Equilibria of second-order NL systems

Duffing model: global behavior

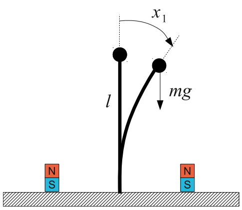


Plots obtained with the MatLab program *pplane*

http://math.rice.edu/~dfield/

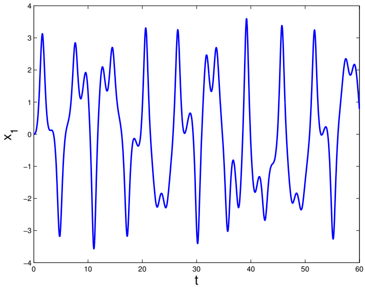
Subharmonics

Subharmonics



Duffing model with input

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_1 - x_1^3 - \eta x_2 + u$$
$$\eta = 0.025, \;\; u(t) = 7.5\sin(t)$$



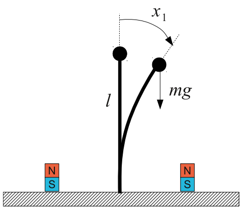
Subharmonics

Harmonics that are NOT present in the input appear in the output (even in the asymptotic régime)

- Impossible for asymptotically stable LTI systems (because of the frequency response theorem)

Chaos

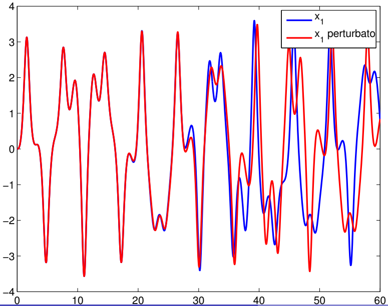
Chaos



Duffing model with input

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_1 - x_1^3 - \eta x_2 + u$$
$$\eta = 0.025, \quad u(t) = 7.5\sin(t)$$

State trajectory x_1 when $x(0) = 0$ and $x(0) = 0 + \epsilon$



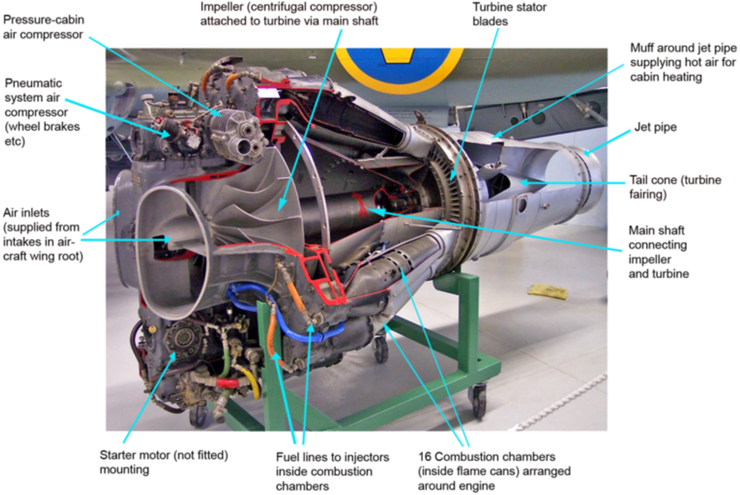
Chaos

- Huge sensitivity to initial states.
- Simulations might be meaningless

Limit cycles

Surge and rotating stall in jet engine compressors

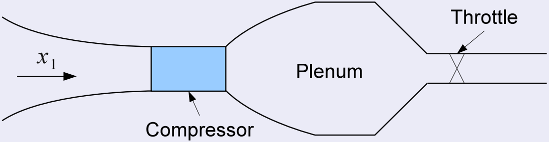
Engine “de Havilland Goblin II”[1]



[1] Picture from Wikipedia

Surge and rotating stall in jet engine compressors

Jet engine



NL model (dimensionless units)

$$\dot{x}_1 = B\,(C(x_1) - x_2)$$
$$\dot{x}_2 = \frac{1}{B}(x_1 - F_\alpha^{-1}(x_2))$$

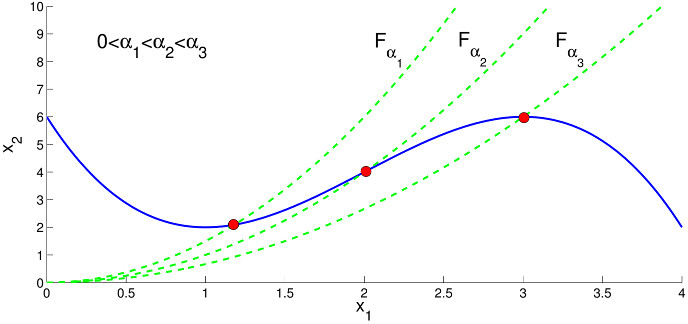
- $B > 0$ compressor angular speed (rotor)
- x_1: compressor mass flow
- x_2: plenum pressure rise
- α: throttle angle
- $C(\cdot)$: compressor charachteristic
- $F_\alpha(\cdot)$: throttle charachteristic

Analysis of equilibria

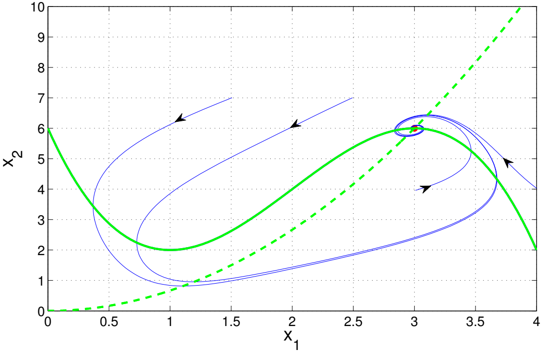
Computation of equilibria

$$\begin{cases} 0 = B\left(C(\bar{x}_1) - \bar{x}_2\right) \\ 0 = \dfrac{1}{B}\left(\bar{x}_1 - F_\alpha^{-1}(\bar{x}_2)\right) \end{cases} \Rightarrow \begin{cases} \bar{x}_2 = C(\bar{x}_1) \\ \bar{x}_1 = F_\alpha^{-1}(\bar{x}_2) \end{cases} \Rightarrow F_\alpha(\bar{x}_1) = C(\bar{x}_1)$$

Equilibria for various throttle angle

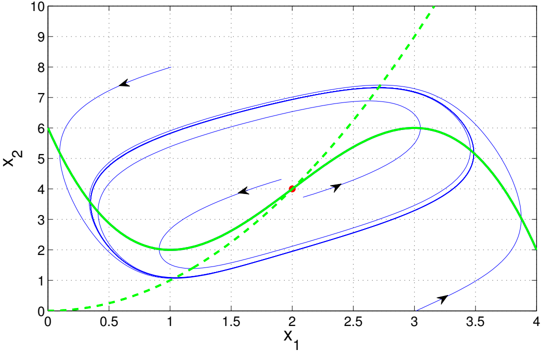


Unstalled operating point



This is the desired behavior: stable equilibrium

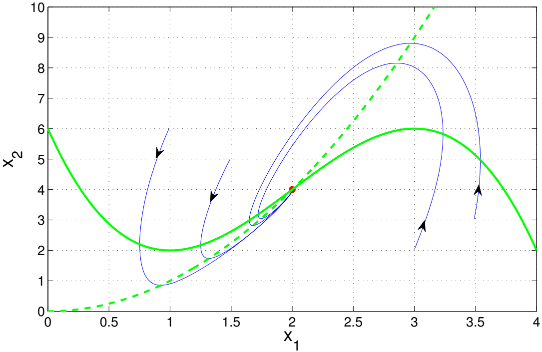
Surge



Perturbation of the throttle charachteristic

- Unstable equilibrium and stable limit cycle
- Surge ⇒ dangerous pressure waves

Rotating stall



Perturbation of the throttle charachteristic + decrease of the angular speed B of the compressor

- Stable equilibrium but insufficient pressure ⇒ rotating stall !