Network models and graph theory

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Industrial Automation

Outline

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Optimization on networks

Methods for solving decision problems where the unknowns can only take finitely many values

Focus on problems that can be represented as a graph or a network such as

- logistic or transport problems
- network design problems
- project management problems

- Several optimization problems on networks are computationally *very demanding*
- However, there are interesting industrial problems that can be solved in an efficient fashion

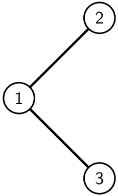
Outline

Basic definitions

Undirected graph

A graph is *undirected* if $G = (V, E)$ is a pair comprising a finite set of *vertices (or nodes)* $V = \{1, 2, \ldots, n\}$ and a set of $E \subset V \times V$ *unordered* pairs called *edges (or arcs)*.

Example: $V = \{1, 2, 3\}$, $E = \{(1, 2), (1, 3)\}$



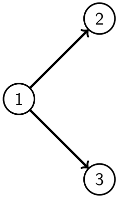
Undirected edges : $(1, 2) = (2, 1)$, $(1, 3) = (3, 1)$

Basic definitions

Directed graph

A graph G is *directed* (or digraph) if all pairs in E are *ordered* .

Example:: $V = \{1, 2, 3\}$, $E = \{(1, 2), (1, 3)\}$



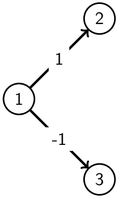
Ordered edges: $(1, 2) \neq (2, 1)$, $(1, 3) \neq (3, 1)$

Basic definitions

Network

If a function $c : E \to \mathbb{R}$ is specified, a graph is called *weighted* or *network*. If the graph is directed, one has a directed network.

Example of directed network: $V = \{1, 2, 3\}$, $E = \{(1, 2), (1, 3)\}$
$c(1, 2) = 1$ e $c(1, 3) = -1$

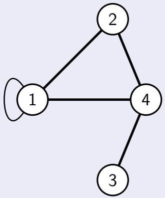


Basic definitions

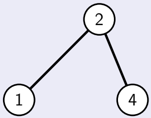
Subgraph

The graph $H = (U, F)$ is a *subgraph* of $G = (V, E)$ if $U \subseteq V$, $F \subseteq E$ and edges in F connect only vertices in U.

Graph $G = (V, E)$



Graph $H = (U, F)$



$H = (U, F)$ is a subgraph of G because $U = \{1, 2, 4\} \subseteq V$,
$F = \{(1, 2), (2, 4)\} \subseteq E$ and edges in F connects only vertices in U.

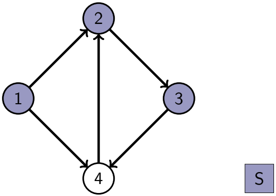
Cuts

Directed cuts

Let $G = (V, E)$ be a digraph and $S \subseteq V$. The *directed cuts* associated to S are the sets of edges

$$\delta^+(S) = \{(i,j) \in E : i \in S,\ j \notin S\} \qquad \text{(edges leaving } S\text{)}$$
$$\delta^-(S) = \{(i,j) \in E : i \notin S,\ j \in S\} \qquad \text{(edges entering in } S\text{)}$$

For an undirected graph $\delta^+(S) = \delta^-(S)$.



$$\delta^+(\{1,2,3\}) = \{(1,4),(3,4)\}$$
$$\delta^-(\{1,2,3\}) = \{(4,2)\}$$

$$\delta^-(\{3,4\}) = \{(1,4),(2,3)\}$$

Graph connectivity

Path

A sequence of arcs $e_1 e_2 \cdots e_k$ such that

$$e_1 = (v_1, v_2), e_2 = (v_2, v_3), \ldots, e_k = (v_k, v_{k+1})$$

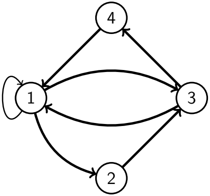
is a *path* from v_1 to v_{k+1}.

Notation: $v_1 v_2 \cdots v_{k+1}$

Path classification

- A path from a vertex to itself is a *cycle*.
- A path is *elementary* if it does not contain the same *edge* twice.
- A path is *simple* if it does not not pass through the same *vertex* twice (with the exception of the starting vertex for a cycle).

Graph connectivity



- The path 12313 is elementary but not simple.
- The path 1234 is simple and elementary.
- The paths 1231 and 11 are cycles.

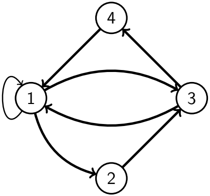
Remarks

- All simple paths are also elementary (if the same vertex is not crossed twice, the path cannot contain the same edge twice).

Graph connectivity

Hamiltonian paths

A path is *Hamiltonian* if it simple and contains all vertices (except the starting vertex for a cycle).



The path 1234 is Hamiltonian. The path 11234 is not Hamiltonian. The cycle 12341 is Hamiltonian.

Remark

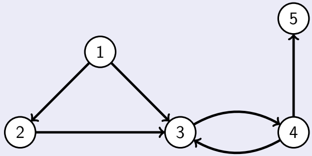
There is no simple algorithm for checking if a graph contains a Hamiltonian cycle ...

Graph connectivity

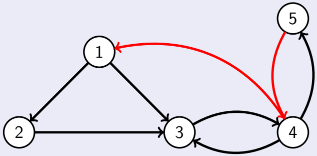
Connectivity and completeness

A vertex v_2 is *connected* to v_1 if there is a path from v_1 to v_2.
- A graph is *connected* if all pairs of vertices are connected.
- A graph $G = (V, E)$ is *complete* if $E = V \times V$

Disconnected graph



Connected graph

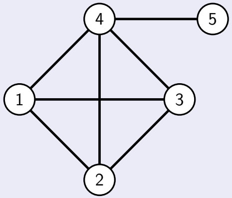


Graph connectivity

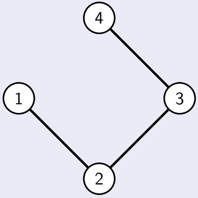
Tree

If $G = (V, E)$ is undirected, a connected subgraph with k vertices and $k - 1$ edges is a *tree*. A tree is *spanning* if $k = n$, with $n = |V|$.

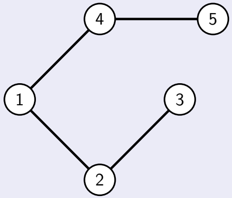
Graph



Tree



Spanning tree

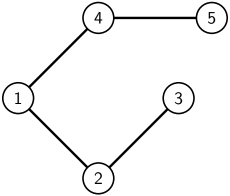


Graph connectivity

Tree theorem

Let T be an undirected graph. The following conditions are equivalent:

- T is a tree
- T is a connected acyclic graph
- T is acyclic and the addition of any arc produces a simple cycle
- T is connected and the deletion of any arc makes T disconnected
- Every pair of vertices of T is connected by a unique simple path



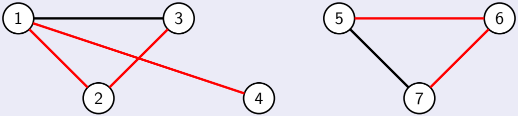
Graph connectivity

Forest

A *forest* is an undirected acyclic graph. A subgraph of G is a *maximal* forest if the addition of any edge produces a cycle.

Example

Red edges define a maximal forest



Remarks

A forest is always the union of disjoint trees. A spanning tree is a maximal forest.

Optimization on networks

Some interesting problems

Problem A Is an undirected graph connected ?

Problem B Given a digraph and two nodes v_1 and v_2, check if v_2 is connected to v_1.

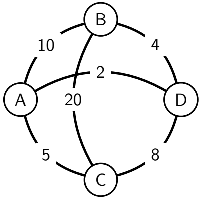
Problem C Does an undirected graph contain a Hamiltonian cycle ?

Problem TSP (Travelling Salesman Problem) Given an undirected complete network and a number $r \in \mathbb{R}$ check if it contains a Hamiltonian cycle of cost less than r.

Problem TSP

- Vertices = towns
- Weights = distances (in Km)

The traveling salesman must visit all town and be back to the first one covering less than r Km



Optimization on networks

Enumeration algorithm

All problems can be solved by computing all paths of length less than (or equal to) $|V|$ contained in the graph
Highly inefficient method: the number of paths explodes with the number of vertices

Which is the computational time needed for solving problems A, B, C and TSP ? It depends upon the adopted algorithm ...
- A rigorous answer is provided by the computational complexity theory

Computational complexity

- Quantify the efficiency of a given algorithms
- Quantify the intrinsic difficulty of a problem