Nonlinear systems

Phase plane analysis

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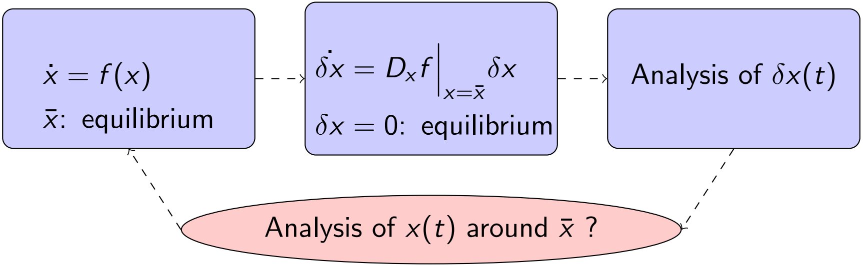
Advanced automation and control

Phase plan analysis

Problem

When $x(t) \in \mathbb{R}^2$, study state trajectories around an equilibrium state



Review: stability of an equilibrium state

Let $\bar{x}$ be an equilibrium state for the NL invariant system $\dot{x} = f(x)$

Ball centered in $\bar{z} \in \mathbb{R}^n$ of radius $\delta > 0$

$B_\delta(\bar{z}) = \{z \in \mathbb{R}^n : \|z - \bar{z}\| < \delta\}$

Definition (Lyapunov stability)

The equilibrium state $\bar{x}$ is

- stable if

$$\forall \epsilon > 0 \ \exists \delta > 0, \ x(0) \in B_\delta(\bar{x}) \Rightarrow x(t) \in B_\epsilon(\bar{x}), \forall t \geq 0$$

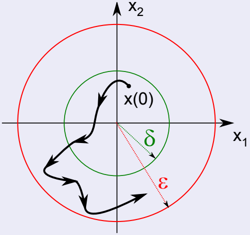
- Asymptotically Stable (AS) if it is stable and $\exists \gamma > 0$ such that

$$x(0) \in B_\gamma(\bar{x}) \Rightarrow \lim_{t \to +\infty} \|\phi(t, x(0)) - \bar{x}\| = 0$$

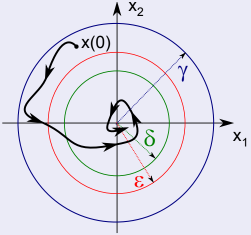
- unstable if it is not stable

Remarks

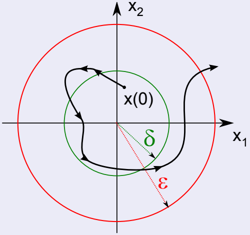
$\bar{x} = 0$ stable



$\bar{x} = 0$ AS



$\bar{x} = 0$ unstable



Regions of attraction of $\bar{x}$ AS

- $X \subseteq \mathbb{R}^n$ is a region of attraction of $\bar{x}$ if

$$x(0) \in X \Rightarrow \lim_{t \to +\infty} \|\phi(t, x(0)) - \bar{x}\| = 0$$

  Example: $B_\gamma(\bar{x})$ is a region of attraction

- THE region of attraction of $\bar{x}$ is the union of all regions of attraction of $\bar{x}$ (i.e. it is maximal)

Review: stability tests for LTI systems

LTI system

$$\dot{x} = Ax, \quad x(t) \in \mathbb{R}^n$$

System eigenvalues = eigenvalues of the matrix A

Theorem

The equilibrium state $\bar{x} = 0$ of a linear system is

- AS $\Leftrightarrow$ all system eigenvalues have real part < 0
- unstable if at least a system eigenvalue has real part > 0
- stable if all system eigenvalues have real part ≤ 0, at least one has zero real part and all eigenvalues with zero real part are simple

When all eigenvalues have real part ≤ 0 and there are multiple eigenvalues with zero real part, the equilibrium state can be either stable or unstable and more advanced tools are needed for reaching a conclusion.

Review: stability test for the equilibrium states of an NL system

NL system

$$\text{NL} : \dot{x} = f(x)$$

$\bar{x}$: equilibrium state

Linearized system around $\bar{x}$

$$\text{LIN} : \dot{\delta x} = A(\bar{x})\delta x$$

$$A(\bar{x}) = D_x f(x)\Big|_{x=\bar{x}}$$

Theorem

The equilibrium state $\bar{x}$ of NL

- is AS if all eigenvalues of LIN have real part < 0
- is unstable if at least an eigenvalue of LIN has real part > 0

No conclusion if all eigenvalues of LIN have real part ≤ 0 and at least an eigenvalue has zero real part

Invariant regions

Definition

A set $G \subseteq \mathbb{R}^n$ is (positively) invariant for $\dot{x} = f(x)$ if

$$x(0) \in G \Rightarrow \phi(t, x(0)) \in G, \ \forall t \geq 0$$

Examples

- $G = \{\bar{x}\}$, $\bar{x}$ equilibrium state
- $G = \mathbb{R}^n$

Review: equivalent LTI systems

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

- Change of coordinates $\hat{x}(t) = Tx(t)$, $T \in \mathbb{R}^{n\times n}$ invertible.

$$\begin{aligned}
\dot{\hat{x}}(t) &= T\dot{x}(t) = T(Ax(t) + Bu(t)) = T(AT^{-1}\hat{x}(t) + Bu(t)) \\
&= TAT^{-1}\hat{x}(t) + TBu(t) = \hat{A}\hat{x}(t) + \hat{B}u(t) \\
&\quad \hat{A} = TAT^{-1}, \quad \hat{B} = TB \\
y(t) &= Cx(t) + Du(t) = CT^{-1}\hat{x}(t) + Du(t) = \hat{C}\hat{x}(t) + \hat{D}u(t) \\
&\quad \hat{C} = CT^{-1}, \quad \hat{D} = D
\end{aligned}$$

Review: equivalent LTI systems

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u$$
$$y = \hat{C}\hat{x} + \hat{D}u$$

Definition

The system $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ is *equivalent* to the system (A, B, C, D) in the sense that for an input $u(t)$, $t \geq 0$ and two initial states x_0 e $\hat{x}_0$ verifying $\hat{x}_0 = Tx_0$, the state trajectories verify $\hat{x}(t) = Tx(t)$, $t \geq 0$, and outputs are identical

Remark

A and $\hat{A}$ are similar $\Rightarrow$ they have the same eigenvalues

LTI systems in the phase plane

$$\dot{x} = Ax, \quad x \in \mathbb{R}^2$$

Change of coordinates: $\hat{x}(t) = Tx(t)$, $T \in \mathbb{R}^{2\times 2}$ invertible. Equivalent system:

$$\dot{\hat{x}} = J\hat{x}, \quad J = TAT^{-1}$$

- One can always choose T such that J is in real Jordan form
  - the new coordinates are called normal

Case 1: A has real eigenvalues $\lambda_1, \ \lambda_2$ and independent eigenvectors (A is diagonalizable)

$$J = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

Phase plan: analysis in normal coordinates

Case 2: A has two real, identical eigenvalues $\lambda_1 = \lambda_2 = \lambda$ and linearly dependent eigenvectors (A is not diagonalizable)

$$J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

Define $V_\lambda = \{v : Av = \lambda v\}$. This case happens ony if $\dim(V_\lambda) = 1$.

Case 3: A has complex conjugate eigenvalues $\lambda_1 = \alpha + j\beta$ $\lambda_2 = \alpha - j\beta$

$$J = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

How T is computed ?

Case 1: $Av_1 = \lambda_1 v_1$, $Av_2 = \lambda_2 v_2 \Rightarrow T^{-1} = \left[\ v_1 \mid v_2\ \right]$

Case 2: $Av = \lambda v$. Compute a generalized eigenvector u verifying $Au = \lambda u + v$. One has

$$A\left[\ v \mid u\ \right] = \left[\ Av \mid Au\ \right] = \left[\ v \mid u\ \right] \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \Rightarrow T^{-1} = \left[\ v \mid u\ \right]$$

Case 3: Let $v_1 = u + jv$, $v_2 = u - jv$ be the eigenvectors associated to the eigenvalues $\lambda_1 = \alpha + j\beta$, $\lambda_2 = \alpha - j\beta$. One has

$$A(u + jv) = (\alpha + j\beta)(u + jv) \quad A(u - jv) = (\alpha - j\beta)(u - jv)$$

Summing and subtracting:

$$Au = \alpha u - \beta v \quad Av = \beta u + \alpha v$$

$$A\left[\ u \mid v\ \right] = \left[\ u \mid v\ \right] \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \Rightarrow T^{-1} = \left[\ u \mid v\ \right]$$

Next

Taxonomy of equilibria

The goal is to study the qualitative behavior of the state trajectories of an LTI system in the phase plane around the equilibrium state $\bar{x} = 0$

- the behavior depends on system eigenvalues
- we use normal coordinates to ease the analysis

Analysis in normal coordinates

Case 1: $J = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad \lambda_1, \lambda_2 \in \mathbb{R}$

$$\dot{\hat{x}}_1 = \lambda_1 \hat{x}_1 \to \hat{x}_1(t) = \hat{x}_1(0) e^{\lambda_1 t}$$
$$\dot{\hat{x}}_2 = \lambda_2 \hat{x}_2 \to \hat{x}_2(t) = \hat{x}_2(0) e^{\lambda_2 t}$$

"Remove" time from the equations. If $\lambda_1 \neq 0$, $\lambda_2 \neq 0$ and $\hat{x}_1(0) \neq 0$ one gets

$$\hat{x}_2(t)^{\frac{\lambda_1}{\lambda_2}} = \hat{x}_2(0)^{\frac{\lambda_1}{\lambda_2}} e^{\frac{\lambda_1 \cancel{\lambda_2}}{\cancel{\lambda_2}} t} = \hat{x}_2(0)^{\frac{\lambda_1}{\lambda_2}} e^{\lambda_1 t} = \frac{\hat{x}_2(0)^{\frac{\lambda_1}{\lambda_2}}}{\hat{x}_1(0)} \hat{x}_1(t)$$

and then

$$\hat{x}_2(t) = \frac{\hat{x}_2(0)}{\hat{x}_1(0)^{\frac{\lambda_2}{\lambda_1}}} \hat{x}_1(t)^{\frac{\lambda_2}{\lambda_1}}$$

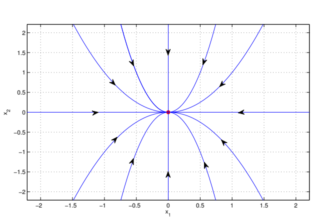
Stable node

$$\hat{x}_2(t) = \frac{\hat{x}_2(0)}{\hat{x}_1(0)^{\frac{\lambda_2}{\lambda_1}}} \hat{x}_1(t)^{\frac{\lambda_2}{\lambda_1}}, \quad \lambda_1 \neq 0,\ \lambda_2 \neq 0$$

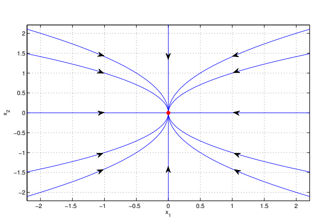
Case 1a: $\lambda_1, \lambda_2 < 0$

The origin is called stable node

$\frac{\lambda_2}{\lambda_1} > 1$



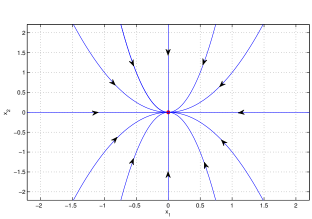
$\frac{\lambda_2}{\lambda_1} < 1$



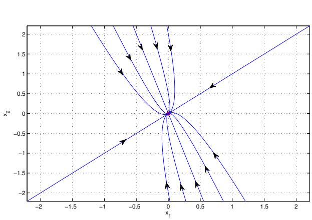
Both axes are invariant sets

Role played by the change of coordinates

In normal coordinates



In the original coordinates



Key remark

Same qualitative behavior, up to a coordinate change.

- The origin of the system in the original coordinates is also termed stable node
- Same remark for all the cases we will study in the sequel !

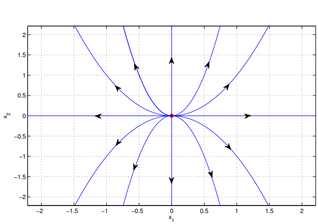
Unstable node

$$\hat{x}_2(t) = \frac{\hat{x}_2(0)}{\hat{x}_1(0)^{\frac{\lambda_2}{\lambda_1}}} \hat{x}_1(t)^{\frac{\lambda_2}{\lambda_1}}, \quad \lambda_1 \neq 0,\ \lambda_2 \neq 0$$

Case 1b: $\lambda_1, \lambda_2 > 0$

The origin is called unstable node

Example: $\frac{\lambda_2}{\lambda_1} > 1$



Both axes are invariant sets

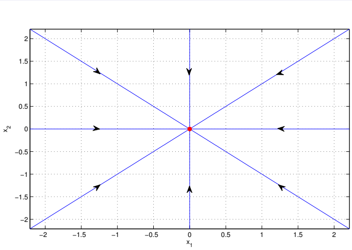
Degenerate node

$$\hat{x}_2(t) = \frac{\hat{x}_2(0)}{\hat{x}_1(0)^{\frac{\lambda_2}{\lambda_1}}}\hat{x}_1(t)^{\frac{\lambda_2}{\lambda_1}}, \quad \lambda_1 \neq 0,\ \lambda_2 \neq 0$$

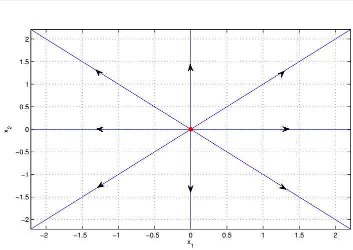
Case 1c: $\lambda_1 = \lambda_2$

The origin is called stable/unstable degenerate node

Stable degenerate node
$\lambda_1 = \lambda_2 < 0$



Unstable degenerate node
$\lambda_1 = \lambda_2 > 0$

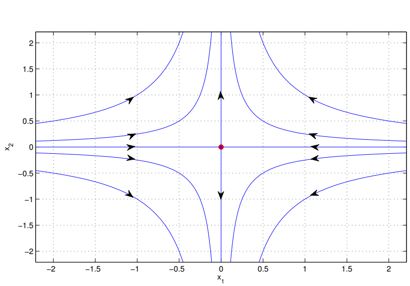


Saddle

$$\hat{x}_2(t) = \frac{\hat{x}_2(0)}{\hat{x}_1(0)^{\frac{\lambda_2}{\lambda_1}}} \hat{x}_1(t)^{\frac{\lambda_2}{\lambda_1}}, \quad \lambda_1 \neq 0,\ \lambda_2 \neq 0$$

Case 1d: $\lambda_1 < 0 < \lambda_2$

The origin is called saddle



Both axes are invariant regions

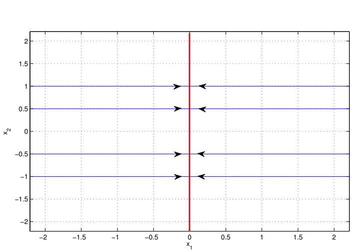
Saddle

$$\dot{\hat{x}}_1 = \lambda_1 \hat{x}_1 \rightarrow \hat{x}_1(t) = \hat{x}_1(0)e^{\lambda_1 t}$$
$$\dot{\hat{x}}_2 = \lambda_2 \hat{x}_2 \rightarrow \hat{x}_2(t) = \hat{x}_2(0)e^{\lambda_2 t}$$

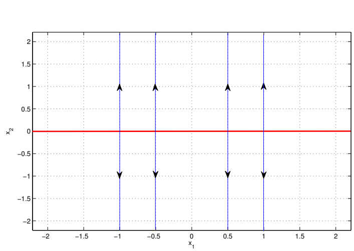
Case 1e: degenerate saddle

- $\lambda_1 < \lambda_2 = 0 \rightarrow$ all states on the $\hat{x}_2$ axis are equilibrium states.
- $0 = \lambda_1 < \lambda_2 \rightarrow$ all states on the $\hat{x}_1$ axis are equilibrium states.

$\lambda_1 < \lambda_2 = 0$



$0 = \lambda_1 < \lambda_2$



Analysis in the normal coordinates

Case 2: $J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \quad \lambda \in \mathbb{R}$

One can show that the state trajectories are given by

$$\hat{x}_1(t) = \hat{x}_1(0)e^{\lambda t} + \hat{x}_2(0)te^{\lambda t} \quad (1)$$

$$\hat{x}_2(t) = \hat{x}_2(0)e^{\lambda t} \quad (2)$$

Assume $\hat{x}_2(0) \neq 0$ and "remove" time. If $\lambda \neq 0$, from (2) one gets

$$e^{\lambda t} = \frac{\hat{x}_2(t)}{\hat{x}_2(0)}, \quad t = \frac{1}{\lambda}\ln\left(\frac{\hat{x}_2(t)}{\hat{x}_2(0)}\right)$$

and using (1) one obtains

$$\hat{x}_1(t) = \hat{x}_1(0)\frac{\hat{x}_2(t)}{\hat{x}_2(0)} + \frac{1}{\lambda}\ln\left(\frac{\hat{x}_2(t)}{\hat{x}_2(0)}\right)\hat{x}_2(t)$$

Improper nodes

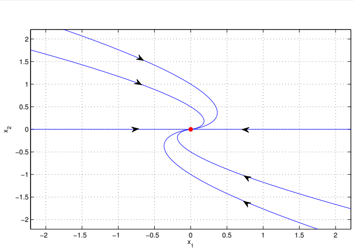
$$\hat{x}_1(t) = \hat{x}_1(0)\frac{\hat{x}_2(t)}{\hat{x}_2(0)} + \frac{1}{\lambda}\ln\left(\frac{\hat{x}_2(t)}{\hat{x}_2(0)}\right)\hat{x}_2(t)$$

Case 2a: $\lambda \neq 0$

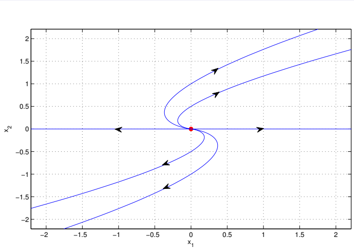
The origin is called stable/unstable improper node

- only the $\hat{x}_1$ axis is invariant

$\lambda < 0$



$\lambda > 0$

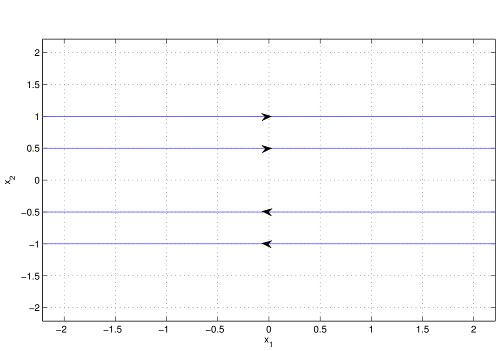


Improper nodes

$$\hat{x}_1(t) = \hat{x}_1(0)e^{\lambda t} + \hat{x}_2(0)te^{\lambda t}$$
$$\hat{x}_2(t) = \hat{x}_2(0)e^{\lambda t}$$

Case 2a: $\lambda = 0$

The system is unstable



Analysis in normal coordinates

Case 3: $J = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$ $\alpha, \beta \in \mathbb{R}$ (eigenvalues: $\alpha \pm j\beta$)

Using polar coordinates

$$r = \sqrt{\hat{x}_1^2 + \hat{x}_2^2}$$

$$\phi = \tan^{-1}\left(\frac{\hat{x}_2}{\hat{x}_1}\right)$$

one can show that

$$\dot{r} = \alpha r \rightarrow r(t) = r(0)e^{\alpha t}$$

$$\dot{\phi} = -\beta \rightarrow \phi(t) = \phi(0) - \beta t$$

State trajectories spiraling clockwise !

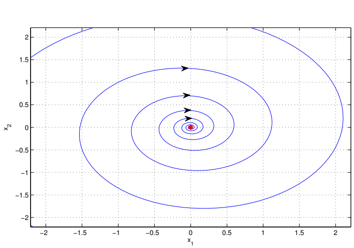
Foci

$$\dot{r} = \alpha r \to r(t) = r(0)e^{\alpha t}$$
$$\dot{\phi} = -\beta \to \phi(t) = \phi(0) - \beta t$$

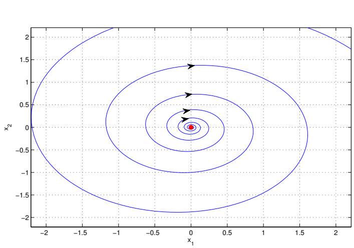
Case 3a: eigenvalues with nonzero real part ($\alpha \neq 0$)

The origin is called stable/unstable focus

Stable focus ($\alpha < 0$)



Unstable focus ($\alpha > 0$)

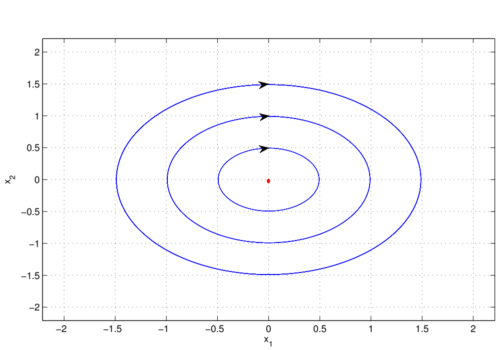


Center

$$\dot{r} = \alpha r \to r(t) = r(0)e^{\alpha t}$$
$$\dot{\phi} = -\beta \to \phi(t) = \phi(0) - \beta t$$

Case 3b: eigenvalues with zero real part ($\alpha = 0$)

The origin is called center



Generalization to nonlinear systems

$$\text{NL}: \begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases}$$

LInearized systems around the equilibrium state $\bar{x} = \begin{bmatrix} \bar{x}_1 & \bar{x}_2 \end{bmatrix}^{\mathrm{T}}$

$$\text{LIN}: \begin{bmatrix} \dot{\delta x_1} \\ \dot{\delta x_2} \end{bmatrix} = A(\bar{x}) \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} \qquad A(\bar{x}) = D_x f(x)\Big|_{x=\bar{x}} = \begin{bmatrix} \frac{\partial f_1(x)}{\partial x_1} & \frac{\partial f_1(x)}{\partial x_2} \\ \frac{\partial f_2(x)}{\partial x_1} & \frac{\partial f_2(x)}{\partial x_2} \end{bmatrix}_{x=\bar{x}}$$

Generalization to nonlinear systems

Definition

The equilibrium state $\bar{x}$ is hyperbolic if LIN does not have eigenvalues on the imaginary axis

Hartman-Grobman theorem

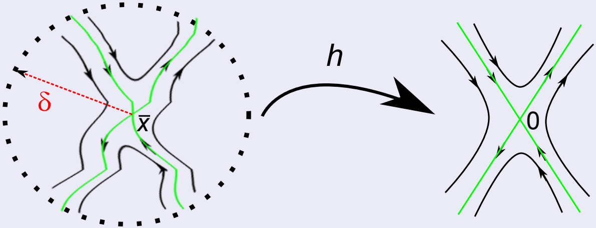
If $f_1, f_2 \in \mathcal{C}^1$ and $\bar{x}$ is hyperbolic, then there exist $\delta > 0$ and an homeomorphism $h : B_\delta(\bar{x}) \mapsto \mathbb{R}^2$ that maps state trajectories of NL into state trajectories of LIN and verifies $h(\bar{x}) = 0$.

Remarks

- Homeomorphism: continuous function with a continuous inverse (i.e. a change of coordinates)
  - Example: $h(x) = Tx$, $\det(T) \neq 0$ is an homeomorphism
- The change of coordinates is unique for all state trajectories until they stay in $B_\delta(\bar{x})$

Remarks on Hartman-Grobman theorem

- Intuitively, h is a distorting lens



- The qualitative behavior of the state trajectories of NL around $\bar{x}$ and of LIN around $\delta x = 0$ is identical

- When the theorem can be applied the equilibria of NL are classified as those of LIN
- However we are classifying only local behaviors

Remarks on Hartman-Grobman theorem

It is important that $A(\bar{x})$ does not have eigenvalues with zero real part

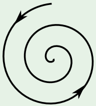
Example

$$\text{NL}: \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - \epsilon x_1^2 x_2 \end{cases} \qquad \bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

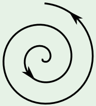
$$\text{LIN}: \begin{cases} \dot{\delta x}_1 = \delta x_2 \\ \dot{\delta x}_2 = -\delta x_1 \end{cases} \qquad A(\bar{x}) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \text{Eigenvalues: } 0 \pm j$$

The origin of LIN is a center but the origin of NL is more like

- a "stable focus" if $\epsilon > 0$



- an "unstable focus" if $\epsilon < 0$



Example

Duffing model

NL system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_1 - x_1^3 - \eta x_2,\ \eta = 1$$

Linearized system

$$\dot{\delta x}_1 = \delta x_2$$
$$\dot{\delta x}_2 = \delta x_1 - 3\bar{x}_1^2 \delta x_1 - \delta x_2$$

Around $p_1 = \begin{bmatrix} -1 & 0 \end{bmatrix}^{\mathrm{T}}$ and $p_3 = \begin{bmatrix} 1 & 0 \end{bmatrix}^{\mathrm{T}}$

$$D_x f = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix} \Rightarrow \text{ Eigenvalues: } -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$$

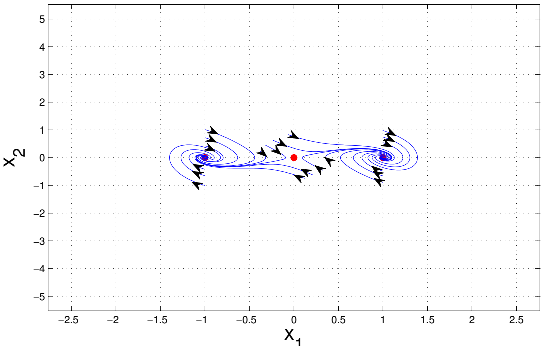
Hartman-Grobman theorem can be applied $\rightarrow$ p_1 and p_3 are stable foci

Example

Around $p_2 = \begin{bmatrix} 0 & 0 \end{bmatrix}^{\mathrm{T}}$

$$D_x f = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow \text{ Eigenvalues: } -1 \pm \frac{\sqrt{5}}{2}$$

Hartman-Grobman theorem can be applied → p_2 is a saddle



Phase plane - conclusions

Analysis around an equilibrium

When $x(t) \in \mathbb{R}^2$, one can study the qualitative behavior of state trajectories around an equilibrium state