Nonlinear systems

Closed orbits and limit cycles

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Closed orbits

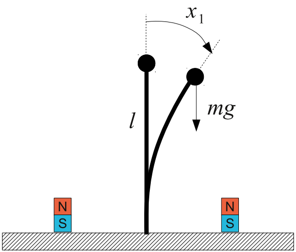
We still analyze NL systems in the phase plane

Definition: closed orbits

A closed orbit $\gamma \subset \mathbb{R}^2$ of the NL system $\dot{x} = f(x)$, $x(t) \in \mathbb{R}^2$ is the projection on the phase plane of a periodic and non constant state trajectory. Equivalently, γ is a closed orbit if

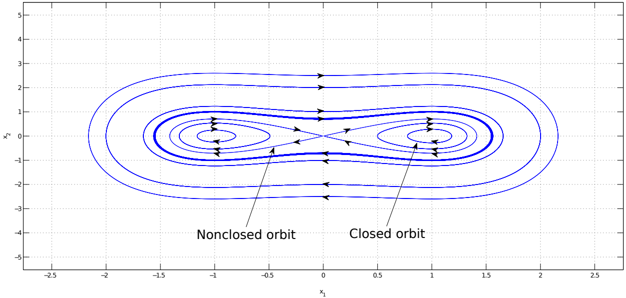
- γ is not an equilibrium point
- $\exists T > 0, \forall x_0 \in \gamma, x(nT) = x_0, \ \forall n \in \mathbb{N}$

Example: undamped Duffing oscillator

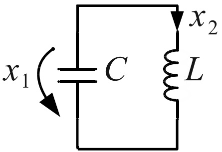


System

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_1 - x_1^3 - \eta x_2, \ \eta = 0$$

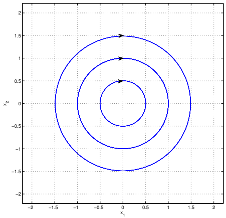


Example: harmonic oscillator



Model $(C = 1,\ L = 1)$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1$$



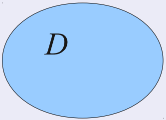
Every state trajectory $\phi(t, x_0)$, $x_0 \neq 0$ produces a closed orbit that depends upon the initial state

How to check if closed orbits exist ?

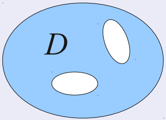
Bendixson criterion (absence of closed orbits)

Simply connected sets

A set is simply connected if it does not have "holes"



This set is simply connected



This set is not simply connected

Bendixson criterion

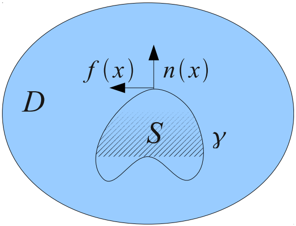
Assume that $f \in \mathcal{C}^1$ and let $D \subseteq \mathbb{R}^2$ be a simply connected set such that

$$\mathrm{div}(f) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$$

is not identically zero and does not change sign in D. Then, there are no closed orbits in D.

Proof of Bendixson criterion

- By contradiction, assume there is a closed orbit γ in D



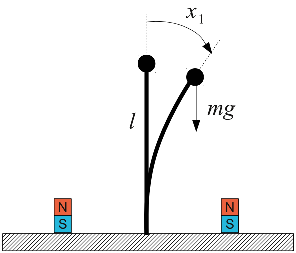
- S: interior of γ
- D simply connected $\Rightarrow S \subseteq D$
- $n(x)$: outgoing normal from S
- by construction $f(x) \cdot n(x) = 0$

- From divergence theorem

$$0 = \int_\gamma f(x) \cdot n(x) = \iint_S \mathrm{div}(f) dx_1\, dx_2$$

This is a contradiction since, in S, $\mathrm{div}(f)$ is not identically zero and it does not change sign

Example: damped Duffing oscillator



System

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_1 - x_1^3 - \eta x_2, \quad \eta > 0$$

Application of Bendixson criterion

$$\mathrm{div}(f) = \frac{\partial x_2}{\partial x_1} + \frac{\partial}{\partial x_2}(x_1 - x_1^3 - \eta x_2) = -\eta < 0, \quad \forall (x_1, x_2) \in \mathbb{R}^2$$

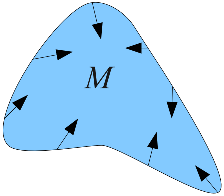
Pick $D = \mathbb{R}^2$. Bendixson $\Rightarrow$ no closed orbit in $\mathbb{R}^2$.

Invariant sets

Definition

A set $M \subseteq \mathbb{R}^n$ is (positively) invariant for $\dot{x} = f(x),\ x(t) \in \mathbb{R}^n$ if

$$\forall x_0 \in M \text{ it holds } \phi(t, x_0) \in M,\ \forall t \geq 0$$



Theorem

If $V : \mathbb{R}^n \to \mathbb{R}$ is of class $\mathcal{C}^1$ and $M = \{x : V(x) \leq c\}$, then M is invariant if

$$f(x) \cdot D_x V(x) \leq 0,\ \forall x : V(x) = c$$

i.e. if x is on the boundary of M, then the vector $f(x)$ points into M.

Remark ($f(x) \in \mathbb{R}^2,\ V(x) \in \mathbb{R}$)

$$f(x) \cdot D_x V(x) = \frac{\partial V}{\partial x_1}(x) f_1(x) + \frac{\partial V}{\partial x_2}(x) f_2(x)$$

Poincaré - Bendixson theorem

Theorem (presence of closed orbits)

Consider the NL system

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

and let $M \subseteq \mathbb{R}^2$ be a nonempty, closed and bounded set such that the following conditions simultaneously hold

(a) M is invariant;

(b) M does not contain equilibrium states or it contains a single equilibrium state $\bar{x}$ that is either an unstable focus or an unstable node.

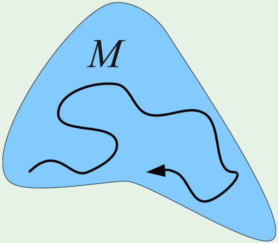
Then M contains a closed orbit.

Remark

(b) $\Leftrightarrow D_x f \Big|_{x=\bar{x}}$ has only eigenvalues with real part > 0.

Poincaré - Bendixson theorem

Remark



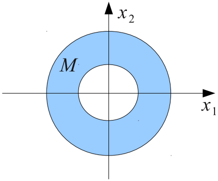
State trajectories cannot converge to an equilibrium state in M and they must lie forever in $M \rightarrow$ closed orbits exist.

Example: harmonic oscillator

System

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1$$

Let $M = \{x : c_1 \leq V(x) \leq c_2\}$ where $V(x) = x_1^2 + x_2^2$, $c_1, c_2 > 0$. Is there any closed orbit in M ?



- M is nonempty closed and bounded.
- $f(x) \cdot \nabla V(x) = 2x_2x_1 - 2x_1x_2 = 0$
  - M is invariant (check it at home !!)

There are no equilibrium states in M (because $c_2 \geq c_1 > 0$)
From the Poincaré-Bendixson theorem there is a closed orbit in M[a]

[a] In this case there are infinite closed orbits.

Example

System

$$\dot{x}_1 = x_1 + x_2 - x_1\left(x_1^2 + x_2^2\right)$$
$$\dot{x}_2 = -2x_1 + x_2 - x_2\left(x_1^2 + x_2^2\right)$$

Let $M = \{x : V(x) \leq c\}$ where $V(x) = x_1^2 + x_2^2$. For $c \geq 2$ is there any closed orbit in M ?

Solution

Only one equilibrium state $\bar{x} = \begin{bmatrix} 0 & 0 \end{bmatrix}^{\mathrm{T}}$.

$$D_x f\Big|_{x=\bar{x}} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \Rightarrow \text{ Eigenvalues: } 1 \pm j\sqrt{2}$$

M is nonempty, closed, bounded and contains an unstable focus.

Example

Compute for which values of c the set M is invariant

$V(x) \in \mathcal{C}^1$. At points x such that $V(x) = c$ one has

$$\begin{aligned}
&\frac{\partial V}{\partial x_1} f_1 + \frac{\partial V}{\partial x_2} f_2 = \\
&= 2x_1 \left(x_1 + x_2 - x_1 \left(x_1^2 + x_2^2\right)\right) + 2x_2 \left(-2x_1 + x_2 - x_2 \left(x_1^2 + x_2^2\right)\right) = \\
&= 2\left(x_1^2 + x_2^2\right) - 2\left(x_1^2 + x_2^2\right)^2 - 2x_1 x_2 \leq \\
&\underbrace{\leq}_{|2x_1 x_2| \leq x_1^2 + x_2^2} \quad 2\left(x_1^2 + x_2^2\right) - 2\left(x_1^2 + x_2^2\right)^2 + \left(x_1^2 + x_2^2\right) = \\
&= 3c - 2c^2
\end{aligned}$$

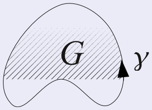
Therefore M is invariant for $c \geq 1.5$ and, from Poincaré-Bendixson theorem, there is a closed orbit in M.

Closed orbits and equilibria

Index theorem

Let γ be a closed orbit for

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$



and let G be the set of points in the interior of γ. Then,

(a) there is at least an equilibrium state in G

(b) if the equilibrium states in G are hyperbolic [a], then $N - S = 1$, where

- N is the number of nodes and foci in G
- S is the number of saddles in G

.

[a] This means the linearized system does not have eigenvalues with zero real part.

Example

Problem

$$\dot{x}_1 = -x_1 + x_1 x_2$$
$$\dot{x}_2 = x_1 + x_2 - 2x_1 x_2$$

Does it exist a closed orbit that embraces all equilibrium states ?

Two equilibrium states:

$$p_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow D_x f\Big|_{x=p_1} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \Rightarrow \text{saddle}$$
$$p_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow D_x f\Big|_{x=p_2} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \Rightarrow \text{stable focus}$$

In order to have $N - S = 1$, a closed orbit can only embrace p_2.

Limit cycles

Definition

A limit cycle is a closed orbit γ such that there is $\tilde{x} \notin \gamma$ and

$$\phi(t, \tilde{x}) \to \gamma \text{ for } t \to +\infty \text{ or for } t \to -\infty$$

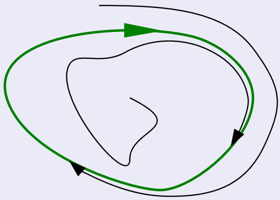
Example

The harmonic oscillator has closed orbits but no limit cycles

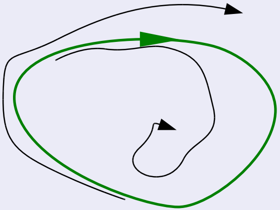
- Limit cycles cannot be generated by LTI systems !
- There are tools for checking if a closed orbit is a limit cycle (not in this course)

Examples of limit cycles

Asymptotically stable limit cycle



Unstable limit cycle



Semistable limit cycle