Nonlinear systems

Lyapunov stability theory

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Advanced automation and control

Aleksandr Mikhailovich Lyapunov (1857-1918)

Stability of differential equations

Torricelli's principle (1644). A mechanical system composed by two rigid bodies subject to the gravitational force is at a stable equilibrium if the total energy is (locally) minimal.

Laplace (1784), Lagrange (1788). If the total energy of a system of masses is conserved, then a state corresponding to zero kinetic energy is stable.

Lyapunov (1892). Student of Chebyshev at the St. Petersburg university. 1892 PhD "The general problem of the stability of motion".

In the 1960's. Kalman brings Lyapunov theory to the field of automatic control (Kalman and Bertram "Control system analysis and design via the second method of Lyapunov")

Stability of the origin

System

We consider the autonomous NL time-invariant system

$$\dot{x} = f(x) \tag{1}$$

Let $\bar{x}$ be an equilibrium state. Hereafter we assume, $f \in \mathcal{C}^1$ and, without loss of generality, $\bar{x} = 0$

If $x^* \neq 0$ is an equilibrium state, set $z = x - x^*$ and, from (1), obtain

$$\dot{z} = f(z + x^*) \tag{2}$$

- $\bar{z} = 0$ is an equilibrium state for (2)
- $x(t) = \phi(t, x_0)$ is a state trajectory for (1) $\Leftrightarrow$ $z(t) = x(t) - x^*$ is a state trajectory of (2) with $z(0) = x_0 - x^*$

Review: stability of an equilibrium state

Let $\bar{x} = 0$ be an equilibrium state for the NL time-invariant system $\dot{x} = f(x)$

Ball centered in $\bar{z} \in \mathbb{R}^n$ of radius $\delta > 0$

$B_\delta(\bar{z}) = \{z \in \mathbb{R}^n : \|z - \bar{z}\| < \delta\}$

Definition (Lyapunov stability)

The equilibrium state $\bar{x} = 0$ is

- Stable if

$$\forall \epsilon > 0\ \exists \delta > 0,\ x(0) \in B_\delta(0) \Rightarrow x(t) \in B_\epsilon(0), \forall t \geq 0$$

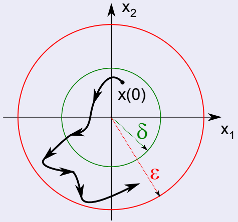
- Asymptotically Stable (AS) if it is stable and $\exists \gamma > 0$ such that

$$x(0) \in B_\gamma(0) \Rightarrow \lim_{t \to +\infty} \|\phi(t, x(0))\| = 0$$

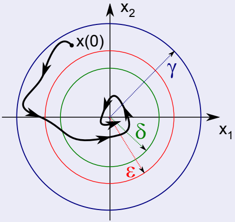
- Unstable if it is not stable

Review: stability of an equilibrium state

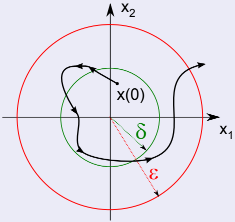
$\bar{x} = 0$ stable



$\bar{x} = 0$ AS



$\bar{x} = 0$ unstable



If $\bar{x} = 0$ is AS, $B_\gamma(0)$ is a *region of attraction*, i.e. a set $X \subseteq \mathbb{R}^n$ such that

$$x(0) \in X \Rightarrow \lim_{t \to +\infty} \|\phi(t, x(0))\| = 0$$

- *THE* region of attraction of $\bar{x} = 0$ is the maximal region of attraction

Remark

Even if $\bar{x} = 0$ is AS, $\phi(t, x(0))$ might converge very slowly to $\bar{x}$

Exponential stability

Definition

$\bar{x} = 0$ is Exponentially Stable (ES) if there are $\delta, \alpha, \lambda > 0$ such that

$$x(0) \in B_\delta(0) \ \Rightarrow \ \|x(t)\| \leq \alpha e^{-\lambda t} \|x(0)\|$$

The quantity λ is an estimate of the exponential convergence rate.

Remark

ES $\Rightarrow$ AS $\Rightarrow$ stability. All the opposite implications are false.

Example:

$$\dot{x} = -x^2 \ \Rightarrow \ x(t) = \frac{x(0)}{1 + tx(0)}$$

$\bar{x} = 0$ is AS (check at home !) but not ES.

Global stability

Remarks

Stability, AS, ES: local concepts ("for $x(0)$ sufficiently close to $\bar{x}$")

Global asymptotic stability

If $\bar{x} = 0$ is stable and

$$x(0) \in \mathbb{R}^n \Rightarrow \lim_{t \to +\infty} \|\phi(t, x(0)\| = 0$$

then $\bar{x} = 0$ is Globally AS (GAS)

Global exponential stability

If there are $\alpha, \lambda > 0$such that

$$x(0) \in \mathbb{R}^n \;\Rightarrow\; \|x(t)\| \leq \alpha e^{-\lambda t} \|x(0)\|$$

then $\bar{x} = 0$ is Globally ES (GES)

Remarks on stability

Example:

$$\dot{x} = -x^2 \;\Rightarrow\; x(t) = \frac{x(0)}{1 + tx(0)}$$

$\bar{x} = 0$ is GAS (but not GES).

Problems

- How to check the stability properties of $\bar{x} = 0$ WITHOUT computing state trajectories ?
- If $\bar{x} = 0$ is AS, how to compute a region of attraction ?
- The analysis of the linearized system around $\bar{x} = 0$ MIGHT allow one to check local stability. How to proceeed when no conclusion can be drawn using the linearized system ?

Need of a more complete approach: Lyapunov direct method

Remarks on stability

Example

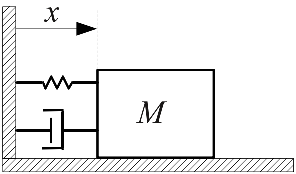
$$\dot{x} = ax - x^5$$

$\bar{x} = 0$ is an equilibrium state.
Linearized system: $\dot{\delta x} = a\delta x$

- $a < 0 \Rightarrow \bar{x} = 0$ is AS
- $a > 0 \Rightarrow \bar{x} = 0$ is unstable
- $a = 0$?

For $a = 0$ one has $\dot{x} = -x^5$ and with the Lyapunov direct method one can show that $\bar{x} = 0$ is AS.

Lyapunov direct method: a first example



Model

$$M\ddot{x} = -\underbrace{b\dot{x}|\dot{x}|}_{\text{NL damping}} - \underbrace{\left(k_0 x + k_1 x^3\right)}_{\text{NL elastic force}}$$

$b, k_0, k_1 > 0$

Defining $x_1 = x$, $x_2 = \dot{x}_1$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{b}{M}x_2|x_2| - \frac{k_0}{M}x_1 - \frac{k_1}{M}x_1^3 \end{cases} \Rightarrow \bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ is stab./AS/ES ?}$$

$$D_x f = \begin{bmatrix} 0 & 1 \\ -\frac{k_0}{M} - \frac{3k_1}{M}x_1^2 & -\frac{2b}{M}x_2 \operatorname{sgn}(x_2) \end{bmatrix} \Rightarrow D_x f\Big|_{x=\bar{x}} = \begin{bmatrix} 0 & 1 \\ -\frac{k_0}{M} & 0 \end{bmatrix}$$

Eigenvalues: $\lambda_{1,2} = \pm j\sqrt{k_0/M}$

No conclusion on $\bar{x} = 0$ using the linearized system

Lyapunov direct method: a first example

Consider the total energy of the system:

$$V(x) = \underbrace{\frac{1}{2}Mx_2^2}_{\text{kinetic}} + \underbrace{\int_0^{x_1} k_0\xi + k_1\xi^3 \, d\xi}_{\textit{potential}} = \frac{1}{2}Mx_2^2 + \frac{1}{2}k_0x_1^2 + \frac{1}{4}k_1x_1^4$$

Remark: zero energy $\Leftrightarrow x_1 = x_2 = 0$ (equilibrium state)

Instantaneous energy change:

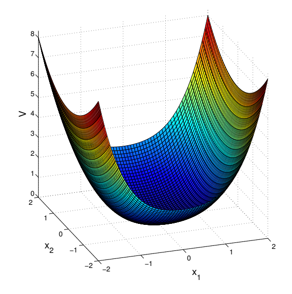
$$\dot{V}(x(t)) = D_xV \cdot \frac{dx}{dt} = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{bmatrix} \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \left(k_0x_1 + k_1x_1^3\right)\dot{x}_1 + Mx_2\dot{x}_2 =$$

$$= \left(k_0x_1 + k_1x_1^3\right)x_2 + Mx_2\left(-\frac{b}{M}x_2|x_2| - \frac{k_0}{M}x_1 - \frac{k_1}{M}x_1^3\right) = -bx_2^2|x_2|$$

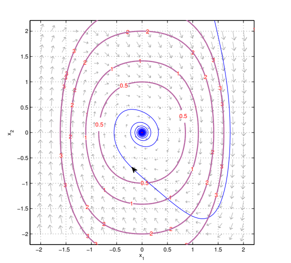
$-bx_2^2(t)|x_2(t)| \leq 0$ *independently of* $x(0) \Rightarrow$ the energy can only decrease with time *independently of* $x(0)$

Lyapunov direct method: a first example

Energy $V(x)$



Phase plane



- Energy is a “measure” of the distance of x from the origin
  - if it can only decrease, then $\bar{x} = 0$ should be stable
- Lyapunov direct method is based on energy-like functions $V(x)$ and the analysis of the function $t \mapsto V(x(t))$

Positive definite functions

In the previous example, $V(x)$ has two key properties

- $V(x) > 0$, $\forall x \neq 0$ and $V(0) = 0$
- $t \mapsto V(x(t))$ decreases for $x(t) = \phi(t, x_0)$

Definition

A scalar and continuous function $V(x)$ is

- *positive definite (pd)* in $B_\delta(0)$ if $V(0) = 0$ and

$$x \in B_\delta(0) \setminus \{0\} \Rightarrow V(x) > 0$$

- *positive semidefinite (psd)* in $B_\delta(0)$ if $V(0) = 0$ and

$$V(x) \geq 0,\ \forall x \in B_\delta(0)$$

- *negative definite (nd)* [resp. *negative semidefinite (nsd)*] if $-V(x)$ is pd [resp. psd]

Globally positive definite functions

Definition

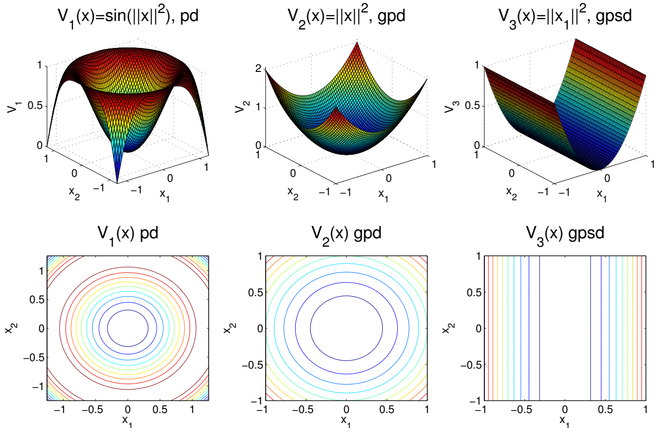
The scalar and continuous function $V(x)$ is pd/psd/nd/nsd if it has the same property in some $B_\delta(0)$

Definition

If, in the previous definitions $B_\delta(0)$ is replaced with $\mathbb{R}^n$, one defines functions

- globally positive definite (gpd)
- globally positive semidefinite (gpsd)
- globally negative definite (gnd)
- globally negative semidefinite (gnsd)

Examples of positive definite functions



Level surfaces: $V_\alpha = \{x \in \mathbb{R}^n : V(x) = \alpha\}$

- they do not intersect

Computation of $\dot{V}$

Given $\dot{x} = f(x)$ and a scalar function $V(x)$ of class $\mathcal{C}^1$

$$\dot{V} = \frac{dV(x(t))}{dt} = D_x V \cdot \dot{x} = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \cdots & \frac{\partial V}{\partial x_n} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) \quad (3)$$

Remark

(3) is called Lie derivative of V along f

- it measures how much V decreases along state trajectories

Lyapunov theory

Lyapunov theorems

The typical statement has the following skeleton:

- If there exists a function $V : \mathbb{R}^n \to \mathbb{R}$ such that V and $\dot{V}$ verify suitable conditions, then the equilibrium $\bar{x} = 0$ has some key properties

If such a function V exists, it is called the *Lyapunov function* that certifies the properties of $\bar{x} = 0$

Remark

As it will be clear in the sequel, Lyapunov can be viewed as generalized energy functions

Invariance Lyapunov Theorem

Theorem

Let $V(x) \in \mathcal{C}^1$ be pd in $B_\delta(0)$. If $\dot{V}(x)$ is nsd in $B_\delta(0)$, then the level sets

$$\mathcal{V}_m = \{x : V(x) < m\}, \quad m > 0 \tag{4}$$

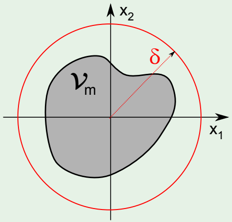
included in $B_\delta(0)$ are invariant.
If, in addition $V(x)$ is gpd and $\dot{V}(x)$ is gnsd then the sets (4) are invariant for all $m > 0$.

Proof

By contradiciton, assume $x(0) \in \mathcal{V}_m$ and let $t_m > 0$ be the first instant such that $V(x(t_m)) \geq m$ [a]. One has

$$V(x(t_m)) \geq m > V(x(0))$$

that is a contradition because $\dot{V}(x(t)) \leq 0$ for $t \in [0, t_m)$. The proof of the second part of the theorem is identical.



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[a] Such an instant exists because $x(t)$ is continuous in t.

Stability Lyapunov theorem

Theorem

If there is $V(x) \in \mathcal{C}^1$ such that it is pd in $B_\delta(0)$ and $\dot{V}$ is nsd in $B_\delta(0)$, then $\bar{x} = 0$ is stable. If in addition $\dot{V}(x)$ is nd in $B_\delta(0)$ then $\bar{x} = 0$ is AS.

Extremely useful !

- Not necessary to compute state trajectories: it is enough to check the sign of V and $\dot{V}$ in a neighborhood of the origin
- The theorem precises the properties that an energy function must have
- Several results in nonlinear control are based on this theorem
- There are several versions of the theorem, e.g. for discrete-time systems, when f is piecewise continuous, when is V continuous only in $x = 0$, ...

Key issue:

- Cheking if $\bar{x} = 0$ is AS has been just recast into the problem of finding a suitable Lyapunov function. How to compute it ?

Proof of the theorem

Theorem

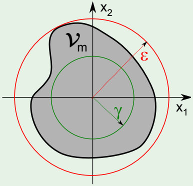
If there is $V(x) \in \mathcal{C}^1$ such that it is pd in $B_\delta(0)$ and $\dot{V}$ is nsd in $B_\delta(0)$, then $\bar{x} = 0$ is stable. If in addition $\dot{V}(x)$ is nd in $B_\delta(0)$ then $\bar{x} = 0$ is AS.

Proof of stability $(\forall \epsilon > 0\ \exists \gamma > 0 : x(0) \in B_\gamma(0) \Rightarrow \forall t \geq 0\ x(t) \in B_\epsilon(0))$

Without loss of generality, assume $\epsilon < \delta$. Let

$$m = \min_{\|x\|=\epsilon} V(x)^a$$

Let $\mathcal{V}_m = \{x : V(x) < m\}^b$. There is $\gamma > 0 : B_\gamma(0) \subset \mathcal{V}_m{}^c$.



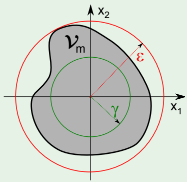
[a] m exists since V is continuous and the boundary of $B_\epsilon(0)$ is bounded and closed.

[b] $m > 0$ because V is pd. In particular, $\mathcal{V}_m$ is not empty.

[c] Continuity of $V(x)$ in $x = 0$:
$\forall \epsilon' > 0, \exists \delta' > 0 : x \in B_{\delta'}(0) \Rightarrow |V(x) - 0| < \epsilon'$. Pick $\epsilon' < m$ and set $\gamma = \delta'$.

Proof of the theorem

From the Lyapunov invariance theorem, one has that $\mathcal{V}_m$ is invariant. Since $B_\gamma(0) \subseteq \mathcal{V}_m$, one has that $\|x(0)\| < \gamma$ implies $\|x(t)\| < \epsilon$, $\forall t \geq 0$.



Proof of AS

Let ϵ and γ be chosen as in the proof of stability (in particular, $\epsilon < \delta$).
One has

$$x(0) \in B_\gamma(0) \Rightarrow \phi(t, x(0)) \in B_\epsilon(0), \ \forall t \geq 0$$

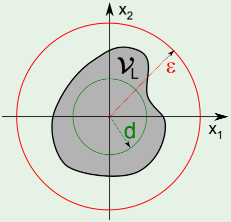
We now show that $B_\gamma(0)$ is a region of attraction for $\bar{x} = 0$.
If $x(0) \in B_\gamma(0)$, since $\dot{V}$ is nd in $B_\epsilon(0)$ one has that $V(x(t))$ is decreasing. $V(x(t))$ is also lower bounded by 0. Therefore $V(x(t))$ has a limit $L \geq 0$ for $t \to +\infty$.
Since $V(x) = 0 \Rightarrow x = 0$, in order to conclude we have only to show that $L = 0$.

Proof of the theorem

By contradiction, assume that $L > 0$.
V continuous $\Rightarrow \exists d > 0 : B_d(0) \subseteq \mathcal{V}_L$ and since $V(x(t)) \geq L$, $\forall t \geq 0$ one has that $x(t)$ never enters in $B_d(0)$.
Let $-\eta = \max_{d \leq \|x\| \leq \epsilon} \dot{V}(x)$[a]. Since $\dot{V}$ is nd, it follows $-\eta < 0$. One has [b]

$$V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(\tau))\, d\tau \leq V(x(0)) - \eta t$$

Therefore, for a sufficiently large t one has $V(x(t)) < L$ that is a contradiction.



[a] η exists because $\dot{V}$ is continuous and the constraints of the maximum problem define a bounded and closed set.

[b] For the fundamental theorem of calculus and the fact that $x(t) \in B_\epsilon(0)$, $\forall t \geq 0$.

Example

System

$$\dot{x}_1 = x_1\left(x_1^2 + x_2^2 - 2\right) - 4x_1x_2^2$$
$$\dot{x}_2 = 4x_1^2x_2 + x_2\left(x_1^2 + x_2^2 - 2\right)$$

Study the stability of the equilibrium state $\bar{x} = 0$.

Candidate Lyapunov function: $V(x) = x_1^2 + x_2^2$ (pd in $\mathbb{R}^2$)

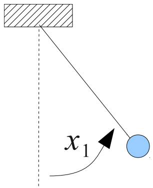
$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial x_1}f_1(x) + \frac{\partial V}{\partial x_2}f_2(x) = \\ &= 2x_1\left(x_1\left(x_1^2 + x_2^2 - 2\right) - 4x_1x_2^2\right) + 2x_2\left(4x_1^2x_2 + x_2\left(x_1^2 + x_2^2 - 2\right)\right) = \\ &= 2\left(x_1^2 + x_2^2\right)\left(x_1^2 + x_2^2 - 2\right)\end{aligned}$$

In $B_{\sqrt{2}}(0)$ one has $\left(x_1^2 + x_2^2 - 2\right) < 0$ and therefore $\dot{V}$ is nd in $B_{\sqrt{2}}(0) \Rightarrow$ $\bar{x} = 0$ is AS.

Key remark

The choice of the Lyapunov function is not unique ! For instance $V(x) = \alpha\left(x_1^2 + x_2^2\right)$, $\alpha > 0$ are all Lyapunov functions for the example.

Example: damped pendolum



System

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_2 - \sin(x_1)$$

Study the stability of $\bar{x} = \begin{bmatrix} 0 & 0 \end{bmatrix}^{\mathrm{T}}$.

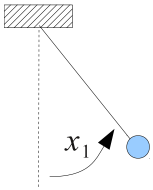
Lyapunov candidate function: $V(x) = \underbrace{(1 - \cos(x_1))}_{\text{potential en.}} + \underbrace{\frac{x_2^2}{2}}_{\text{kinetic en.}}$

V is pd ? In which region ?

- $V(0) = 0$
- $V(x) > 0$ if $x_1 \in (-2\pi, 2\pi)$, $x_1 \neq 0$

Then, V is pd in $B_{2\pi}(0)$

Example: damped pendolum



System

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Lyapunov candidate function: $V(x) = \underbrace{(1 - \cos(x_1))}_{\text{potential en.}} + \underbrace{\frac{x_2^2}{2}}_{\text{kinetic en.}}$

$\dot{V}$ is nsd ? In which region ?

$$\dot{V} = \frac{\partial V}{\partial x_1} f_1(x) + \frac{\partial V}{\partial x_2} f_2(x) = \sin(x_1)x_2 + x_2\left(-x_2 - \sin(x_1)\right) = -x_2^2$$

$\dot{V}$ is nsd in $\mathbb{R}^2$ (and then in $B_{2\pi}(0)$) $\Rightarrow \bar{x} = 0$ is stable.

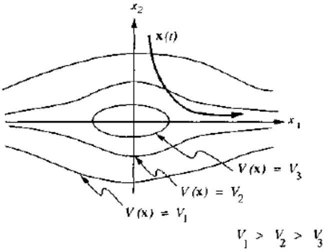
Physical intuition tells us the equilibrium is AS but the chosen Lyapunov function certifies only stability

Global asymptotic stability

If $V(x)$ is globally pd and $\dot{V}$ is globally nd, is it possible to deduce that $\bar{x} = 0$ is GAS ?

NO !

Example: $V(x) = \dfrac{x_1^2}{1 + x_1^2} + x_2^2$ is globally pd but the level surfaces V_α are not bounded for $\alpha > 1$. Then $V(x(t))$ can decrease but $x(t)$ can diverge.



Lyapunov theorem for global asymptotic stability

Definition

A function $V(x)$ pd is radially unbounded if $V(x) \to +\infty$ for $\|x\| \to +\infty$.

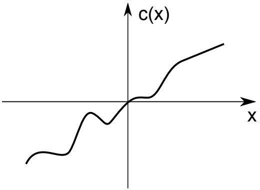
Remark

The behavior in the previous example cannot happen if V is radially unbounded

Theorem (GAS)

If there is $V(x) \in \mathcal{C}^1$ gpd and radially unbounded such that $\dot{V}$ is gnd then $\bar{x} = 0$ is GAS.

Example



First-order system

$$\dot{x} = -c(x)$$

where $c(x) \in \mathcal{C}^1$ and $x \neq 0 \Rightarrow xc(x) > 0$

Study the stability of $\bar{x} = 0$.

Candidate Lyapunov function: $V(x) = x^2$ (gpd and radially unbounded)

$$\dot{V}(x) = 2x\dot{x} = -2xc(x)$$

$\dot{V}$ is gnd $\Rightarrow \bar{x} = 0$ is GAS

Lyapunov theorem for exponential stability

Theorem (ES/GES)

If there exists a ball $B_\delta(0)$ and a function $V(x) \in \mathcal{C}^1$ such that, for all $x \in B_\delta(0)$ one has

$$k_1\|x\|^a \leq V(x) \leq k_2\|x\|^a \tag{5}$$

$$\dot{V}(x(t)) \leq -k_3\|x\|^a \tag{6}$$

where $k_1, k_2, k_3, a > 0$ are suitable constants, then $\bar{x} = 0$ is ES. Moreover, if (5) and (6) hold for all $x \in \mathbb{R}^n$, then $\bar{x} = 0$ is GES.

Remarks

- (5) $\Rightarrow$ $V(x)$ is pd. The opposite implication does not hold.
- (5) + (6) $\Rightarrow$ $\dot{V}$ is nd. The opposite implication does not hold.

Lyapunov theorem for exponential stability

Proof of ES

From the Lyapunov theorem, (5) and (6) imply that $\bar{x} = 0$ is AS. Moreover

$$\dot{V}(x(t)) \underbrace{\leq}_{\text{from (6)}} -k_3\|x\|^a \underbrace{\leq}_{\text{from (5)}} -\frac{k_3}{k_2}V(x(t))$$

In particular, the last inequality follows from $\dfrac{V(x)}{k_2} \leq \|x\|^a$.

If equalities hold, one gets the LTI system $\dot{V} = -\dfrac{k_3}{k_2}V$ and therefore

$$V(x(t)) = V(x(0))e^{-\frac{k_3}{k_2}t}$$

It is possible to show that if $\dot{V} \leq -\dfrac{k_3}{k_2}V$ then $V(x(t)) \leq V(x(0))e^{-\frac{k_3}{k_2}t}$

Lyapunov theorem for exponential stability

Therefore

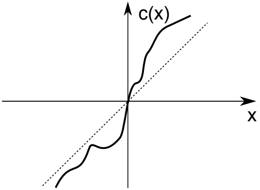
$$\|x(t)\| \underbrace{\leq}_{\text{from (5)}} \left[\frac{V(x(t))}{k_1}\right]^{\frac{1}{a}} \leq \left[\frac{V(x(0))e^{-\frac{k_3}{k_2}t}}{k_1}\right]^{\frac{1}{a}} \underbrace{\leq}_{\text{from (5)}} \left[\frac{k_2\|x(0)\|^a e^{-\frac{k_3}{k_2}t}}{k_1}\right]^{\frac{1}{a}} =$$

$$= \left(\frac{k_2}{k_1}\right)^{\frac{1}{a}} \|x(0)\| e^{-\frac{k_3}{ak_2}t}$$

Remark

$\lambda = \frac{k_3}{ak_2}$ is an estimate of the exponential convergence rate

Example



System

$$\dot{x} = -c(x)$$

where $c(x) \in \mathcal{C}^1$ and verifies $xc(x) \geq x^2$

Study the stability of $\bar{x} = 0$.

Candidate Lyapunov function: $V(x) = x^2$

- $k_1\|x\|^a \leq V(x) \leq k_2\|x\|^a$ with $k_1 = k_2 = 1$ and $a = 2$, $\forall x \in \mathbb{R}$
- $\dot{V}(x) = 2x\dot{x} = -2xc(x) \leq -k_3\|x\|^a$ with $k_3 = a = 2$ $\forall x \in \mathbb{R}$

Then, $\bar{x} = 0$ is GES.

Estimate of the exponential convergence rate: $\lambda = \frac{k_3}{ak_2} = 1$

Lyapunov instability theorem

Theorem (instability)

If in a set $B_\delta(0)$ there is a scalar function $V(x) \in \mathcal{C}^1$ such that V is pd in $B_\delta(0)$ and $\dot{V}$ is pd in $B_\delta(0)$, then $\bar{x} = 0$ is unstable.

Lyapunov instability theorem

Example

$$\dot{x}_1 = 2x_2 + x_1\left(x_1^2 + x_2^4\right)$$
$$\dot{x}_2 = -2x_1 + x_2\left(x_1^2 + x_2^4\right)$$

Study the stability of $\bar{x} = 0$.

Linearized system around $\bar{x} = 0$

$$\Sigma : \begin{cases} \delta\dot{x}_1 = 2\delta x_2 \\ \delta\dot{x}_2 = -2\delta x_1 \end{cases} \quad \text{eigenvalues: } \pm 2j$$

No conclusion on stability of $\bar{x} = 0$ using Σ.

Candidate Lyapunov function: $V(x) = \frac{1}{2}\left(x_1^2 + x_2^2\right)$ (pd in $\mathbb{R}^2$)

$$\dot{V} = x_1\left(2x_2 + x_1\left(x_1^2 + x_2^4\right)\right) + x_2\left(-2x_1 + x_2\left(x_1^2 + x_2^4\right)\right) =$$
$$= \left(x_1^2 + x_2^2\right)\left(x_1^2 + x_2^4\right) \rightarrow \text{ pd in } \mathbb{R}^2$$

Then, $\bar{x} = 0$ is unstable.

Conclusions

Summary of Lyapunov theorems

Conditions on $V(x) \in \mathcal{C}^1$	Conditions on $\dot{V}$	Stab. of $\bar{x} = 0$
pd in $B_\delta(0)$	nsd in $B_\delta(0)$	stable
pd in $B_\delta(0)$	nd in $B_\delta(0)$	AS
pd in $\mathbb{R}^n$ and rad. unbounded	nd in $\mathbb{R}^n$	GAS
$k_1\|x\|^\alpha \leq V(x) \leq k_2\|x\|^\alpha$ in $B_\delta(0)$	$\dot{V} \leq -k_3\|x\|^\alpha$ in $B_\delta(0)$	ES
$k_1\|x\|^\alpha \leq V(x) \leq k_2\|x\|^\alpha$ in $\mathbb{R}^n$	$\dot{V} \leq -k_3\|x\|^\alpha$ in $\mathbb{R}^n$	GES
pd in $B_\delta(0)$	pd in $B_\delta(0)$	unstable

Remarks

All Lyapunov theorems are only *sufficient* conditions. Moreover for a given system there can be multiple Lyapunov functions certifying different stability properties of $\bar{x} = 0$