Nonlinear systems

Lyapunov stability theory - part 2

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Advanced automation and control

Invariant set theorems

In a neighborhood of $\bar{x} = 0$ it can happen that $\dot{V}$ is only nsd even if $\bar{x} = 0$ is AS (see, e.g., the pendulum example). In this case, one can use theorems attibuted to LaSalle that characterize attractivity of invariant sets.

Review

A set $G \subseteq \mathbb{R}^n$ is (positively) invariant for $\dot{x} = f(x)$ if

$$x(0) \in G \Rightarrow \phi(t, x(0)) \in G, \ \forall t \geq 0$$

Examples

- $G = \{\bar{x}\}$, $\bar{x}$ equilibrium state
- $G = \{p_1, p_2, p_3\}$ if $p_i \in \mathbb{R}^n, \ i = 1, 2, 3$ are equilibrium states
- Periodic orbits and limit cycles
- $G = \mathbb{R}^n$

Local LaSalle theorem

Theorem

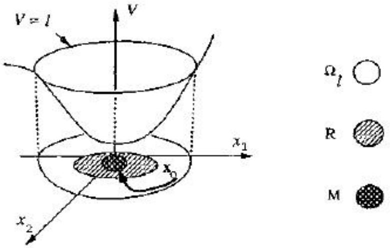
Let $V(x) \in \mathcal{C}^1$ be a scalar function with the following properties:

- $\exists l > 0$ such that the level set $\Omega_l = \{x : V(x) < l\}$ is bounded
- $\dot{V}(x) \leq 0 \ \forall x \in \Omega_l$

Let $R = \left\{x : \dot{V}(x) = 0\right\} \cap \Omega_l$ and let M the largest invariant set in R.

Then, $\forall x(0) \in \Omega_l$ one has $\phi(t, x(t)) \to M$ for $t \to +\infty$[a]

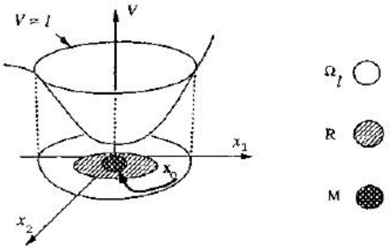
[a] In the sense that $\lim_{t\to+\infty} \inf_{z\in M} \|\phi(t, x(0)) - z\| = 0$



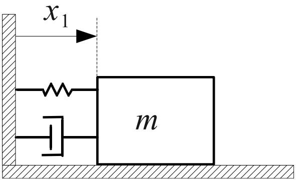
Local LaSalle theorem

Remarks

- The theorem provides sufficient conditions for Ω_l to be a region of attraction for the set M
- It is not necessary for V to be pd. If it happens and $\dot{V}(x) \leq 0$ $\forall x \in \Omega_l$ Lyapunov stability theorem holds as well and therefore $\bar{x} = 0$ is stable.
- Notable case: when $M = \{0\}$ the theorem gives a region of attraction for the equilibrium state $\bar{x} = 0$



Example: how to prove that $\bar{x} = 0$ is AS when $\dot{V}$ is sdn



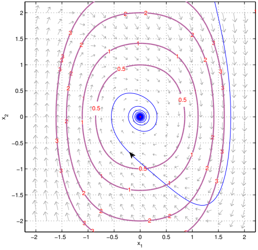
Model

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{b}{m}x_2|x_2| - \frac{k_0}{m}x_1 - \frac{k_1}{m}x_1^3$$
$$m, b, k_0, k_1 > 0$$

We have already shown that

- $\bar{x} = 0$ is an equilibrium state
- $V(x) = \frac{1}{2}mx_2^2 + \frac{1}{2}k_0x_1^2 + \frac{1}{4}k_1x_1^4$ is pd in $\mathbb{R}^2$
- $\dot{V}(x(t)) = -bx_2^2|x_2|$ is nsd in $\mathbb{R}^2$

Lyapunov stability theorem $\Rightarrow \bar{x} = 0$ is stable



Example: how to prove that $\bar{x} = 0$ is AS when $\dot{V}$ is sdn

Now we use LaSalle theorem

- Find for which $l > 0$ the set Ω_l is bounded
  Remark: if $\tilde{V}(x) \leq V(x)$ then $\Omega_l \subseteq \left\{ x : \tilde{V}(x) < l \right\}$

  Idea: remove positive terms from V in order to get a function $\tilde{V}(x)$ for which is easy to show that level sets are bounded (at least for a sufficiently small l)

  - often one tries to get a function $\tilde{V}(x)$ that depends on $\|x\|$ and not on variables $x_1 \ldots, x_n$ separately.

$$V(x) = \frac{1}{2} m x_2^2 + \frac{1}{2} k_0 x_1^2 + \frac{1}{4} k_1 x_1^4 \geq \frac{1}{2} m x_2^2 + \frac{1}{2} k_0 x_1^2 \geq c \|x\|^2$$

for $c = \min \left\{ \frac{1}{2} m, \frac{1}{2} k_0 \right\}$. Then since $\{x : c\|x\| < l\}$ is bounded $\forall l > 0$, also Ω_l is bounded.

- $R = \left\{ x : x_2^2 |x_2| = 0 \right\} \cap \Omega_l = \left\{ \begin{bmatrix} x_1 & 0 \end{bmatrix}^{\mathrm{T}}, x_1 \in \mathbb{R} \right\} \cap \Omega_l$

Example: how to prove that $\bar{x} = 0$ is AS when $\dot{V}$ is sdn

- Now we show that the largest invariant set in R is $M = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$
  - By contradiction, if M contains $p = \begin{bmatrix} \alpha & 0 \end{bmatrix}^{\mathrm{T}}$, $\alpha \neq 0$, then for $x(0) = p$ one has
    $$\dot{x}_2(0) = -\frac{k_0}{m} x_1(0) - \frac{k_1}{m} x_1(0)^3$$
    and therefore $x_2(0 + \epsilon) \neq 0$

From LaSalle theorem, $\forall x(0) \in \Omega_l$ one has

$$\lim_{t \to +\infty} \phi(t, x(0)) = \begin{bmatrix} 0 & 0 \end{bmatrix}^{\mathrm{T}}$$

Then $\bar{x} = 0$ is stable (from Lyapunov theorem) and Ω_l is a region of attraction $\Rightarrow$ $\bar{x} = 0$ is AS

Example: how to estimate a region of attraction of $\bar{x} = 0$

System

$$\dot{x}_1 = x_1 \left(x_1^2 + x_2^2 - 2 \right) - 4 x_1 x_2^2$$
$$\dot{x}_2 = 4 x_1^2 x_2 + x_2 \left(x_1^2 + x_2^2 - 2 \right)$$

We have already shown that

- $\bar{x} = 0$ is an equilibrium state
- $V(x) = x_1^2 + x_2^2$ is pd in $B_{\sqrt{2}}(0)$
- $\dot{V}(x) = 2 \left(x_1^2 + x_2^2 \right) \left(x_1^2 + x_2^2 - 2 \right)$ is nd in $B_{\sqrt{2}}(0)$
- ... therefore $\bar{x} = 0$ is AS

Example: how to estimate a region of attraction of $\bar{x} = 0$

Now we use LaSalle theorem

- Consider $\Omega_1 = \left\{x : x_1^2 + x_2^2 < 1\right\}$ that is bounded
- $R = \left\{x : 2(x_1^2 + x_2^2)(x_1^2 + x_2^2 - 2) = 0\right\} \cap \Omega_1 = \left\{\begin{bmatrix}0 & 0\end{bmatrix}^{\mathrm{T}}\right\}$
- $M = R$

From LaSalle theorem, $\Omega_1$is a region of attraction of $\bar{x} = 0$

Remarl

Also Ω_l, $l \in [1, 2]$ is a region of attraction of $\bar{x} = 0$. However, Ω_l, $l > 2$ is not a regione of attraction of $\bar{x} = 0$ (check it at home !)

Global LaSalle theorem

Theorem

Let $V(x) \in \mathcal{C}^1$ be a scalar function such that

- $V(x)$ is radially unbounded
- $\dot{V}(x) \leq 0$, $\forall x \in \mathbb{R}^n$

Define $R = \left\{ x : \dot{V}(x) = 0 \right\}$ and let M be the largest invariant set in R.
Then, $\forall x(0) \in \mathbb{R}^n$ one has $\phi(t, x(t)) \to M$ for $t \to +\infty$

Remark

V radially unbounded $\Rightarrow$ sets Ω_l are bounded, $\forall l > 0$

Example

System

$$\dot{x}_1 = -(x_2 - a)\, x_1 \quad a \in \mathbb{R}$$
$$\dot{x}_2 = \gamma x_1^2 \quad \gamma > 0$$

Computation of equilibrium states

$$\begin{cases} 0 = -(\bar{x}_2 - a)\, \bar{x}_1 \\ 0 = \gamma \bar{x}_1^2 \end{cases} \Rightarrow \text{ all } \bar{x} \in X = \left\{ \begin{bmatrix} 0 \\ \bar{x}_2 \end{bmatrix}, \bar{x}_2 \in \mathbb{R} \right\} \text{ are equilibria}$$

Problem

Show that all state trajectories converge to X

Example

System

$$\dot{x}_1 = -\left(x_2 - a\right)x_1 \quad a \in \mathbb{R}$$
$$\dot{x}_2 = \gamma x_1^2 \quad \gamma > 0$$

Candidate function V

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2\gamma}(x_2 - b)^2 \text{ for } b > a.$$

- It is radially unbounded
- It is not psd since $V(0) \neq 0$

$$\dot{V}(x) = x_1\left(-\left(x_2 - a\right)x_1\right)\right) + \frac{x_2 - b}{\gamma}\gamma x_1^2 =$$
$$= -x_1^2\left(x_2 - a\right) + x_1^2\left(x_2 - b\right) = -x_1^2(b - a) \leq 0,\ \forall x \in \mathbb{R}^2$$

Esempio

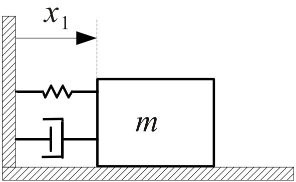
Application of LaSalle theorem

$R = \left\{ x : \dot{V}(x) = 0 \right\} = \left\{ \begin{bmatrix} x_1 & x_2 \end{bmatrix}^{\mathrm{T}} : -x_1^2 = 0 \right\} = \left\{ \begin{bmatrix} 0 & x_2 \end{bmatrix}^{\mathrm{T}}, x_2 \in \mathbb{R} \right\} = X$

Then $M = R$.

LaSalle theorem (global version) $\Rightarrow$ all state trajectories converge to X as $t \to +\infty$

Example: how to prove that $\bar{x} = 0$ is GAS when $\dot{V} \leq 0$



Model

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \underbrace{-b(x_2)}_{\text{damping}} \underbrace{-x_1^3}_{\text{elastic force}}$$

where b is a $\mathcal{C}^1$ function verifying

$$x_2 b(x_2) > 0 \text{ for } x_2 \neq 0 \qquad (1)$$

From (1) one has $b(0) = 0 \rightarrow \bar{x} = 0$ is an equilibrium state

Candidate Lyapunov function: $V(x) = \frac{1}{2}x_2^2 + \int_0^{x_1} y^3\, dy$

- $V(x)$ is pd, radially unbounded and of class $\mathcal{C}^1$
- $\dot{V}(x) = x_1^3 x_2 + x_2\left(-b(x_2) - x_1^3\right) = -x_2 b(x_2)$. From (1), $\dot{V}$ is gnsd.

Example: how to prove that $\bar{x} = 0$ is GAS when $\dot{V} \leq 0$

Application of Lyapunov and LaSalle theorems

- $R = \left\{ x : \dot{V}(x) = 0 \right\} = \left\{ \begin{bmatrix} x_1 & 0 \end{bmatrix}^{\mathrm{T}}, x_1 \in \mathbb{R} \right\} = X.$
- Computation of M (the largest invariant set in R)
  If $x(0) \in R$ then $\dot{x}_2(0) = -x_1(0)^3$. Therefore
  $x_2(t) = 0, \forall t \geq 0 \Leftrightarrow x_1(0) = 0$. It follows that $M = \left\{ \begin{bmatrix} 0 & 0 \end{bmatrix}^{\mathrm{T}} \right\}$

Conclusions:

- V is pd and $\dot{V}$ is nsd in all $B_\delta(0)$. From Lyapunov stability theorem, $\bar{x} = 0$ is stable
- V is radially unbounded and $\dot{V} \leq 0$, $\forall x \in \mathbb{R}^2$. From LaSalle theorem (global version) the region of attraction of $\bar{x} = 0$ is $\mathbb{R}^2$

We conclude that $\bar{x} = 0$ is GAS

Lyapunov theory for LTI systems

LTI system

$$\dot{x} = Ax + Bu \quad x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^m \tag{2}$$

Review

- Stability of $\bar{x} = 0$ does not depend upon inputs and one can study the stability of the origin of
$$\dot{x} = Ax \tag{3}$$
- If $\bar{x} = 0$ of (3) is stable/AS/unstable, all equilibrium states of (2) due to a constant input have the same property

For LTI systems one says that "the system is stable".

Next

Study how Lyapunov theory looks like when applied to linear systems

Review: positive definite matrices

Definition

A matrix $M \in \mathbb{R}^{n\times n}$ is

(a) positive definite (pd) if $x \neq 0 \Rightarrow x^{\mathrm{T}}Mx > 0$. Notation: $M > 0$

(b) positive semidefinite (psd) if $x^{\mathrm{T}}Mx \geq 0,\ \forall x \in \mathbb{R}^n$. Notation: $M \geq 0$

(c) negative definite/semidefinite (nd/nsd) if $-M$ is pd/psd. Notation: $M < 0 / M \leq 0$

Notation: $M_1 > M_2$ means $M_1 - M_2 > 0$

Quadratic functions $x^{\mathrm{T}}Mx$

Decomposition

$$M = \underbrace{\frac{M + M^{\mathrm{T}}}{2}}_{\text{symmetric part}} + \underbrace{\frac{M - M^{\mathrm{T}}}{2}}_{\text{antisymmetric part}}$$

Review: quadratic functions

Remark: the quadratic form of the antisymmetric part is zero

$$x^{\mathrm{T}}\frac{M-M^{\mathrm{T}}}{2}x = x^{\mathrm{T}}\frac{M}{2}x - x^{\mathrm{T}}\frac{M^{\mathrm{T}}}{2}x = x^{\mathrm{T}}\frac{M}{2}x - \left(x^{\mathrm{T}}\frac{M^{\mathrm{T}}}{2}x\right)^{\mathrm{T}} = 0$$

Since

$$x^{\mathrm{T}}Mx = x^{\mathrm{T}}\frac{M+M^{\mathrm{T}}}{2}x + x^{\mathrm{T}}\frac{M-M^{\mathrm{T}}}{2}x$$

one can assume without loss of generality that M is symmetric.

Properties of the quadratic function $x^{\mathrm{T}}Mx$

- from (a) and (b) one has
  - if $M > 0$, $V(x) = x^{\mathrm{T}}Mx$ is a pd function
  - if $M \geq 0$, $V(x) = x^{\mathrm{T}}Mx$ is a psd function

Review: quadratic functions

Properties of the quadratic function $x^{\mathrm{T}} Mx$

- a symmetric matrix M has real eigenvalues
- If $M > 0$, defining $\lambda_{min}(M)$ and $\lambda_{max}(M)$ as the minimum and maximum eigenvalue of M, respectively, one has

$$\lambda_{min}(M)\|x\|^2 \leq x^{\mathrm{T}} Mx \leq \lambda_{max}(M)\|x\|^2$$

- More in general, $\|M\| = \lambda_{max}(M)$ is a norm in the space of positive-definite symmetric matrices. Moreover $\|Mx\| \leq \|M\|\|x\|$

Lyapunov functions for LTI systems

$$\dot{x} = Ax \tag{4}$$

Candidate Lyapunov function: $V(x) = x^{\mathrm{T}} P x$, $P > 0$ (P symmetric)

- $V(x)$ is quadratic, gpd and radially unbounded
- $\dot{V} = \dot{x}^{\mathrm{T}} P x + x^{\mathrm{T}} P \dot{x} = x^{\mathrm{T}} \left(A^{\mathrm{T}} P + PA \right) x$

If $A^{\mathrm{T}} P + PA < 0$, i.e. there is $Q > 0$ symmetric such that

$$A^{\mathrm{T}} P + PA = -Q \tag{5}$$

then $\dot{V}(x)$ is gnd for (4) and $\bar{x} = 0$ is GAS.

Definition

(5) is called Lyapunov equation

Problem

Is it also true that if $\bar{x} = 0$ is AS then there is P such that $A^{\mathrm{T}} P + PA < 0$?

Lyapunov theorem for LTI systems

Theorem

A necessary and sufficient condition for the asymptotic stability of an LTI system is that for all symmetric matrices $Q > 0$ the only matrix P that solves the Lyapunov equation is symmetric and $P > 0$. In this case $V(x) = x^{\mathrm{T}} P x$ is a Lyapunov function and $\dot{V}(x) = -x^{\mathrm{T}} Q x$.

Remarks

- For LTI systems it is enough to consider quadratic Lyapunov functions !
- Example of "converse Lyapunov theorem": if $\bar{x} = 0$ is AS, then there is a Lyapunov function
  - converse Lyapunov theorems exisit just for special classes of NL systems
- Algorithm:
  - choose $Q > 0$ (e.g. $Q = I$)
  - solve $A^{\mathrm{T}} P + PA = -Q$ (linear systems in the entries of the symmetric matrix P)
  - the LTI system is AS if and only if $P > 0$

Lyapunov theorem for LTI systems

Proof

We have already seen that for a given P, if there is Q such that $A^{\mathrm{T}}P + PA = -Q$ the system is AS.
We now show that if the system is AS then, given a symmetric matrix $Q > 0$, the equation $A^{\mathrm{T}}P + PA = -Q$ has only one solution P and $P > 0$.
Define

$$P = \int_0^{+\infty} e^{A^{\mathrm{T}}t} Q e^{At}\, dt$$

Since $\det(e^{At}) \neq 0$, for all $A \in \mathbb{R}^{n\times n}$, one has that P is pd. Moreover P is symmetric (since $e^{A^{\mathrm{T}}t} = \left(e^{At}\right)^{\mathrm{T}}$ and $Q = Q^{\mathrm{T}}$). One has

$$A^{\mathrm{T}}P + PA = \int_0^{+\infty} \left(A^{\mathrm{T}} e^{A^{\mathrm{T}}t} Q e^{At} + e^{A^{\mathrm{T}}t} Q e^{At} A\right) dt =$$

$$= \int_0^{+\infty} \frac{d}{dt}\left(e^{A^{\mathrm{T}}t} Q e^{At}\right) dt = \left[e^{A^{\mathrm{T}}t} Q e^{At}\right]_0^{+\infty}$$

Lyapunov theorem for LTI systems

Since all eigenvalues of A have strictly negative real part, $e^{At} \to 0$ as $t \to +\infty$ and then $\left[e^{A^{\mathrm{T}}t} Q e^{At}\right]_0^{+\infty} = -Q$ that shows $A^{\mathrm{T}}P + PA = -Q$.

For proving that P is the only solution to the Lyapunov equation, assume that P_1 is another solution. One has

$$
\begin{aligned}
P_1 &= -\left[e^{A^{\mathrm{T}}t} P_1 e^{At}\right]_0^{+\infty} = -\int_0^{+\infty} \frac{d}{dt}\left(e^{A^{\mathrm{T}}t} P_1 e^{At}\right) dt = \\
&= -\int_0^{+\infty} e^{A^{\mathrm{T}}t} \left(A^{\mathrm{T}} P_1 + P_1 A\right) e^{At}\, dt = \\
&= \int_0^{+\infty} e^{A^{\mathrm{T}}t} Q e^{At}\, dt = P
\end{aligned}
$$

Example

$$\begin{cases} \dot{x}_1 = 4x_2 + u \\ \dot{x}_2 = -8x_1 - 12x_2 \end{cases} \quad \Rightarrow A = \begin{bmatrix} 0 & 4 \\ -8 & -12 \end{bmatrix}$$

Check if the systems is AS using the Lyapunov theorem

Choose $Q = I$. Let $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$ where $p_{12} = p_{21}$.

Solve the Lyapunov equation: $A^{\mathrm{T}}P + PA = -Q$

$$\begin{bmatrix} 0 & -8 \\ 4 & -12 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 4 \\ -8 & -12 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -8p_{12} & -8p_{22} \\ 4p_{11} - 12p_{12} & 4p_{12} - 12p_{22} \end{bmatrix} + \begin{bmatrix} -8p_{12} & 4p_{11} - 12p_{12} \\ -8p_{22} & 4p_{12} - 12p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Example

Imposing identity between corresponding elements in the matrices one gets the linear system

$$\begin{cases} -8p_{12} - 8p_{12} = -1 \\ -8p_{22} + 4p_{11} - 12p_{12} = 0 \\ 4p_{11} - 12p_{12} - 8p_{22} = 0 \\ 4p_{12} - 12p_{22} + 4p_{12} - 12p_{22} = -1 \end{cases}$$

Remark: the third equation is redundant because matrices $A^{\mathrm{T}}P + PA$ and Q are symmetric.

Solving the linear systems one gets

$$P = \frac{1}{16}\begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow \text{ eigenvalues: } 0.0477,\ 0.3273$$

Since $P > 0$ the systems is AS.

LTI systems: GAS and GES

If $V(x) = x^{\mathrm{T}}Px$, $P > 0$ is a Lyapunov function certifying the asymptotic stability if $\bar{x} = 0$ and $\dot{V}(x) = -x^{\mathrm{T}}Qx$, $Q > 0$, then

(a) $\lambda_{min}(P)\|x\|^2 \leq V(x) \leq \lambda_{max}(P)\|x\|^2, \forall x \in \mathbb{R}^n$

(b) $\dot{V}(x(t)) \leq -\lambda_{min}(Q)\|x\|^2, \forall x \in \mathbb{R}^n$

From Lyapunov theorem on global exponential stability, (a)+(b)$\Rightarrow$ $\bar{x} = 0$ is GES

Corollary

An LTI system is AS if and only if it is GES. Moreover, if for a symmetric $Q > 0$, the symmetric matrix $P > 0$ solves $A^{\mathrm{T}}P + PA = -Q$, then the scalar $\gamma = \dfrac{\lambda_{min}(Q)}{2\lambda_{max}(P)}$ is an estimate of the convergence rate.
The best estimate of the convergence rate can be obtained setting $Q = I$.

Previous example

$$\begin{cases} \dot{x}_1 = 4x_2 + u \\ \dot{x}_2 = -8x_1 - 12x_2 \end{cases} \Rightarrow A = \begin{bmatrix} 0 & 4 \\ -8 & -12 \end{bmatrix}$$

For $Q = I$ we have found

$$P = \frac{1}{16}\begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix}$$

Eigenvalues of P are 0.0477 and 0.3273. Then $\gamma = \frac{\lambda_{min}(Q)}{2\lambda_{max}(P)} = 1.5276$

This implies, $\forall x(0) \in \mathbb{R}^2$, $\|x(t)\| \leq c\|x(0)\|e^{-1.5276t}$, where $c > 0$ is a suitable constant.

How one can build Lyapunov functions for NL systems ?

$$\dot{x} = f(x), \quad f \in \mathcal{C}^1, \quad \bar{x} = 0: \text{ equilibrium state}$$

There is no general procedure

Common alternatives:

- physical energy of the system, possibly leaving free parameters that can be tuned to make $\dot{V}$ nd/nsd
- try with $V(x) = x^{\mathrm{T}} P x$, where P is symmetric and pd
- *ad hoc* methods for special classes of systems

The indirect Lyapunov method

Is the test for the stability of an equilibrium based on the linearized system

Nonlinear system

$$\Sigma : \dot{x} = f(x)$$

$\bar{x} = 0$: equilibrium state, $f \in \mathcal{C}^1$

Linearized system

$$\Sigma_l : \dot{\delta x} = A\delta x$$

$$A = D_x f(x)\Big|_{x=0}$$

Theorem

The equilibrium state $\bar{x}$ of Σ

- is AS if all eigenvalues of Σ_l have real part < 0
- is unstable if at least an eigenvalue of Σ_l has real part > 0

hHen the origin is AS, the proof of the theorem provides an estimate of the region of attraction

Estimation of the region of attraction

Partial proof

From the Lyapunov theorem for LTI systems we know that if the system is AS, then given the symmetric matrix $Q > 0$, the Lyapunov equation

$$A^{\mathrm{T}} P + PA = -Q$$

has only one solution $P > 0$ where P is symmetric. Let $V(x) = x^{\mathrm{T}} P x$ be a candidate Lyapunov function for the NL system $\dot{x} = f(x)$

$$\dot{V}(x) = x^{\mathrm{T}} P \dot{x} + \dot{x}^{\mathrm{T}} P x = x^{\mathrm{T}} P f(x) + f(x)^{\mathrm{T}} P x$$

Write f as

$$f(x) = Ax + g(x)$$

where $A = D_x f(x)\Big|_{x=0}$ and $g(x)$ is the remainder of the Taylor series of f around the origin.

Estimation of the region of attraction

Then,

$$\frac{\|g(x)\|}{\|x\|} \to 0 \text{ per } \|x\| \to 0 \tag{6}$$

Let us consider again $\dot{V}$

$$\begin{aligned}\dot{V}(x) &= x^{\mathrm{T}} P \left(Ax + g(x)\right) + \left(x^{\mathrm{T}} A^{\mathrm{T}} + g(x)^{\mathrm{T}}\right) Px = \\ &= x^{\mathrm{T}} \left(A^{\mathrm{T}} P + PA\right) x + 2x^{\mathrm{T}} Pg(x) = -x^{\mathrm{T}} Qx + 2x^{\mathrm{T}} Pg(x)\end{aligned}$$

From (6) one has

$$\forall \gamma > 0, \exists r > 0 : \|g(x)\| < \gamma \|x\|, \ \forall x \in B_r(0) \tag{7}$$

and then, for $x \in B_r(0)$

$$\begin{aligned}\dot{V}(x) &\leq -x^{\mathrm{T}} Qx + |2x^{\mathrm{T}} Pg(x)| \leq -x^{\mathrm{T}} Qx + 2\|x\|\|P\|\|g(x)\| \\ &\underbrace{<}_{x \in B_r(0)} -x^{\mathrm{T}} Qx + 2\gamma \lambda_{max}(P)\|x\|^2 \leq -\left[\lambda_{min}(Q) - 2\gamma\lambda_{max}(P)\right] \|x\|^2\end{aligned}$$

Estimation of the region of attraction

For a sufficiently small γ, one has $\lambda_{min}(Q) - 2\gamma\lambda_{max}(P) > 0$. We have then shown that in $B_r(0)$ (where r is such that (7) holds) V is pd and $\dot{V}$ is nd. Form LaSalle thereom, we can show that any level set $\Omega_l = \{x : V(x) < l\}$ included in $B_r(0)$ is a region of attraction for $\bar{x} = 0$.

Algorithm

- decompose f as $f(x) = Ax + g(x)$, $A = D_x f(x)\Big|_{x=0}$
- pick $Q > 0$ and solve $A^{\mathrm{T}}P + PA = -Q$
- compute γ such that $\dfrac{\lambda_{min}(Q)}{2\lambda_{max}(P)} > \gamma$

Let $B_r(0)$ such that $x \in B_r(0) \Rightarrow \|g(x)\| < \gamma\|x\|$. Any level set $\Omega_l = \left\{x : x^{\mathrm{T}}Px < l\right\}$ included in $B_r(0)$ is a region of attraction.
Remark: the estimation of the maximal region of attraction is *conservative*

Example

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - x_1^3 - 2x_2 \end{cases}$$

Show that the origin is AS and compute a region of attraction.

Stability analysis using the linearized system

$$\begin{cases} \dot{\delta x}_1 = \delta x_2 \\ \dot{\delta x}_2 = -\delta x_1 - 3\bar{x}_1^2 \delta x_1 - 2\delta x_2 \end{cases} \Rightarrow A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

$$\det(\lambda I - A) = 0 \;\Rightarrow\; \det\left(\begin{bmatrix} \lambda & 1 \\ 1 & \lambda + 2 \end{bmatrix}\right) = 0 \;\Rightarrow\; \lambda^2 + 2\lambda + 1 = 0$$

Eigenvalues: $\lambda_1 = \lambda_2 = -1$. From the Lyapunov indirect method, the origin is AS[1]

[1] In particular, from Hartman-Grobman theorem, the origin is a stable node.

Example

Estimate of the region of attraction

$$f(x) = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -x_1^3 \end{bmatrix} \Rightarrow g(x) = \begin{bmatrix} 0 \\ -x_1^3 \end{bmatrix}$$

Choose (arbitrarily) $Q = I$ and solve the Lyapunov equation
$A^{\mathrm{T}} P + PA = -Q$

Let $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$ with $p_{12} = p_{21}$.

$$\begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -12 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -p_{12} & -p_{22} \\ p_{11} - 2p_{12} & p_{12} - 2p_{22} \end{bmatrix} + \begin{bmatrix} -p_{12} & p_{11} - 2p_{12} \\ -p_{22} & p_{12} - 2p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Imposing identity between corresponding elements in the matrices one gets the linear system

Example

$$\begin{cases} -2p_{12} = -1 \\ -p_{22} + p_{11} - 2p_{12} = 0 \\ 2p_{12} - 4p_{22} = -1 \end{cases}$$

and therefore

$$P = \frac{1}{2}\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \;\Rightarrow\; \text{eigenvalues: } 1 \pm \frac{\sqrt{2}}{2}$$

Choose γ such that $\dfrac{\lambda_{min}(Q)}{2\lambda_{max}(P)} > \gamma$

$\lambda_{max}(P) = 1 + \dfrac{\sqrt{2}}{2}$, $\lambda_{min}(Q) = 1$

$$\frac{\lambda_{min}(Q)}{2\lambda_{max}(P)} = \frac{1}{2+\sqrt{2}} > \gamma \;\Rightarrow\; \text{set (arbitrarily) } \gamma = \frac{1}{4}$$

Example

Conclusion

Let r be such that $x \in B_r(0) \Rightarrow \|g(x)\| < \gamma \|x\|$.

$$\|g(x)\| < \gamma \|x\| \;\Rightarrow\; \left\| \begin{bmatrix} 0 \\ x_1^3 \end{bmatrix} \right\| < \frac{1}{4}\|x\| \;\Rightarrow\; |x_1^3| < \frac{1}{4}\sqrt{x_1^2 + x_2^2}$$

Any ellipsoid

$$\Omega_l = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : \begin{bmatrix} x_1 & x_2 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} < l \right\}$$

included in $B_r(0)$ is a region of attraction.