Nonlinear systems

State feedback control

G. Ferrari Trecate

Dipartimento di Ingegneria Industriale e dell'Informazione
Università degli Studi di Pavia

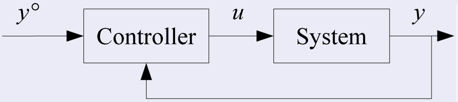
Advanced automation and control

Control schemes: output feedback

NL system

$$\dot{x} = f(x, u)$$
$$y = h(x, u)$$

Output feedback

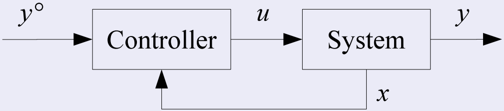


- $y^o(t)$: setpoint
- $u(t)$: control variable

Output feedback: the controller uses the setpoint and a measure of the output to compute the control variable.

Control schemes: state feedback

State feedback



State feedback: the controller uses the setpoint and a measure of the state for computing the control variable.

Pros

Since $y = h(x, u)$ the output can only contain less information than the state. Therefore, state feedback usually guarantees better performances.

Cons

The state must be measured and this is not always the case. Otherwise the state must be estimated from measurements of u and y.

Control schemes

Control problems:

- Regulation: make a desired equilibrium state AS
- Tracking: make the system output track, according to given criteria, special classes of setpoints y^o

In both problems disturbances must be also attenuated or rejected.

Taxonomy of controllers

- Static: the controller is a static system (e.g. proportional control $u(t) = k(y(t) - y^o(t))$
- Dynamic: the controller is a dynamic system (e.g. PID controllers)

Topics that will be covered in this class

Mainly static state-feedback controllers for NL invariant and SISO systems

Stabilization of the origin

Regulation problem

System

$$\dot{x} = f(x, u)$$

Design the control law $u(t) = k(x(t))$ $k : \mathbb{R}^n \to \mathbb{R}$ such that the origin of the closed-loop system

$$\dot{x} = f(x, k(x))$$

is an AS equilibrium state

Stabilization of a generic equilibrium $(\bar{x}, \bar{u})$

$$0 = f(\bar{x}, \bar{u})$$

Define the variables $\tilde{x} = x - \bar{x}$ and $\tilde{u} = u - \bar{u}$. Define also $\tilde{f}(\tilde{x}, \tilde{u}) = f(\bar{x} + \tilde{x}, \bar{u} + \tilde{u})$. Then one has

$$\dot{\tilde{x}} = \tilde{f}(\tilde{x}, \tilde{u})$$

where $\tilde{f}(0, 0) = 0$ (i.e. we are in the previous case)

Stabilization of the origin

If we design $\tilde{u}(t) = k(\tilde{x}(t))$ stabilizing the origin of the system in the new variables, the controller

$$u = \bar{u} + \tilde{u} = \bar{u} + k(\tilde{x}) = \bar{u} + k(x - \bar{x})$$

stabilizes the equilibrium state $\bar{x}$ of the original system

Remarks

- Several industrial systems are designed to work around a *nominal operation point* $(\bar{x}, \bar{u})$ that must be stabilized by the controller
- Stabilization of the origin is also at the core of the design of controllers for tracking problems
- For the sake of simplicity, in most cases we will neglect the presence of disturbances

State-feedback controllers - LTI systems

Multi-input LTI system

$$\dot{x} = Ax + Bu, \quad x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^m$$

Control law (stabilizing $\bar{x} = 0$)

$$u(t) = Kx(t), \quad K \in \mathbb{R}^{m \times n}: \text{ to be designed}$$

Closed-loop system: $\dot{x} = (A + BK)\,x$

Eigenvalue Assignment (EA) problem

Compute, if possible, K such that the eigenvalues of $A + BK$ take prescribed values (real or in complex conjugate pairs)

Solution to the EA problem

Theorem

The EA problem can be solved if and only if the LTI system is reachable

Review

The system $\dot{x} = Ax + Bu$ is reachable if and only if the matrix

$$M_r = \left[\; B \; \middle| \; AB \; \middle| \; A^2B \; \middle| \; \cdots \; \middle| \; A^{n-1}B \; \right]$$

has maximal rank.

- M_r: reachability matrix
- Terminology: the pair (A, B) is reachable

Solution to the EA problem - single input

Definition

Let $u(t) \in \mathbb{R}$. The pair (A, B) is in the canonical controllability form if

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b \end{bmatrix}, \;\; b \neq 0$$

Remarks

- If (A, B) is the canonical controllability form, then M_r has maximal rank by construction
- Let $p_A(\lambda)$ be the charachteristic polynomial of A. By construction, one has

$$p_A(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

Solution to the EA problem - single input

- Structure of the canonical controllability form

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \end{aligned}\right\} \leftarrow \text{ chain of } n-1 \text{ integrators}$$

$$\dot{x}_n = a(x) + bu \leftarrow \text{ the input acts on } \dot{x}_n$$

where $a(x) = -a_0 x_1 - a_1 x_2 - \ldots - a_{n-1} x_n$

Idea

If the LTI system is in the canonical controllability form, choose

$$u = \underbrace{\frac{1}{b}(-a(x))}_{\text{this cancels } a(x)} + \frac{1}{b}\tilde{u}$$

such that the auxiliary input $\tilde{u}$ assigns the closed-loop eigenvalues

Solution to the EA problem - single input

Algorithm

Let (A, B) be in canonical controllability form

- For given desired closed-loop eigenvalues $\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_n$, build up the polynomial

$$p^D(\lambda) = (\lambda - \tilde{\lambda}_1)(\lambda - \tilde{\lambda}_2) \cdots (\lambda - \tilde{\lambda}_n) = \lambda^n + \tilde{a}_{n-1}\lambda^{n-1} + \cdots + \tilde{a}_1\lambda + \tilde{a}_0$$

- Use

$$u = \frac{1}{b}(-a(x) + \tilde{a}(x))$$

where $\tilde{a}(x) = -\tilde{a}_0 x_1 - \tilde{a}_1 x_2 - \ldots - \tilde{a}_{n-1} x_n$.

Solution to the EA problem - single input

Closed-loop system

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \end{aligned}\right\} \text{chain of } n-1 \text{ integrators}$$

$$\dot{x}_n = \tilde{a}(x)$$

The matrix A is in the canonical controllability form: by construction $p^D(\lambda)$ is the closed-loop characteristic polynomial

Matrix K (gain matrix)

$$u = \frac{1}{b}(-a(x) + \tilde{a}(x)) =$$

$$= \frac{1}{b}((a_0 - \tilde{a}_0)\,x_1 + (a_1 - \tilde{a}_1)\,x_2 + \cdots + (a_{n-1} - \tilde{a}_{n-1})\,x_n) = Kx$$

with $K = \frac{1}{b}\begin{bmatrix}(a_0 - \tilde{a}_0) & (a_1 - \tilde{a}_1) & \cdots & (a_{n-1} - \tilde{a}_{n-1})\end{bmatrix}$

Solution to the EA problem - single input

How to solve the EA problem if the LTI system is not in the canonical controllability form ?

Lemma

If (A, B) is reachable, there is an invertible matrix T such that the equivalent system

$$\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u, \quad \hat{A} = TAT^{-1}, \hat{B} = TB$$

where $\hat{x} = Tx$, is in the canonical controllability form with $b = 1$.

Computation of T

$$\left.\begin{aligned} M_r &= \left[\begin{array}{c|c|c|c|c} B & AB & A^2B & \cdots & A^{n-1}B \end{array}\right] \\ \hat{M}_r &= \left[\begin{array}{c|c|c|c|c} \hat{B} & \hat{A}\hat{B} & \hat{A}^2\hat{B} & \cdots & \hat{A}^{n-1}\hat{B} \end{array}\right] = TM_r \end{aligned}\right\} \rightarrow T = \hat{M}_r M_r^{-1}$$

Solution to the EA problem - single input

Algorithm

Given A, B and the desired closed-loop charachteristic polynomial

$$p^D(\lambda) = \lambda^n + \tilde{a}_{n-1}\lambda^{n-1} + \cdots + \tilde{a}_1\lambda + \tilde{a}_0$$

1. compute M_r and verify that (A, B) is reachable
2. compute
$$p_A(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$
3. build[a] $\hat{A}$, $\hat{B}$ and $\hat{M}_r$. Compute $T = \hat{M}_r M_r^{-1}$
4. build[b] $\hat{K} = \begin{bmatrix}(a_0 - \tilde{a}_0) & (a_1 - \tilde{a}_1) & \cdots & (a_{n-1} - \tilde{a}_{n-1})\end{bmatrix}$
5. compute $K = \hat{K}T$ an set $u = Kx$

[a] $\hat{A}$ and $\hat{B}$ are in the canonical controllability form with $b = 1$. For the computation it is enough to know $p_A(\lambda)$.

[b] Controller design in the coordinates $\hat{x}$.

Ackermann's formula

In the previous algorithm one can avoid the use of $\hat{x}$ coordinates and design directly the controller K as a function of A and B.

Theorem

Let (A, B) be a reachable pair and let

$$p^D(\lambda) = \lambda^n + \tilde{a}_{n-1}\lambda^{n-1} + \cdots + \tilde{a}_1\lambda + \tilde{a}_0$$

be the desired closed-loop polynomial. Then, the controller $u = Kx$ such that the charachteristic polynomial of $A + BK$ is $p^D(\lambda)$ is given by

$$K = -\begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix} M_r^{-1} p^D(A) \tag{1}$$

Equation (1) is called the Ackermann's formula

Proof of the Ackermann's formula

Being $\hat{A}$ in in the canonical controllability form, one can verify that the first row of $\hat{A}^i$, $1 \leq i < n$ is composed by zero entries except the entry in position $(1, i+1)$ that is 1

$$\hat{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ x & x & x & \cdots & x & x \end{bmatrix} \quad \hat{A}^2 = \begin{bmatrix} 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ x & x & x & \cdots & x & x \\ x & x & x & \cdots & x & x \end{bmatrix}$$

$$\hat{A}^{n-1} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ x & x & x & \cdots & x & x \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x & x & x & \cdots & x & x \\ x & x & x & \cdots & x & x \\ x & x & x & \cdots & x & x \end{bmatrix}$$

Proof of the Ackermann's formula

Since from the Caley-Hamilton theorem one has
$\hat{A}^n + a_{n-1}\hat{A}^{n-1} + \cdots + a_1\hat{A} + a_0 I = 0$, it follows that

$$p^D(\hat{A}) = \begin{bmatrix} (\tilde{a}_0 - a_0) & (\tilde{a}_1 - a_1) & (\tilde{a}_2 - a_2) & \cdots & (\tilde{a}_{n-1} - a_{n-1}) \\ x & x & x & \cdots & x \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x & x & x & \cdots & x \\ x & x & x & \cdots & x \\ x & x & x & \cdots & x \end{bmatrix}$$

and therefore the controller $\hat{K}$ we have computed before is given by

$$\hat{K} = -\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} p^D(\hat{A})$$

Proof of the Ackermann's formula

Since $\hat{A} = TAT^{-1}$, $T = \hat{M}_r M_r^{-1}$, $K = \hat{K}T$ one has

$$K = -\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} p^D(\hat{A})T = \tag{2}$$
$$= -\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} Tp^D(A)T^{-1}T = \tag{3}$$
$$= -\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \hat{M}_r M_r^{-1} p^D(A) \tag{4}$$

For getting rid of $\hat{M}_r$, we observe that, since $\hat{A}$ and $\hat{B}$ are in canonical controllability form, one has

$$\hat{M}_r = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & x \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 1 & \cdots & x & x \\ 0 & 1 & x & \cdots & x & x \\ 1 & x & x & \cdots & x & x \end{bmatrix}$$

Therefore, $-\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \hat{M}_r = -\begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}$.

Solution to the EA problem

Remarks

1. The EA algorithm can be generalized to MIMO systems.
2. Closed-loop eigenvalues are usually chosen on the basis of performance requirements (for instance raising time, settling time and maximal overshoot of the closed-loop step response)

Generalization

If (A, B) is not reachable, using a controller $u = Kx$ only reachable eigenvalues are modified. Therefore, in order to guarantee closed-loop asymptotic stability, it is necessary that unreachable eigenvalues have real part < 0.

Example

Problem

$$\dot{x}_1 = x_1 + x_2 + u$$
$$\dot{x}_2 = u$$

Compute a state-feedback controller such that the closed-loop system has all eigenvalues equal to -2

Desired closed-loop charachteristic polynomial

$$p^D(\lambda) = (\lambda + 2)^2 = \lambda^2 + \underbrace{4}_{\tilde{a}_1}\lambda + \underbrace{4}_{\tilde{a}_0}$$

Computation of M_r

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \; B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow M_r = \left[\; B \mid AB \; \right] = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

M_r is full rank $\Rightarrow$ EA problem can be solved

Example

Computation of $p_A(\lambda)$

$$p_A(\lambda) = \det\left(\begin{bmatrix} \lambda - 1 & -1 \\ 0 & \lambda \end{bmatrix}\right) = \lambda^2 + \underbrace{(-1)}_{a_1}\lambda + \underbrace{0}_{a_0}$$

Build $\hat{A}$, $\hat{B}$, $\hat{M}_r$ and T

$$\hat{A} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \;\; \hat{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \hat{M}_r = \left[\; \hat{B} \;\middle|\; \hat{A}\hat{B} \;\right] = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$T = \hat{M}_r M_r^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Build $\hat{K}$

$$\hat{K} = \begin{bmatrix} (a_0 - \tilde{a}_0) & (a_1 - \tilde{a}_1) \end{bmatrix} = \begin{bmatrix} 0 - 4 & -1 - 4 \end{bmatrix} = \begin{bmatrix} -4 & -5 \end{bmatrix}$$

Example

Build K

$$K = \hat{K}T = \begin{bmatrix} -\frac{9}{2} & -\frac{1}{2} \end{bmatrix}$$

Check the result

$$A + BK = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} -\frac{9}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -3.5 & 0.5 \\ -4.5 & -0.5 \end{bmatrix}$$

Eigenvalues of $A + BK$: $\lambda_1 = \lambda_2 = -2$

Example

Using Ackermann's formula

$$K = -\begin{bmatrix} 0 & 1 \end{bmatrix} M_r^{-1} p^D(A)$$

$$p^D(A) = A^2 + 4A + 4I = \begin{bmatrix} 9 & 5 \\ 0 & 4 \end{bmatrix}$$

$$M_r = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \quad \Rightarrow \quad M_r^{-1} = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$K = -\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 4 \\ \frac{9}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{9}{2} & -\frac{1}{2} \end{bmatrix}$$

Regulation based on linearization

Regulation problem

System

$$\Sigma: \ \dot{x} = f(x,u) \quad f(0,0) = 0, \ f \in \mathcal{C}^1$$

Design the control law $u(t) = k(x(t))$ such that $(\bar{x}, \bar{u}) = (0,0)$ is an AS equilibrium for the closed-loop system.

Controller design based on linearization

Linarization of Σ about $\bar{x} = 0$, $\bar{u} = 0$

$$\Sigma_{lin}: \ \dot{\delta x} = A\delta x + B\delta u$$

If (A, B) is reachable, design $\delta u = K\delta x$ that stabilizes the origin of Σ_{lin}.
For the closed-loop system

$$\Sigma_{cl}: \ \dot{x} = f_{cl}(x) = f(x, K\delta x + \bar{u}) = f(x, Kx)$$

$\bar{x} = 0$ is an equilibrium state.

Regulation based on linearization

Stability analysis for the closed-loop system

Linearization of Σ_{cl} about $\bar{x} = 0$

$$\Sigma_{cl,lin}: \; \dot{\delta x} = D_x f(x,u)\Big|_{\substack{x=0\\u=0}} \delta x + D_u f(x,u)\Big|_{\substack{x=0\\u=0}} K\delta x = (A + BK)\,\delta x$$

All eigenvalues of $A + BK$ have real part $< 0 \Rightarrow$ the origin of $\Sigma_{cl,lin}$ is AS $\Rightarrow$ the origin of Σ_{cl} is AS.

Computation of a region of attraction of $\bar{x} = 0$: as in the indirect Lyapunov method

Flowchart of the procedure

Open-loop system

$$\dot{x} = f(x, u), \ f(0,0) = 0$$

↓

Linearized open-loop system

$$\dot{\delta x} = A\delta x + B\delta u$$

↓

Controller design: $\delta u = K\delta x$

↓

Closed-loop system

$$\dot{x} = f(x, Kx)$$

↓

Linearized closed-loop system

$$\dot{\delta x} = (A + BK)\delta x$$

→

Analysis of the region of attraction of $\bar{x} = 0$ for the closed-loop system

Example

Problem

$$\dot{x} = x^2 + u$$

Design a state-feedback controller that makes $\bar{x} = 0$ AS and compute a region of attraction

Linearization about the equilibrium $\bar{x} = 0$ and $\bar{u} = 0$

$$\Sigma_{lin}: \ \dot{\delta x} = 2\bar{x}\delta x + \delta u \quad \Rightarrow \quad \dot{\delta x} = \delta u$$

Σ_{lin} is reachable: design $\delta u = K\delta x$ such that Σ_{lin} is AS.

$$\dot{\delta x} = K\delta x \quad \Rightarrow \quad \text{pick } K < 0, \text{ e.g. } K = -1$$

Example

Computation of a region of attraction

Closed-loop system

$$\Sigma_{cl}: \ \dot{x} = f_{cl}(x) = f(x, Kx) = x^2 - x$$

Decomposition:

$$\Sigma_{cl}: \ \dot{x} = (A + BK)x + g(x), \text{ where } A + BK = -1, \ g(x) = x^2$$

Choose (arbitrarily) $Q = 1$ and solve $(A + BK)^{\mathrm{T}} P + P(A + BK) = -Q$

$$-2P = -1 \quad \Rightarrow \quad P = \frac{1}{2}$$

Choose (arbitrarily) $\gamma < \dfrac{\lambda_{min}(Q)}{2\lambda_{max}(P)} = 1$. For instance $\gamma = \dfrac{1}{2}$

Example

Let r be such that $x \in B_r(0) \Rightarrow \|g(x)\| < \gamma\|x\|$.

$$\|g(x)\| < \gamma\|x\| \;\Rightarrow\; x^2 < \frac{1}{2}|x| \;\Rightarrow\; r \leq \frac{1}{2}$$

Every interval

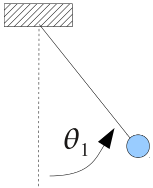
$$\Omega_l = \left\{ x : \frac{1}{2}x^2 < l \right\}$$

included in $B_r(0)$ is a region of attraction

One has $\Omega_l = (-\sqrt{2l}, \sqrt{2l})$ and, for $r = \frac{1}{2}$, one has the constraint $\sqrt{2l} \leq r = \frac{1}{2}$. Then, for $l = \frac{1}{8}$, a region of attraction is $\Omega_l = (-\frac{1}{2}, \frac{1}{2})$

Since we are dealing with a first-order system, from the graph of $f_{cl}(x)$ one can easily show that $(-\infty, 1)$ is the maximal region of attraction

Example



Damped pendulum

$$\dot{\theta}_1 = \theta_2$$
$$\dot{\theta}_2 = -\theta_2 - \sin(\theta_1) + \tau, \quad \tau = \text{input}$$

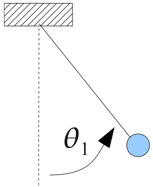
Problem

Design a state-feedback controller such that

- the state $\bar{x} = \begin{bmatrix} \frac{\pi}{3} & 0 \end{bmatrix}^{\mathrm{T}}$ is an AS equilibrium for the closed-loop system
- the linearized system about $\bar{x}$ has two eigenvalues equal to -1

Compute also a region of attraction of the equilibrium.

Example



Damped pendolum

$$\dot\theta_1 = \theta_2$$
$$\dot\theta_2 = -\sin(\theta_1) - \theta_2 + \tau, \quad \tau = \text{input}$$

Equilibrium input $\bar\tau$

$$0 = -\sin\left(\frac{\pi}{3}\right) + \bar\tau \quad \Rightarrow \quad \bar\tau = \sin\left(\frac{\pi}{3}\right)$$

Change of variables such that the origin is an equilibrium state for zero input: $x_1 = \theta_1 - \frac{\pi}{3}$, $x_2 = \theta_2$, $u = \tau - \bar\tau$

$$\Sigma : \begin{cases} \dot x_1 = x_2 \\ \dot x_2 = -\sin(x_1 + \frac{\pi}{3}) - x_2 + u + \sin(\frac{\pi}{3}) \end{cases}$$

Example

Linearization of Σ about the origin

$$\Sigma_{lin}: \begin{cases} \dot{\delta x}_1 = \delta x_2 \\ \dot{\delta x}_2 = -\cos(\frac{\pi}{3})\delta x_1 - \delta x_2 + \delta u \end{cases} \Rightarrow A = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & -1 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Design of the EA controller for Σ_{lin}

- Desired closed-loop charachteristic polynomial

$$p^D(\lambda) = (\lambda+1)^2 = \lambda^2 + \underbrace{2}_{\tilde{a}_1}\lambda + \underbrace{1}_{\tilde{a}_0}$$

- Computation of M_r

$$M_r = \left[\ B \mid AB\ \right] = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

M_r is full rank $\Rightarrow$ the EA problem can be solved

Example

- Computation of $p_A(\lambda)$

$$p_A(\lambda) = \det\left(\begin{bmatrix} \lambda & -1 \\ \frac{1}{2} & \lambda + 1 \end{bmatrix}\right) = \lambda^2 + \underbrace{(1)}_{a_1}\lambda + \underbrace{\frac{1}{2}}_{a_0}$$

- Canonical controllability form: build $\hat{A}$, $\hat{B}$, $\hat{M}_r$ and T

$$\hat{A} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & -1 \end{bmatrix} = A,\ \hat{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = B \Rightarrow \hat{M}_r = M_r$$

$$T = \hat{M}_r M_r^{-1} = I$$

- Design of $\hat{K}$ and $K = \hat{K}T$ (verify @ home with Ackermann's formula)

$$\hat{K} = \begin{bmatrix} (a_0 - \tilde{a}_0) & (a_1 - \tilde{a}_1) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - 1 & 1 - 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -1 \end{bmatrix} \quad (5)$$

$$K = \hat{K} \quad (6)$$

- Controller

$$\delta u = K\delta x = -\frac{1}{2}\delta x_1 - \delta x_2 \ \Rightarrow\ u = \delta u + \bar{u} = -\frac{1}{2}x_1 - x_2$$

Example

Computation of the region of attraction

Closed-loop system

$$\Sigma_{cl}:\begin{cases}\dot{x}_1 = x_2\\ \dot{x}_2 = -\sin(x_1+\frac{\pi}{3}) - x_2 - \frac{1}{2}x_1 - x_2 + \sin(\frac{\pi}{3}) = \\ \quad = -\sin(x_1+\frac{\pi}{3}) - \frac{1}{2}x_1 - 2x_2 + \sin(\frac{\pi}{3})\end{cases}$$

Decomposition: $\Sigma_{cl}: \ \dot{x} = (A+BK)x + g(x)$ where

$$A+BK = \begin{bmatrix}0 & 1\\ -1 & -2\end{bmatrix}, \quad g(x) = \begin{bmatrix}0\\ -\sin(x_1+\frac{\pi}{3}) + \frac{1}{2}x_1 + \sin(\frac{\pi}{3})\end{bmatrix}$$

Choose (arbitrarily) $Q = I$ and solve $(A+BK)^{\mathrm{T}}P + P(A+BK) = -Q$

$$\begin{bmatrix}0 & -1\\ 1 & -2\end{bmatrix}\begin{bmatrix}p_{11} & p_{12}\\ p_{12} & p_{22}\end{bmatrix} + \begin{bmatrix}p_{11} & p_{12}\\ p_{12} & p_{22}\end{bmatrix}\begin{bmatrix}0 & 1\\ -1 & -2\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$$

(... computed previously ...) $P = \frac{1}{2}\begin{bmatrix}3 & 1\\ 1 & 1\end{bmatrix} \ \Rightarrow$ eigenvalues: $1 \pm \frac{\sqrt{2}}{2}$

Example

Choose (arbitrarily) $\gamma < \dfrac{\lambda_{min}(Q)}{2\lambda_{max}(P)} = \dfrac{1}{2+\sqrt{2}}$. For instance $\gamma = \dfrac{1}{4}$.

Conclusions

Let r be such that $x \in B_r(0) \Rightarrow \|g(x)\| < \gamma\|x\|$.

$$\|g(x)\| < \gamma\|x\| \;\Rightarrow\; \left\| \begin{bmatrix} 0 \\ -\sin(x_1 + \frac{\pi}{3}) + \frac{1}{2}x_1 + \sin(\frac{\pi}{3}) \end{bmatrix} \right\| < \frac{1}{4}\|x\|$$

Every ellipsoid

$$\Omega_l = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : \begin{bmatrix} x_1 & x_2 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} < l \right\}$$

included in $B_r(0)$ is a region of attraction

Example

Control law for the original system

$$\tau = u + \bar{\tau} = Kx + \sin\left(\frac{\pi}{3}\right) = \begin{bmatrix} -\frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} \theta_1 - \frac{\pi}{3} \\ \theta_2 \end{bmatrix} + \sin\left(\frac{\pi}{3}\right)$$