Nonlinear systems

Integral control

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Advanced automation and control

Integral control

NL SISO system with disturbances

$$\Sigma : \begin{cases} \dot{x} = f(x, u, w), & w(t) \in \mathbb{R}^{n_w} \text{ disturbance} \\ y = h(x, w), & (u \text{ does not affect the output directly}) \end{cases}$$

Problema: how to track a *costant* setpoint y^o in presence of *costant* disturbances $w(t) = \bar{w}$?

Idea

Step 1 augment the system with the state equation $\dot{\sigma} = e$, where $e(t) = y(t) - y^o$ is the error

$$\Sigma_a : \begin{cases} \dot{x} = f(x, u, \bar{w}) \\ \dot{\sigma} = h(x, \bar{w}) - y^o \end{cases}$$

Step 2 design a controller using $x(t)$ and $\sigma(t)$ for stabilizing Σ_a around a suitable equilibrium corresponding to zero error

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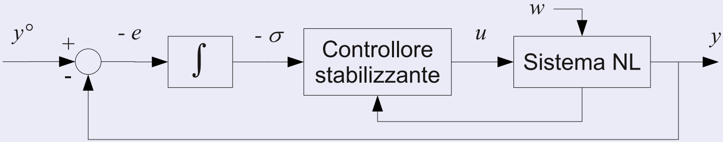
Control scheme



Step 1: analogous to the design of the static part of the controller in loopshaping for LTI systems

- project of the static part of the controller: "add an integrator to drive the error to zero when the setpoint is constant"

Step 2: analogous to the design of the dynamic part of the regulator in loopshaping. This design step considers the cascade of the system under control and the integrator

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Assumption: uniqueness of the nominal operation point

For constant signals y^o, $\bar{w}$ there is a single equilibrium point $(\bar{x}, \bar{u})$ of Σ verifying

$$\begin{aligned} 0 &= f(\bar{x}, \bar{u}, \bar{w}) \\ y^o &= h(\bar{x}, \bar{w}) \end{aligned}$$

Remarks

For the input $u(t) = \bar{u}$, equilibrium states $\begin{bmatrix} \hat{x} & \hat{\sigma} \end{bmatrix}^{\mathrm{T}}$ of Σ_a verify

$$0 = f(\hat{x}, \bar{u}, \bar{w}) \quad (1)$$

$$0 = h(\hat{x}, \bar{w}) - y^o \quad (2)$$

Then, $\hat{x} = \bar{x}$ and $\hat{\sigma}$ is an arbitrary vector.

- If Σ_a is in an equilibrium state, the error is zero
- Idea: design a controller that stabilizes an equilibrium state of Σ_a

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Control scheme

State-feedback control

$$u = K_1 x + K_2 \sigma$$

Closed-loop augmented system

$$\Sigma_{a,cl} : \begin{cases} \dot{x} = f(x, K_1 x + K_2 \sigma, \bar{w}) \\ \dot{\sigma} = h(x, \bar{w}) - y^o \end{cases}$$

Controller synthesis: design K_1 and K_2 such that

(a) for a given $(\bar{x}, \bar{u})$ there is only one $\bar{\sigma}$ such that $\bar{u} = K_1 \bar{x} + K_2 \bar{\sigma}$
- the only equilibrium state of $\Sigma_{a,cl}$ is $\begin{bmatrix} \bar{x} & \bar{\sigma} \end{bmatrix}^{\mathrm{T}}$

(b) the equilibrium state $\begin{bmatrix} \bar{x} & \bar{\sigma} \end{bmatrix}^{\mathrm{T}}$ is AS

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Solution

Requirement (a): whatever $K_2 \neq 0$ gives

$$\bar{\sigma} = \frac{1}{K_2}(\bar{u} - K_1\bar{x})$$

Requirement (b): design K_1 and K_2 based on the linearization of Σ_a around $\begin{bmatrix}\bar{x} & \bar{\sigma}\end{bmatrix}^{\mathrm{T}}$ (for instance, design an EA controller)

$$\Sigma_a : \begin{cases} \dot{x} = f(x, u, \bar{w}) \\ \dot{\sigma} = h(x, \bar{w}) - y^o \end{cases}$$

Let $\delta x = x - \bar{x}$, $\delta\sigma = \sigma - \bar{\sigma}$, $\delta u = u - \bar{u}$

$$\Sigma_{a,lin} : \begin{cases} \dot{\delta x} = A\delta x + B\delta u, \quad A = D_x f(x, u, \bar{w})\Big|_{\substack{x=\bar{x}\\ u=\bar{u}}}, \; B = D_u f(x, u, \bar{w})\Big|_{\substack{x=\bar{x}\\ u=\bar{u}}} \\ \dot{\delta\sigma} = C\delta x, \quad C = D_x\left(h(x, \bar{w}) - y^o\right)\Big|_{x=\bar{x}} = D_x h(x, \bar{w})\Big|_{x=\bar{x}} \end{cases}$$

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In matrix form

$$\Sigma_{a,lin} : \begin{bmatrix} \dot{\delta x} \\ \dot{\delta \sigma} \end{bmatrix} = \mathcal{A} \begin{bmatrix} \delta x \\ \delta \sigma \end{bmatrix} + \mathcal{B} \delta u, \quad \mathcal{A} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}, \ \mathcal{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$$

The EA problem can be solved only if $(\mathcal{A}, \mathcal{B})$ is reachable.

Lemma

$(\mathcal{A}, \mathcal{B})$ is reachable if and only if (A, B) is reachable and

$$\mathrm{rank}\left(\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \right) = n + 1$$

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Design of the EA controller

Compute $\mathcal{K} = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$ with $K_2 \neq 0$ such that $\Sigma_{a,lin}$ with input $\delta u = \mathcal{K} \begin{bmatrix} x \\ \sigma \end{bmatrix}$ has prescribed eigenvalues with real part < 0

Conclusion

The closed-loop system

$$\Sigma_{cl} : \begin{cases} \dot{x} = f(x, u, \bar{w}) \\ \dot{\sigma} = h(x, \bar{w}) - y^o, \quad u = \begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} x \\ \sigma \end{bmatrix} \end{cases}$$

Has an equilibrium state $\begin{bmatrix} \bar{x} \\ \bar{\sigma} \end{bmatrix}$, $\bar{\sigma} = \frac{1}{K_2}(\bar{u} - K_1 \bar{x})$ that is AS and, for all initial states in its region of attraction, one has $|y(t) - y^o| \to 0$ as $t \to +\infty$

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Algorithm

1. given $\bar{w}$ and y^o compute $\bar{x}$ and $\bar{u}$ such that

$$
\begin{aligned}
0 &= f(\bar{x}, \bar{u}, \bar{w}) \\
y^o &= h(\bar{x}, \bar{w})
\end{aligned}
$$

2. compute

$$
\begin{aligned}
A &= D_x f(x, u, \bar{w})\Big|_{\substack{x=\bar{x} \\ u=\bar{u}}}, \; B = D_u f(x, u, \bar{w})\Big|_{\substack{x=\bar{x} \\ u=\bar{u}}}, \\
C &= D_x h(x, \bar{w})\Big|_{x=\bar{x}}
\end{aligned}
$$

3. If (A, B) is reachable and $\mathrm{rango}\left(\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}\right) = n+1$, then compute $\mathcal{K} = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$ such that $\mathcal{A} + \mathcal{B}\mathcal{K}$ has prescribed eigenvalues[a] with real part < 0, where $\mathcal{A} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}$ and $\mathcal{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$

[a] One can show that, by construction, it holds $K_2 \neq 0$.

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Algorithm - conclusions

The dynamic state-feedbak controller

$$\Sigma_c : \begin{cases} \dot{\sigma} = h(x, \bar{w}) - y^o \\ u = K_1 x + K_2 \sigma \end{cases}$$

guarantees $|y(t) - y^o| \to 0$ as $t \to +\infty$ if the initial state is in the region of attraction of $\begin{bmatrix} \bar{x} \\ \frac{1}{K_2}(\bar{u} - K_1 \bar{x}) \end{bmatrix}$

Integral control

How to deal with unknown $\bar{w}$ and/or y^o ?

- unknown $\bar{w}$ $\Rightarrow$ not measurable (but constant) disturbance
  - usually, bounds on $\bar{w}$ are known
- unknown y^o $\Rightarrow$ tracking for classes of constant setpoints
  - usually, bounds on y^o are known

Treat $\bar{w}$ and y^o as known parameters and solve a *robust stabilization problem* by computing $\mathcal{K}$ stabilizing $\mathcal{A} + \mathcal{B}\mathcal{K}$ for all admissible valued of $\bar{w}$ e y^o

Example

Problem

$$\dot{x} = \bar{w}x^2 + u, \quad x(t) \in \mathbb{R},\ \bar{w} \text{ constant and } \bar{w} \in [-1, 1]$$
$$y = -x$$

Design a controller that guarantees the tracking of constant setpoints $y^o \in [1, 2]$

1. compute $\bar{x}$ and $\bar{u}$ such that $0 = f(\bar{x}, \bar{u}, \bar{w})$ e $y^o = h(\bar{x}, \bar{w})$

$$\begin{cases} 0 = \bar{w}\bar{x}^2 + \bar{u} \\ y^o = -\bar{x} \end{cases} \Rightarrow \begin{cases} \bar{u} = -\bar{w}\left(y^0\right)^2 \\ \bar{x} = -y^o \end{cases}$$

Example

2. Linearized system

$$A = D_x f(x, u, \bar{w})\Big|_{\substack{x=\bar{x}\\ u=\bar{u}}} = 2\bar{w}\bar{x}$$

$$B = D_u f(x, u, \bar{w})\Big|_{\substack{x=\bar{x}\\ u=\bar{u}}} = 1$$

$$C = D_x h(x, \bar{w})\Big|_{x=\bar{x}} = -1$$

3. Reachability of the augmented linearized system and controller design

The pair (A, B) is reachable ($M_r = 1$).

$$\mathrm{rank}\left(\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}\right) = \mathrm{rank}\left(\begin{bmatrix} -2\bar{w}y^o & 1 \\ -1 & 0 \end{bmatrix}\right) = 2,\ \forall \bar{w} \in [-1, 1],\ y^o \in [1, 2]$$

Example

Design $\mathcal{K} = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$ such that $\mathcal{A} + \mathcal{B}\mathcal{K}$ has eigenvalues with real part < 0

$$\mathcal{A} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} = \begin{bmatrix} 2\bar{w}\bar{x} & 0 \\ -1 & 0 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} B \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\mathcal{A} + \mathcal{B}\mathcal{K} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} + \begin{bmatrix} K_1 & K_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2\bar{w}\bar{x} + K_1 & K_2 \\ -1 & 0 \end{bmatrix}$$

Charachteristic polynomial of $\mathcal{A} + \mathcal{B}\mathcal{K}$: $\chi(\lambda) = \lambda^2 + \lambda\left(-2\bar{w}\bar{x} - K_1\right) + K_2$
A necessary and sufficient condition such that $\chi(\lambda)$ has eigenvalues with real part < 0 is

$$\begin{cases} -2\bar{w}\bar{x} - K_1 > 0 \\ K_2 > 0 \end{cases} \Rightarrow \begin{cases} 2\bar{w}y^o > K_1 \\ K_2 > 0 \end{cases}$$

The worst case corresponds to $\bar{w} = -1$ and $y^o = 2 \Rightarrow -4 > K_1$.
Pick $K_1 = -5$ and $K_2 = 1$

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Conclusion

The dynamic state-feedback controller

$$\Sigma_c : \begin{cases} \dot{\sigma} = -x - y^o \\ u = -5x + \sigma \end{cases}$$

guarantees $|y(t) - y^o| \to 0$ for $t \to +\infty$ if the initial state $\begin{bmatrix} x(0) \\ \sigma(0) \end{bmatrix}$ is in the region of attraction of $\begin{bmatrix} \bar{x} \\ \frac{1}{K_2}(\bar{u} - K_1\bar{x}) \end{bmatrix} = \begin{bmatrix} -y^o \\ \bar{w}(y^o)^2 + 5y^o \end{bmatrix}$