Nonlinear systems

Design of nonlinear controllers

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Introduction

Nonlinear control

Motivation: overcome limitations of controllers based on linearization about an equilibrium (e.g. limited region of attraction)

Idea: Design *nonlinear* controllers

Some design procedures for NL controllers

- Methods based on Lyapunov functions
- Backstepping
- Gain scheduling
- Sliding mode control

Introduction

Challenges of NL control

Differently from the linear case:

- Methods tailored to classes of NL systems with specific structure
- Methods do not always guarantee closed-loop global stability of a desired equilibrium
- It can be difficult to analyze controller robustness against disturbances and/or parametric uncertainty

Methods based on Lyapunov functions

Problem

$$\dot{x} = f(x,u), \quad f(0,0) = 0$$

Regulation problem: design $u = k(x)$, $k(0) = 0$ such that the origin of the closed-loop system is AS/GAS

$$\dot{x} = f(x, k(x))$$

Two possibilities

1. Fix $u = k(x)$ and look for a Lyapunov function $V(x)$ certifying AS/GAS of the origin of the closed-loop system
2. Fix a candidate Lyapunov function $V(x,u)$ and compute $u = k(x)$ such that $V(x, k(x))$ certifies AS/GAS of the origin of the closed-loop system

Both approaches do not guarantee to find an appropriate control law $\Rightarrow$ trial and error

Example

Problem

$$\dot{x}_1 = -3x_1 + 2x_1x_2^2 + u$$
$$\dot{x}_2 = -x_2^3 - x_2$$

Design a regulator such that the origin of the closed-loop system is AS

Approach 2: candidate Lyapunov function

$$V(x) = \frac{1}{2}\left(x_1^2 + x_2^2\right) \quad \text{(is gpd, } \mathcal{C}^1 \text{ and radially unbounded)}$$
$$\dot{V}(x) = x_1\dot{x}_1 + x_2\dot{x}_2 = x_1\left(-3x_1 + 2x_1x_2^2 + u\right) + x_2\left(-x_2^3 - x_2\right) =$$
$$= -3x_1^2 - x_2^2 + \underbrace{ux_1}_{\text{effect of the input}} + \underbrace{2x_1^2x_2^2}_{\text{positive term}} - x_2^4$$

Look for $u = k(x)$ such that $\dot{V}$ is nd/gnd

Example

Solution 1 (completing the square): $u = k(x) = -x_1^3$

$$\begin{aligned}\dot{V}(x) &= -3x_1^2 - x_2^2 + ux_1 + 2x_1^2x_2^2 - x_2^4 = \\ &= -3x_1^2 - x_2^2 - x_1^4 + 2x_1^2x_2^2 - x_2^4 = \\ &= -3x_1^2 - x_2^2 - \left(x_1^2 - x_2^2\right)^2 < 0,\ x \neq 0\end{aligned}$$

Solution 2 (cancel terms without sign or with positive sign):
$u = k(x) = -2x_1x_2^2$

$$\begin{aligned}\dot{V}(x) &= -3x_1^2 - x_2^2 + ux_1 + 2x_1^2x_2^2 - x_2^4 = \\ &= -3x_1^2 - x_2^2 - 2x_1^2x_2^2 + 2x_1^2x_2^2 - x_2^4 = \\ &= -3x_1^2 - x_2^2 - x_2^4 < 0,\ x \neq 0\end{aligned}$$

Both control laws make the origin GAS (even GES: verify @ home)

Example

Problem

$$\dot{x}_1 = x_2^3$$
$$\dot{x}_2 = u$$

Design a regulator such that the origin of the closed-loop system is AS

We use the approach 2

1st trial: candidate Lyapunov function

$$V(x) = \frac{1}{2}\left(x_1^2 + x_2^2\right) \quad \text{(is gpd, } \mathcal{C}^1 \text{ and radially unbounded)}$$
$$\dot{V}(x) = x_1\dot{x}_1 + x_2\dot{x}_2 = x_1 x_2^3 + x_2 u = x_2\left(x_2^2 x_1 + u\right)$$

Example

Choosing $u = k(x) = -x_2 - x_2^2 x_1$ one has

$$\dot{V}(x) = x_2 \left(x_2^2 x_1 + u \right) = -x_2^2$$

that is gnsd. Then, $\bar{x} = 0$ is stable. Is it also AS ?
Use global LaSalle theorem

$$R = \left\{ x : \dot{V}(x) = 0 \right\} = \left\{ \begin{bmatrix} x_1 & 0 \end{bmatrix}^{\mathrm{T}}, \forall x_1 \in \mathbb{R} \right\}$$

Compute M, the biggest invariant set in R

$$\Sigma_{cl} : \begin{cases} \dot{x}_1 = x_2^3 \\ \dot{x}_2 = -x_2 - x_2^2 x_1 \end{cases}$$

If $x(0) \in R$ then $\dot{x}_1(0) = \dot{x}_2(0) = 0$. Therefore $M = R$.

We can not conclude that the origin of Σ_{cl} is AS

Example

$$\dot{x}_1 = x_2^3$$
$$\dot{x}_2 = u$$

2nd trial: candidate Lyapunov function

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{4}x_2^4 \quad \text{(is gpd, } \mathcal{C}^1 \text{ and radially unbounded)}$$
$$\dot{V}(x) = x_1\dot{x}_1 + x_2^3\dot{x}_2 = x_2^3\,(x_1 + u)$$

Example

Choosing $u = k(x) = -x_1 - x_2$ one has

$$\dot{V}(x) = x_2^3 (x_1 + u) = -x_2^4$$

that is gnsd. Therefore $\bar{x} = 0$ is stable. Is it also AS ?
Use global LaSalle theorem

$$R = \left\{ x : \dot{V}(x) = 0 \right\} = \left\{ \begin{bmatrix} x_1 & 0 \end{bmatrix}^{\mathrm{T}}, \forall x_1 \in \mathbb{R} \right\}$$

Compute M, the biggest invariant set in R

$$\Sigma_{cl} : \begin{cases} \dot{x}_1 = x_2^3 \\ \dot{x}_2 = -x_1 - x_2 \end{cases}$$

If $x(0) \in R$ then $\dot{x}_1(0) = 0$ and $\dot{x}_2(0) = -x_1(0)$. Therefore $M = \left\{ \begin{bmatrix} 0 & 0 \end{bmatrix}^{\mathrm{T}} \right\}$.

We conclude that the origin of Σ_{cl} is GAS.

Conclusions

Methods based on Lyapunov functions

Approach 2:

- the choice of the Lyapunov function is critical
- it works only in simple cases

Approach 1 suffers from similar drawbacks.

Problem

Is it possible to make the second approach systematic, at least for special classes of NL systems ?

Backstepping procedure

Features

Iterative algorithm for the controller design based on Lyapunov functions for NL systems *with a "triangular" structure*.

At every iteration:

- build a control law that stabilizes the origin of a suitable subsystem
- build a Lyapunov function that certifies AS/GAS for the origin of the subsystem

Single-integrator backstepping

Problem

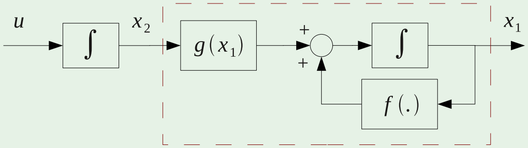
System under control

$$\Sigma : \begin{cases} \dot{x}_1 = f(x_1) + g(x_1)x_2, \quad f(0) = 0, f \text{ and } g \text{ in } \mathcal{C}^1,\ x_1 \in \mathbb{R} \\ \dot{x}_2 = u, \quad x_2 \in \mathbb{R} \ \ \textit{(integrator)} \end{cases}$$

Design $u = k(x)$ such that $\bar{x} = 0$ is an AS equilibrium state of the closed-loop system

Idea

The system is the cascade of two subsystems.



Design first a controller for the system with input x_2 and state x_1.

Single-integrator backstepping

First step: stabilize the inner loop

Assume that for partial system

$$\dot{x}_1 = f(x_1) + g(x_1)\tilde{v}, \quad \tilde{v}\text{: auxiliary input}$$

one knows $\tilde{v} = \phi_1(x_1)$, $\phi \in \mathcal{C}^1$, $\phi_1(0) = 0$ such that $\bar{x}_1 = 0$ is an AS equilibrium state for the closed-loop system

$$\dot{x}_1 = f(x_1) + g(x_1)\phi_1(x_1)$$

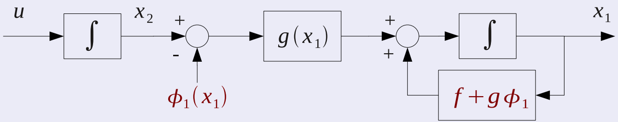
Assume also that a Lyapunov function $V_1(x_1)$ certifying AS of $\bar{x}_1 = 0$ is known.

Single-integrator backstepping

Second step: backstepping

Compute the dynamics of the error in the control variable

$$\Sigma : \begin{cases} \dot{x}_1 = f(x_1) + g(x_1)\phi_1(x_1) + g(x_1)\underbrace{(x_2 - \phi_1(x_1))}_{\text{error}} \\ \dot{x}_2 = u \end{cases}$$



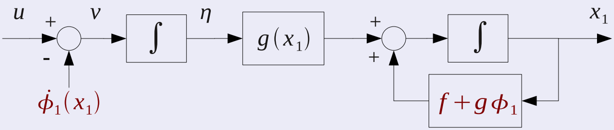
Nonlinear change of coordinates:

$$\eta = x_2 - \phi_1(x_1), \quad \text{error}$$
$$v = u - \dot{\phi}_1(x_1), \quad \text{new input}$$

Single-integrator backstepping

System in the new variables

$$\Sigma_{new} : \begin{cases} \dot{x}_1 = f(x_1) + g(x_1)\phi_1(x_1) + g(x_1)\eta \\ \dot{\eta} = v \end{cases}$$



Remarks

- ϕ_1 has been moved before the integrator = "backstepping"
- v is such that for $\bar{v} = 0$, the state $\begin{bmatrix} \bar{x}_1 \\ \bar{\eta} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is an equilibrium state (because $\phi_1(0) = 0$)

Single-integrator backstepping

Third step: stabilize the origin of the whole system

Build a candidate Lyapunov function $V_2(x_1, \eta)$ for Σ_{new}

$$
\begin{aligned}
V_2(x_1, \eta) &= V_1(x_1) + \frac{\eta^2}{2} \quad \text{(is pd)} \\
\dot{V}_2(x_1, \eta) &= D_{x_1} V_1(x_1) \cdot \dot{x}_1 + \eta\dot{\eta} = \\
&= D_{x_1} V_1(x_1) \cdot (f(x_1) + g(x_1)\phi_1(x_1) + g(x_1)\eta) + \eta v = \\
&= \underbrace{D_{x_1} V_1(x_1) \cdot (f(x_1) + g(x_1)\phi_1(x_1))}_{\dot{V}_1(x_1) \text{ that is nd}} + D_{x_1} V_1(x_1) \cdot g(x_1)\eta + \eta v
\end{aligned}
$$

Choosing $v = -D_{x_1} V_1(x_1) \cdot g_1(x_1) - \tilde{k}\eta$, $\tilde{k} > 0$ one has

$$\dot{V}_2(x, \eta) = \dot{V}_1 - \tilde{k}\eta^2$$

that is nd. Therefore the origin of Σ_{new} is AS.

Single-integrator backstepping

Remark

The origin $\begin{bmatrix}\bar{x}_1 & \bar{\eta}\end{bmatrix}^{\mathrm{T}} = \begin{bmatrix}0 & 0\end{bmatrix}^{\mathrm{T}}$ of Σ_{new} corresponds to the equilibrium $\begin{bmatrix}\bar{x}_1 & \bar{\eta} + \phi_1(\bar{x}_1)\end{bmatrix}^{\mathrm{T}}$ of the original system Σ and since $\phi_1(\bar{x}_1) = 0$ one has $\begin{bmatrix}\bar{x}_1 & \bar{\eta} + \phi_1(\bar{x}_1)\end{bmatrix}^{\mathrm{T}} = \begin{bmatrix}0 & 0\end{bmatrix}^{\mathrm{T}}$.

Conclusion

The origin of the closed-loop system Σ_{new} is AS using the controller $v = -D_{x_1} V_1(x_1) g_1(x_1) - \tilde{k}\eta$. In the original coordinates one gets

$$u - \dot{\phi}_1(x_1) = -D_{x_1} V_1(x_1) \cdot g_1(x_1) - \tilde{k}\,(x_2 - \phi_1(x_1))$$

$$u = \underbrace{D_{x_1}\phi_1(x_1) \cdot (f(x_1) + g(x_1)x_2)}_{\dot{\phi}_1(x_1) = D_{x_1}\phi(x_1)\cdot \dot{x}_1} - D_{x_1} V_1(x_1) \cdot g_1(x_1) - \tilde{k}\,\underbrace{(x_2 - \phi_1(x_1))}_{error}$$

In particular, the last expression highlights it is not necessary to compute derivatives of the function $t \mapsto \phi_1(x_1(t))$. The control law can be computed using measurements of the state only.

We have sketched the proof of the following Lemma ...

Single-integrator backstepping

Backstepping Lemma

Let $z = \begin{bmatrix} x_1 & \cdots & x_{j-1} \end{bmatrix}^{\mathrm{T}}$ and consider the single-input system

$$\Sigma : \begin{cases} \dot{z} = f(z) + g(z)x_j, \quad f(0) = 0, \\ \dot{x}_j = u \end{cases}$$

Let $\phi_{j-1}(z)$ be an auxiliary control law of class $\mathcal{C}^1$ verifying $\phi_{j-1}(0) = 0$ and such that the origin of

$$\dot{z} = f(z) + g(z)\phi_{j-1}(z) \tag{1}$$

is AS. Moreover, let $V_{j-1}(z)$ be a Lyapunov function certifying the AS of $\bar{z} = 0$ in (1). Then, for all $\tilde{k}_j > 0$, the control law

$$u = \phi_j(z, x_j) = \dot{\phi}_{j-1}(z) - D_z V_{j-1}(z) \cdot g(z) - \tilde{k}_j \left(x_j - \phi_{j-1}(z)\right)$$

is such that the origin of the closed-loop system is AS.

Single-integrator backstepping

In addition

$$V_j(z, x_j) = V_{j-1}(z) + \frac{(x_j - \phi_{j-1}(z))^2}{2}$$

is a Lyapunov function certifying the origin of the closed-loop system is AS

$$\Sigma_{cl} : \begin{cases} \dot{z} = f(z) + g(z)x_j \\ \dot{x}_j = \phi_j(z, x_j) \end{cases}$$

Finally, if $V_{j-1}(z)$ is gpd and radially unbounded and $\dot{V}_{j-1}(z)$ is gnd, then $V_j(z, x_j)$ has the same properties and the origin of Σ is GAS.

Remarks

- *Iterative* construction procedure, starting from ϕ_{j-1} and V_{j-1}, of the control law ϕ_j and the Lyapunov function V_j
- It is not necessary to compute the derivative of the signal $t \mapsto \phi_1(x_1(t))$ because

$$\dot{\phi}_j(z) = D_z\phi_j(z) \cdot \dot{z} = D_z\phi_j(z) \cdot (f(z) + g(z)x_j)$$

Example

Problem

$$\Sigma : \begin{cases} \dot{x}_1 = x_1^2 - x_1^3 + x_2 \\ \dot{x}_2 = u \end{cases}$$

Design a regulator $u = k(x)$ that makes the origin of the closed-loop system AS

Is Σ in the right form for applying backstepping ?

$$\Sigma : \begin{cases} \dot{x}_1 = f(x_1) + g(x_1)x_2 \\ \dot{x}_2 = u \end{cases}$$

for $f(x_1) = x_1^2 - x_1^3$ and $g(x_1) = 1$. Moreover $f(0) = 0$.
The assumptions of the backstepping procedure are verified.

Example

$$\Sigma : \begin{cases} \dot{x}_1 = x_1^2 - x_1^3 + x_2 \\ \dot{x}_2 = u \end{cases}$$

Stabilization of the origin of the subsystem

$$\Sigma_1 : \dot{x}_1 = x_1^2 - x_1^3 + \tilde{v}, \quad \tilde{v}\text{: auxiliary input}$$

Choosing $\tilde{v} = \phi_1(x_1) = -x_1^2 - x_1$ such that $\phi_1(0) = 0$ and $\phi_1 \in \mathcal{C}^1$ one has

$$\Sigma_{1,cl} : \dot{x}_1 = -x_1 - x_1^3$$

and $V_1(x_1) = \dfrac{x_1^2}{2}$ (that is gpd, $\mathcal{C}^1$ and radially unbounded) verifies

$$\dot{V}_1 = x_1 \cdot \dot{x}_1 = -x_1^2 - x_1^4$$

Since $\dot{V}_1$ is gnd, the origin of $\Sigma_{1,cl}$ is GAS

Example

$$\Sigma : \begin{cases} \dot{x}_1 = x_1^2 - x_1^3 + x_2 \\ \dot{x}_2 = u \end{cases}$$

Stabilization of the origin of Σ: one applies the backstepping Lemma with $z = x_1$, $j = 2$, $x_j = x_2$. For $\tilde{k}_2 > 0$ one gets

$$u = \phi_2(x_1, x_2) = \dot{\phi}_1(x_1) - D_{x_1} V_1(x_1) \cdot g(x_1) - \tilde{k}_2 \left(x_2 - \phi_1(x_1)\right) =$$

$$= \underbrace{(-2x_1 - 1)\left(x_1^2 - x_1^3 + x_2\right)}_{\dot{\phi}_1(x_1) = D_{x_1}\phi_1(x_1)\cdot(f(x_1)+g(x_1)x_2)} - x_1 - \tilde{k}_2 \left(x_2 + x_1^2 + x_1\right)$$

Conclusions

$u = \phi_2(x_1, x_2)$ is such that the origin of the closed-loop system is GAS. A Lyapunov function that certifies this property is

$$V_2(x_1, x_2) = V_1(x_1) + \frac{\left(x_2 - \phi_1(x_1)\right)^2}{2} = \frac{x_1^2}{2} + \frac{\left(x_2 + x_1^2 + x_1\right)^2}{2}$$

Example: integrator cascade

Problem

$$\Sigma : \begin{cases} \Sigma_1 : \begin{cases} \dot{x}_1 = x_1^2 - x_1^3 + x_2 \\ \dot{x}_2 = x_3 \end{cases} \\ \dot{x}_3 = u \end{cases}$$

Design a regulator such that the origin of the closed-loop system is an AS equilibrium

Remark: Σ_1 is the same system of the previous example (with auxiliary input $\tilde{v} = x_3$).

For $z = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^{\mathrm{T}}$, Σ is in the form

$$\Sigma_1 : \begin{cases} \dot{z} = f(z) + g(z)x_3 \\ \dot{x}_3 = u \end{cases}$$

for $f(z) = \begin{bmatrix} f_1(z) \\ f_2(z) \end{bmatrix} = \begin{bmatrix} x_1^2 - x_1^3 + x_2 \\ 0 \end{bmatrix}$ and $g(z) = \begin{bmatrix} g_1(z) \\ g_2(z) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Moreover $f(0) = 0 \Rightarrow$ the assumptions of the backstepping method are verified.

Example: integrator cascade

$$\Sigma : \begin{cases} \Sigma_1 : \begin{cases} \dot{x}_1 = x_1^2 - x_1^3 + x_2 \\ \dot{x}_2 = x_3 \end{cases} \\ \dot{x}_3 = u \end{cases}$$

After a backstepping we know the origin of

$$\Sigma_{1,cl} : \begin{cases} \dot{x}_1 = x_1^2 - x_1^3 + x_2 \\ \dot{x}_2 = \phi_2(x_1, x_2) = (-2x_1 - 1)\left(x_1^2 - x_1^3 + x_2\right) - x_1 - \tilde{k}_1\left(x_2 + x_1^2 + x_1\right) \end{cases}$$

is GAS, $\forall \tilde{k}_1 > 0$.

Moreover, $\phi_2(0) = 0$ (by construction), $\phi_2 \in \mathcal{C}^1$ and a Lyapunov function that certifies the global asymptotic stability of the origin is

$$V_2(x_1, x_2) = V_1(x_1) + \frac{\left(x_2 - \phi_1(x_1)\right)^2}{2} = \frac{x_1^2}{2} + \frac{\left(x_2 + x_1^2 + x_1\right)^2}{2}$$

Example: integrator cascade

$$\Sigma : \begin{cases} \Sigma_1 : \begin{cases} \dot{x}_1 = x_1^2 - x_1^3 + x_2 \\ \dot{x}_2 = x_3 \end{cases} \\ \dot{x}_3 = u \end{cases}$$

Stabilization of the origin of Σ. Apply the Backstepping Lemma with $z = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^{\mathrm{T}}$, $j = 3$, $x_j = x_3$. For $\tilde{k}_2 > 0$ one gets

$$u = \phi_3(z, x_3) = \dot{\phi}_2(z) - D_z V_2(z) \cdot g(z) - \tilde{k}_2 \left(x_3 - \phi_2(z)\right) =$$

$$= \frac{\partial \phi_2}{\partial x_1} \dot{x}_1 + \frac{\partial \phi_2}{\partial x_2} \dot{x}_2 - \frac{\partial V_2}{\partial x_1} \cdot g_1(z) - \frac{\partial V_2}{\partial x_2} \cdot g_2(z) - \tilde{k}_2 \left(x_3 - \phi_2(z)\right) =$$

$= (\ldots$ computations $\ldots)$

The origin of the closed-loop system is GAS and a Lyapunov function certifying this property is

$$V_3(z, x_3) = V_2(z) + \frac{\left(x_3 - \phi_2(z)\right)^2}{2}$$

Integrator cascade

Generalization of the previous example

Let $z = \begin{bmatrix} x_1 & \cdots & x_{j-1} \end{bmatrix}^{\mathrm{T}}$ and consider the class of NL single-input systems given by

$$\Sigma \begin{cases} \Sigma_{n-1} \begin{cases} \Sigma_{j+1} \begin{cases} \Sigma_j \begin{cases} \dot{z} = f(z) + g(z)x_j, \quad f(0) = 0 \\ \dot{x}_j = x_{j+1} \end{cases} \\ \dot{x}_{j+1} = x_{j+2} \end{cases} \\ \vdots \\ \dot{x}_{n-1} = x_n \end{cases} \\ \dot{x}_n = u \end{cases}$$

One can design a regulator stabilizing the origin of the closed-loop system applying the backstepping method to Σ_j, Σ_{j+1}, Σ_{j+2} etc. in a recursive fashion

Backstepping for more general classes of NL systems

Generalization

Let $z = \begin{bmatrix} x_1 & \cdots & x_{j-1} \end{bmatrix}^{\mathrm{T}}$ and consider the single-input system

$$\Sigma : \begin{cases} \dot{z} = f(z) + g(z)x_j, \quad f(0) = 0, \\ \dot{x}_j = f_j(z, x_j) + g_j(z, x_j)u \end{cases}$$

where $f_j, g_j \in \mathcal{C}^1$ and $g_j \neq 0$. Choosing

$$u = \frac{1}{g_j(z, x_j)}(u_j - f_j(z, x_j))$$

nonlinearities in $\dot{x}_j = \cdots$ disappear and one gets

$$\Sigma_m : \begin{cases} \dot{z} = f(z) + g(z)x_j, \quad f(0) = 0, \\ \dot{x}_j = u_j \end{cases}$$

Using the backstepping method one can design u_j such that the origin of the closed-loop system is AS.

Backstepping for more general classes of NL systems

Therefore the origin of

$$\Sigma : \begin{cases} \dot{z} = f(z) + g(z)x_j, \quad f(0) = 0, \\ \dot{x}_j = f_j(z, x_j) + g_j(z, x_j)u \end{cases}$$

is AS using

$$u = \phi_j(z, x_j) =$$

$$= \frac{1}{g_j(z, x_j)} \left(\underbrace{\dot{\phi}_{j-1}(z) - D_z V_{j-1}(z) \cdot g(z) - \tilde{k}_j \left(x_j - \phi_{j-1}(z)\right)}_{u_j} - f_j(z, x_j) \right)$$

and a Lyapunov function that certifies the AS of the origin of the closed-loop system is

$$V_j(z, x_k) = V_{j-1}(z) + \frac{\left(x_j - \phi_{j-1}(z)\right)^2}{2}$$

Backstepping for more general classes of NL systems

Strict-feedback form

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\
\dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)x_3 \\
&\vdots \\
\dot{x}_n &= f_n(x_1, \ldots, x_n) + g_n(x_1, \ldots, x_n)u
\end{aligned}$$

where $f_i(0) = 0$ and $g_i \neq 0$, $i = 1, \ldots, n$.

Remarks

- $\begin{bmatrix} \bar{x}_1 & \bar{x}_2 & \cdots & \bar{x}_n \end{bmatrix}^{\mathrm{T}} = 0$ is an equilibrium state for $\bar{u} = 0$
- Knowing a state-feedback $\phi_1(x_1)$, $\phi_1 \in \mathcal{C}^1$, $\phi_1(0) = 0$ and a Lyapunov function $V_1(x_1)$ certifying the asymptotic stability of the origin of the closed-loop system

  $$\dot{x}_1 = f_1(x_1) + g_1(x_1)\phi_1(x_1)$$

  one can apply the backstepping procedure in a recursive fashion

Backstepping for more general classes of NL systems

Strict-feedback form

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\
\dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)x_3 \\
&\vdots \\
\dot{x}_n &= f_n(x_1, \dots, x_n) + g_n(x_1, \dots, x_n)u
\end{aligned}$$

where $f_i(0) = 0$ and $g_i \neq 0$, $i = 1, \dots, n$.

One gets for $j > 1$ and $\tilde{k}_j > 0$ the auxiliary control laws

$$\phi_j(x_1, \dots, x_j) = \frac{1}{g_j}\left(\dot{\phi}_{j-1} - D_{x_1,\dots,x_{j-1}} V_{j-1} \cdot \begin{bmatrix} g_1 \\ \vdots \\ g_{j-1} \end{bmatrix} - \tilde{k}_j\,(x_j - \phi_{j-1}) - f_j\right)$$

Backstepping for more general classes of NL systems

and the partial Lyapunov functions

$$V_j(x_1, \ldots, x_j) = V_{j-1}(x_1, \ldots, x_{j-1}) + \frac{(x_j - \phi_{j-1}(x_1, \ldots, x_{j-1}))^2}{2}$$

Finally, if $\phi_2, \ldots, \phi_{n-1}$ are of class $\mathcal{C}^1$, the controller $u = \phi_n(x_1, \ldots, x_n)$is such that the origin of the closed-loop system is AS.

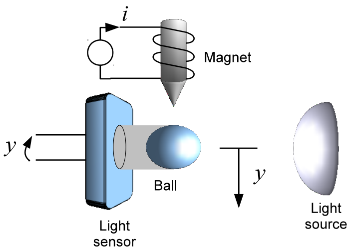
Remark

Systems in strict-feedback form have special "controllability" properties because, under weak assumptions, the regulation problem can be solved.

Limitations

At each step the function ϕ_j must be differentiable with continuity. If in some iteration this does not hold, the method stops and it does not provide a control law.

Example: magnetic levitation



Model

$$m\ddot{y} = -D\dot{y} + mg - \frac{i^2}{2(1+y)^2}$$

- mg: gravity force
- $-D\dot{y}$: damping force ($D > 0$)
- $-\frac{i^2}{2(1+y)^2}$: electromagnetic force

For $m = 1$, $D = 1$, $y_1 = y$, $y_2 = \dot{y}_1$ and input i^2,

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = \mathrm{g} - y_2 - \dfrac{i^2}{2\left(1+y_1\right)^2} \end{cases}$$

Problem

Design a controller such that $\begin{bmatrix}\bar{y}_1 & \bar{y}_2\end{bmatrix}^{\mathrm{T}} = \begin{bmatrix}1 & 0\end{bmatrix}^{\mathrm{T}}$ is an AS equilibrium state of the closed-loop system.

Example: magnetic levitation

Change of coordinates such that the origin is an equilibrium state for zero input

Computation of the equilibrium input

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = \mathrm{g} - y_2 - \dfrac{i^2}{2\left(1+y_1\right)^2} \end{cases} \Rightarrow \begin{cases} 0 = \bar{y}_2 \\ 0 = \mathrm{g} - \bar{y}_2 - \dfrac{\bar{i}^2}{2\left(1+\bar{y}_1\right)^2} \end{cases} \Rightarrow \bar{i}^2 = 8\mathrm{g}$$

Define $x_1 = y_1 - 1$, $x_2 = y_2$, $u = i^2 - 8\mathrm{g}$ and obtain

$$\Sigma : \begin{cases} \dot{x}_1 = x_2 & = f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 = \mathrm{g} - x_2 - \dfrac{u + 8\mathrm{g}}{2\left(1+x_1+1\right)^2} & = f_2(x_1,x_2) + g_2(x_1,x_2)u \end{cases}$$

The system is in strict-feedback form with $f_1(x_1) = 0$, $g_1(x_1) = 1$, $f_2(x_1,x_2) = \mathrm{g} - x_2 - \frac{8\mathrm{g}}{2(2+x_1)^2}$, $g_2(x_1,x_2) = -\frac{1}{2(2+x_1)^2}$. Moreover $g_2 \neq 0$ and $f_1(0) = f_2(0,0) = 0$

Example: magnetic levitation

$$\Sigma : \begin{cases} \dot{x}_1 = x_2 & = f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 = \mathrm{g} - x_2 - \dfrac{u + 8\mathrm{g}}{2\,(2 + x_1)^2} & = f_2(x_1, x_2) + g_2(x_1, x_2)u \end{cases}$$

Stabilization of the origin of $\Sigma_1 : \dot{x}_1 = \tilde{v}_1$

Choose $\tilde{v}_1 = \phi_1(x_1) = -x_1$ (it verifies $\phi_1(0) = 0$ and $\phi_1 \in \mathcal{C}^1$).

The origin of $\dot{x}_1 = -x_1$ is GAS and a Lyapunov function certifying this property is

$$V_1(x_1) = \frac{1}{2}x_1^2 \;\rightarrow\; \dot{V}_1(x_1) = -x_1^2$$

Example: magnetic levitation

$$\Sigma : \begin{cases} \dot{x}_1 = x_2 & = f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 = \mathrm{g} - x_2 - \dfrac{8\mathrm{g}}{2\,(2+x_1)^2} - \dfrac{u}{2\,(2+x_1)^2} & = f_2(x_1,x_2) + g_2(x_1,x_2)u \end{cases}$$

Stabilization of the origin of Σ. If

$$u = \phi_2(x_1,x_2) = -\left(-\mathrm{g} + x_2 + \frac{8\mathrm{g}}{2\,(2+x_1)^2}\right) 2\,(2+x_1)^2 - 2\tilde{v}_2\,(2+x_1)^2 \tag{2}$$

one has $\dot{x}_2 = \tilde{v}_2$. Using the backstepping Lemma one gets, for $\tilde{k}_2 > 0$,

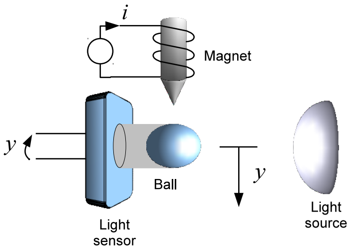
$$\begin{aligned} \tilde{v}_2 &= \dot{\phi}_1(x_1) - D_{x_1}V(x_1)\cdot g_1(x_1) - \tilde{k}_2\,(x_2 - \phi_1(x_1)) = \\ &= -x_2 - x_1 - \tilde{k}_2\,(x_1 + x_2) \end{aligned}$$

and the Lyapunov function

$$V_2(x_1,x_2) = V_1(x_1) + \frac{1}{2}(x_2 - \phi_1(x_1))^2 = \frac{1}{2}x_1^2 + \frac{1}{2}(x_1 + x_2)^2$$

certifies the origin is GAS (Is it also GES ? Check @ home ...)

Example: magnetic levitation



Model

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = \mathrm{g} - y_2 - \dfrac{i^2}{2\left(1+y_1\right)^2} \end{cases}$$

Controller for the original system

Recalling that $i^2 = u + 8\mathrm{g}$, $x_1 = y_1 - 1$, $y_2 = x_2$, from (2) one gets

$$i^2 = 2\left(1+y_1\right)^2 \left(\mathrm{g} - y_2 - \frac{8\mathrm{g}}{2\left(1+y_1\right)^2} + \left(1+\tilde{k}_2\right)\left(y_1+y_2-1\right) \right) + 8\mathrm{g}$$

and for the closed-loop system the desired equilibrium is AS.

Example

Problem - LTI system

$$\Sigma : \begin{cases} \dot{x}_1 = x_2 & = f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 = x_3 & = f_2(x_1, x_2) + g_2(x_1, x_2)x_3 \\ \dot{x}_3 = x_1 + x_2 - x_3 + u & = f_3(x_1, x_2, x_3) + g_3(x_1, x_2, x_3)u \end{cases}$$

Using the backstepping method design a controller such that the origin of the closed-loop system is AS

Remarks The system is in the canonical controllability form. It is also in strict-feedback form with

$$f_1(x_1) = f_2(x_1, x_2) = 0$$
$$f_3(x_1, x_2, x_3) = x_1 + x_2 - x_3$$
$$g_1(x_1) = g_2(x_1, x_2) = g_2(x_1, x_2, x_3) = 1$$

Example

$$\Sigma : \begin{cases} \dot{x}_1 = x_2 & = f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 = x_3 & = f_2(x_1, x_2) + g_2(x_1, x_2)x_3 \\ \dot{x}_3 = x_1 + x_2 - x_3 + u & = f_3(x_1, x_2, x_3) + g_3(x_1, x_2, x_3)u \end{cases}$$

Stabilization of the origin of $\Sigma_1 : \dot{x}_1 = \tilde{v}_1$

Choose $\tilde{v}_1 = \phi_1(x_1) = -\tilde{k}_1 x_1$, $\tilde{k}_1 > 0$ (one has $\phi_1(0) = 0$ and $\phi_1 \in \mathcal{C}^1$)

The origin of $\dot{x}_1 = -\tilde{k}_1 x_1$ is GAS.

Lyapunov function: $V_1(x_1) = \frac{1}{2}x_1^2 \rightarrow \dot{V}_1(x_1) = -\tilde{k}_1 x_1^2$

Example

$$\Sigma : \begin{cases} \dot{x}_1 = x_2 & = f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 = x_3 & = f_2(x_1, x_2) + g_2(x_1, x_2)x_3 \\ \dot{x}_3 = x_1 + x_2 - x_3 + u & = f_3(x_1, x_2, x_3) + g_3(x_1, x_2, x_3)u \end{cases}$$

Stabilization of the origin of Σ_2

$$\Sigma_2 : \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \tilde{v}_2 \end{cases}$$

Applying the backstepping Lemma, for $\tilde{k}_2 > 0$ one gets

$$\tilde{v}_2 = \phi_2(x_1, x_2) = \dot{\phi}_1(x_1) - D_{x_1} V_1(x_1) \cdot g_1(x_1) - \tilde{k}_2 \left(x_2 - \phi_1(x_1)\right) =$$
$$= \underbrace{-\tilde{k}_1 x_2}_{D_{x_1}\phi_1 \cdot \dot{x}_1} - x_1 - \tilde{k}_2 \left(x_2 + \tilde{k}_1 x_1\right) = -\left(\tilde{k}_2 \tilde{k}_1 + 1\right) x_1 - \left(\tilde{k}_1 + \tilde{k}_2\right) x_2$$

and $V_2(x_1, x_2) = V_1(x_1) + \frac{1}{2}\left(x_2 - \phi_1(x_1)\right)^2 = \frac{1}{2}x_1^2 + \frac{1}{2}\left(x_2 + \tilde{k}_1 x_1\right)^2$ certifies the origin of the closed-loop system is GAS.

Example

$$\begin{aligned}
\dot{x}_1 &= x_2 && = f_1(x_1) + g_1(x_1)x_2 \\
\dot{x}_2 &= x_3 && = f_2(x_1, x_2) + g_2(x_1, x_2)x_3 \\
\dot{x}_3 &= x_1 + x_2 - x_3 + u && = f_3(x_1, x_2, x_3) + g_3(x_1, x_2, x_3)u
\end{aligned}$$

Check: closing the loop around Σ_2 one gets a system in canonical controllability form

$$\Sigma_{2,cl} : \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\left(\tilde{k}_2\tilde{k}_1 + 1\right)x_1 - \left(\tilde{k}_1 + \tilde{k}_2\right)x_2 \end{cases}$$

The closed-loop characteristic polynomial is

$$\chi(\lambda) = \lambda^2 + (\tilde{k}_1 + \tilde{k}_2)\lambda + \left(\tilde{k}_2\tilde{k}_1 + 1\right)$$

and it always has roots with real part < 0 if $\tilde{k}_1 > 0$ and $\tilde{k}_2 > 0$ (it is a second-order polynomial and all coefficients are nonzero and with the same sign).

Choosing $\tilde{k}_1$ and $\tilde{k}_2$ one assigns, implicitly, the closed-loop eigenvalues

Example

$$\Sigma : \begin{cases} \dot{x}_1 = x_2 & = f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 = x_3 & = f_2(x_1, x_2) + g_2(x_1, x_2)x_3 \\ \dot{x}_3 = x_1 + x_2 - x_3 + u & = f_3(x_1, x_2, x_3) + g_3(x_1, x_2, x_3)u \end{cases}$$

Stabilization of the origin of Σ. If

$$u = \phi_3(x_1, x_2, x_3) = -x_1 - x_2 + x_3 + \tilde{v}_3 \tag{3}$$

one has $\dot{x}_3 = \tilde{v}_3$. Using the backstepping lemma one gets, for $\tilde{k}_3 > 0$

$$\begin{aligned} \tilde{v}_3 &= \dot{\phi}_2(x_1, x_2) - D_{x_1, x_2} V_2(x_1, x_2) \cdot \begin{bmatrix} g_1(x_1) \\ g_2(x_1, x_2) \end{bmatrix} - \tilde{k}_3 \left(x_3 - \phi_2(x_1, x_2) \right) = \ldots \\ &= -x_1 \left(\tilde{k}_1^2 + \tilde{k}_1 + \tilde{k}_1 \tilde{k}_2 \tilde{k}_3 + \tilde{k}_3 + 1 \right) - x_2 \left(2\tilde{k}_1 + \tilde{k}_2 + \tilde{k}_3 \tilde{k}_1 + \tilde{k}_2 \tilde{k}_1 + 1 \right) + \\ &- x_3 \left(\tilde{k}_1 + \tilde{k}_2 + \tilde{k}_3 \right) \end{aligned}$$

Example

$$\tilde{v}_3 = -x_1 \underbrace{\left(\tilde{k}_1^2 + \tilde{k}_1 + \tilde{k}_1\tilde{k}_2\tilde{k}_3 + \tilde{k}_3 + 1\right)}_{a_0} - x_2 \underbrace{\left(2\tilde{k}_1 + \tilde{k}_2 + \tilde{k}_3\tilde{k}_1 + \tilde{k}_2\tilde{k}_1 + 1\right)}_{a_1} +$$

$$- x_3 \underbrace{\left(\tilde{k}_1 + \tilde{k}_2 + \tilde{k}_3\right)}_{a_2}$$

Closed-loop system

$$\Sigma_{cl} : \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -a_0 x_1 - a_1 x_2 - a_2 x_3 \end{cases}$$

Closed-loop characteristic polynomial: $\chi(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$

Remarks

Closed-loop eigenvalues have real part < 0 but they depend on $\tilde{k}_1$, $\tilde{k}_2$ and $\tilde{k}_3$ in a nontrivial way. This is due to the fact that backstepping partially fixes the shape of the Lyapunov function certifying AS of the origin.

Example

Lyapunov function for the closed-loop system

$$V_3(x_1, x_2, x_3) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 - \phi_1(x_1))^2 + \frac{1}{2}(x_3 - \phi_2(x_1, x_2))^2$$

Conclusions

Controller design through backstepping

- One of the few existing methods for design controllers for NL systems NL *in a systematic way*
- Applicable only to systems with a special structure (strict-feedback)
- There are several generalizations to different classes of systems (e.g. to multi-input systems)

Limitations of backstepping

If the system is in strict-feedback form one has to cancel nonlinearities. If canceling is approximate (because, for instance, the nominal model is not precise) stability of the origin is no longer guaranteed.

Remedies

There are variants of the method that avoid canceling or make it in a "robust" fashion.

Course conclusions - theory

Goals

- Provide basic tools for the analysis of NL systems
  - Second-order systems: analysis of state trajectories in the neighborhood of an equilibrium, closed orbits
  - Stability of an equilibrium: Lyapunov theory
- Provide some methods for controlling NL systems through state-feedback
  - LTI systems: eigenvalue assignment
  - Regulation problems: controller design based on the linearized system
  - Tracking: integral control
  - Regulation: backstepping
- ... but the main aim was to give an idea of the difficulties that one has to face in the analysis and control of nonlinear systems (I hope the whole story was not too boring :-)

Next lectures

Exam preparation exercises !