Branch and Bound for Integer Problems

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Enumerative Approach

The **branch and bound** method belongs to the category of **enumerative approaches** for solving linear integer problems.

It is based on the partially **implicit exploration** of the feasible region, which can be subdivided in smaller subregions which result easier to be solved, following a recursive scheme.

The basic idea is that the more the number of solutions which are implicitly explored, the more efficient is the algorithm.

The Integer Problem

Let consider a *pure integer linear programming problem*

$$\begin{aligned} z_I^* &= \max\ \mathbf{c}'\mathbf{x} \\ s.t.\ \mathbf{A}\mathbf{x} &= \mathbf{b} \\ \mathbf{x} &\in \mathcal{Z}_+^n \end{aligned}$$

With feasible region defined as

$$S = \{\mathbf{x} \in \mathcal{R}^n : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \in \mathcal{Z}_+^n\}$$

Preliminary Definition

Let define a collection of subregions $S_1, S_2, \dots, S_r$ such that: $S = \cup_{k=1}^{r} S_k$

and consider the following linear integer subproblems defined over the different subregions

$$z_{I_k}^* = \max \mathbf{c}'\mathbf{x}$$
$$s.t.\ \mathbf{x} \in S_k$$

Then, it is easy to prove the following expression $z_I^* = \max_{k=1,2,\cdots,r} z_{I_k}^*$

Branch and Bound Idea

The branch and bound algorithm aims at:

- exploring only the promising areas of the feasible region
- storing upper and lower bounds for the optimal value z_I^*
- using these bounds to decide that certain subproblems do not need to be solved.

Lower bound $\boldsymbol{z_L}$: corresponding to the best solution x_L (feasible for the original problem) obtained in the previous iterations. This is also called incumbent solution.

Upper bound $\boldsymbol{z_U(I_k)}$: solution of the linear continuous relaxation problem C_k at iteration k. Note: if $z_{C_k}^* \leq z_L$ it is not useful to explicitly find $z_{I_k}^*$ since the integer solution will be for sure worse than the available incumbent solution.

Branch and Bound Algorithm

The branch and bound algorithm is divided into the following parts:

- Step 1: Initialization
- Step 2: Stopping test
- Step 3: Choice of the subproblem I_k to be solved
- Step 4: Bounding
- Step 5: Branching

Consider L as the list of active problems to be solved, the algorithm stops when L is empty.

Branch and Bound: 1) Initialization

The initialization phase consists in

- Setting $z_L = -\infty$
- Including in the list L the original integer problem I

Branch and Bound: 2) Stopping Test

If L is empty the algorithm stops. In this case, if $z_L = -\infty$ the original problem I is infeasible (remember we are maximizing). Otherwise the lower bound z_L is the optimal solution z_I^* of the original problem.

Branch and Bound: 3) Subproblem Choice

Chose a subproblem I_k from the list L and solve its linear continuous relaxation C_k.

If C_k is not feasible, it is removed from the list and the algorithm proceeds with Step 2.

The criteria of choice of the subproblems (e.g. breadth first of depth first) from the list L is independent from the correct functioning of the algorithm, however it affects its efficiency.

Branch and Bound: 4) Bounding

Compute the upper bound $z_U(I_k)$ of the optimal value $z^*_{I_k}$ of subproblem I_k. There can be 3 different situations:

- $z_U(I_k) \leq z_L$: proceed to Step 2 (stopping test, remember we are maximizing)
- $z_U(I_k) > z_L$ and the optimal solution $x^*_{C_k}$ is integer: before proceeding to Step 2.

  If $z^*_{C_k} = z^*_{I_k} > z_L$ then set $z_L = z^*_{I_k}$ and the integer solution $x^*_{C_k} = x^*_{I_k}$ is kept in memory as incumbent solution.
- $z_U(I_k) > z_L$ and the optimal solution $x^*_{C_k}$ is not integer: I_k is subdivided in subregions as in Step 5. Note that in this case $z_U(I_k)$ represents an upper bound for problem I_k and its subregions.

Branch and Bound: 5) Branching

The feasible region S_k of problem I_k is subdivided into r subregions $S_{k_1}, S_{k_2}, \dots, S_{k_r}$, which constitute a partition of S_k. This is the branching phase.

The subproblems $I_{k_1}, I_{k_2}, \dots, I_{k_r}$ corresponding to the subregions in which S_k has been subdivided are added to the list L and Step 2 is executed.

Branching criteria: consider a variable x_i of $\boldsymbol{x}^*_{C_k}$ that is not integer (with fractionary value f).
We can divide the region S_k into S_{k_1} and S_{k_2} as follows (in this case r =2):

$$S_{k_1} = \{\mathbf{x} \in S_k : x_i \leq \lfloor f \rfloor\} \qquad S_{k_2} = \{\mathbf{x} \in S_k : x_i \geq \lfloor f \rfloor + 1\}$$

Advantages of Branch and Bound

A premature interruption of the branch and bound method provides a feasible approximation of the optimal solution if an incumbent solution has been found.

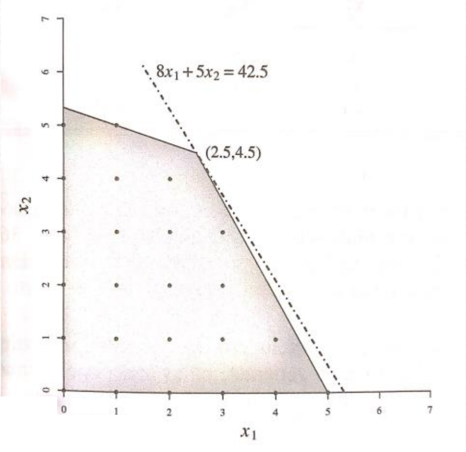
This constitutes one of the most important advantages of the branch and bound with respect to the cutting plane approach. This latter in fact, achieves the optimal solution by a succession of non feasible approximations.

Numerical Examples

Solve the following problem with the branch and bound method.

$$\begin{aligned} z_I^* = \max\ & 8x_1 + 5x_2 \\ s.t.\ & 9x_1 + 5x_2 \leq 45 \\ & x_1 + 3x_2 \leq 16 \\ & x_1, x_2 \in \mathcal{Z}_+^n \end{aligned}$$

The feasible region of the original problem and its continuous relaxation are depicted in the figure.



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Numerical Examples

The solution for the linear relaxation is $\boldsymbol{x}^*_{C_0} = (2.5, 4.5)$ with optimal value $z^*_{C_0} = 42.5$.

Since $\mathrm{z_L} = -\infty$, it has that $z^*_{C_0} > z_L$ with a non integer $\boldsymbol{x}^*_{C_0}$, therefore $z^*_{C_0}$ represents only an upper bound on problem I_0 and we need to proceed with the branching Step.

Selecting the variable x_1 we obtain the subregions S_1 and S_2 (with P_1 and P_2 the corresponding feasible regions of the linear relaxations). Insert in the list L the problems I_1 and I_2 and remove I_0.

$$S_1 = \{\mathbf{x} \in S_0 : \, x_1 \leq 2\} \qquad S_2 = \{\mathbf{x} \in S_0 : \, x_1 \geq 3\}$$

Numerical Examples

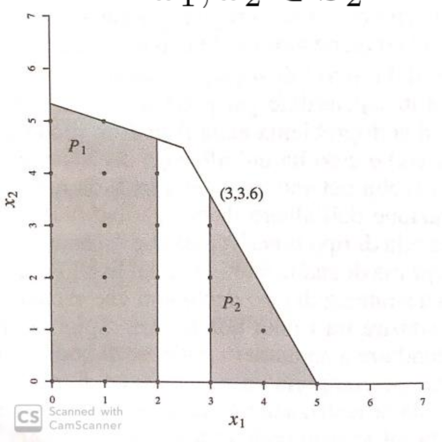
Select now one problem from the list L, e.g. problem I_2.

The choice of which problem to select first can be done by rounding the continuous value obtained before (2.5 was rounded to 3).

$$z^*_{I_2} = \max\ 8x_1 + 5x_2$$
$$s.t.\ 9x_1 + 5x_2 \leq 45$$
$$x_1 + 3x_2 \leq 16$$
$$x_1, x_2 \in S_2$$

In the following the **depth first** choice will be adopted.

The feasible region S_1 and S_2 (black dots) of the problems I_1 and I_2 and their continuous relaxations P_1 and P_2 are depicted in the figure.



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Numerical Examples

The solution for the linear relaxation is $\boldsymbol{x}_{C_2}^* = (3, 3.6)$ with optimal value $z_{C_2}^* = 42$.

Since $z_L = -\infty$, it has that $z_{C_2}^* > z_L$ with a non integer $\boldsymbol{x}_{C_2}^*$, therefore one has to proceed with the branching Step.

Selecting the variable x_2 we obtain the subregions S_3 and S_4 (with P_3 and P_4 the corresponding feasible regions of the linear relaxations). Insert in the list L the problems I_3 and I_4 and remove I_2.

$$S_3 = \{\mathbf{x} \in S_2 : x_2 \leq 3\} \qquad S_4 = \{\mathbf{x} \in S_2 : x_2 \geq 4\}$$

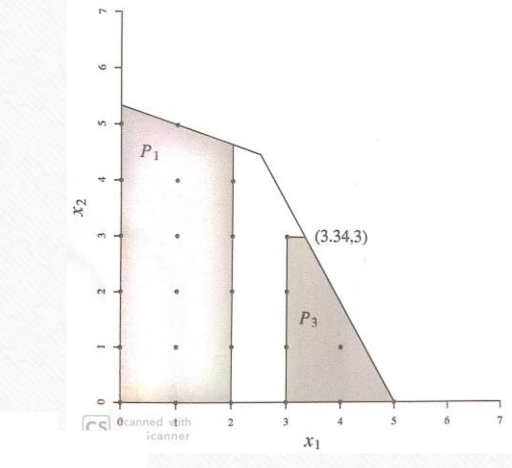
Numerical Examples

Select now from the list L the problem I_4.

$$z^*_{I_4} = \max\ 8x_1 + 5x_2$$
$$s.t.\ 9x_1 + 5x_2 \leq 45$$
$$x_1 + 3x_2 \leq 16$$
$$x_1, x_2 \in S_4$$

The feasible region of the problems I_3 and I_4 and their continuous relaxations are depicted in the figure.

For I_4 the feasible regions P_4 and S_4 are empty.



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Numerical Examples

The relaxation C_4 is infeasible and therefore I_4 is removed from the list L. Then select problem I_3 .

The solution fo the linear relaxation is $\boldsymbol{x}^*_{C_3} = (3.33, 3)$ with optimal value $z^*_{C_3} = 41.67.$

Since $z_L = -\infty$, it has that $z^*_{C_3} > z_L$ with a non integer $\boldsymbol{x}^*_{C_3}$, therefore one has to proceed with the branching Step.

Selecting the variable x_1 we obtain the subregions S_5 and S_6 (with P_5 and P_6 the corresponding feasible regions of the linear relaxations). Insert in the list L the problems I_5 and I_6 and remove I_3.

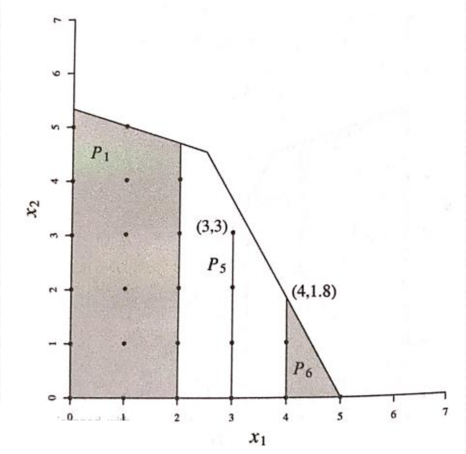
$$S_5 = \{\mathbf{x} \in S_3 : \ x_1 \leq 3\} \qquad S_6 = \{\mathbf{x} \in S_3 : \ x_1 \geq 4\}$$

Numerical Examples

Select now from the list L the problem I_5.

$$
\begin{aligned}
z^*_{I_5} = \max\ & 8x_1 + 5x_2 \\
s.t.\ & 9x_1 + 5x_2 \leq 45 \\
& x_1 + 3x_2 \leq 16 \\
& x_1, x_2 \in S_5
\end{aligned}
$$

The feasible region of the problems I_5 and I_6 and their continuous relaxations are depicted in the figure.



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Numerical Examples

The solution for the linear relaxation is $\boldsymbol{x}^*_{C_5} = \boldsymbol{x}^*_{I_5} = (3, 3)$ with optimal value $z^*_{C_5} = z^*_{I_5} = 39.$

Since $\mathrm{z_L} = -\infty$, it has that $z^*_{C_5} > z_L$ with an integer $\boldsymbol{x}^*_{C_5}$, therefore one has to proceed with the updating of the lower bound $\mathrm{z_L} = z^*_{I_5}$ and the solution $\boldsymbol{x}^*_{I_5}$ is stored as the best known solution (incumbent solution).

Problem I_5 is removed from the list L. Proceed with Step 2.

Select now from the list L the problem I_6.

Numerical Examples

The solution for the linear relaxation is $\boldsymbol{x}^*_{C_6} = (4, 1.8)$ with optimal value $z^*_{C_6} = 41$.

Since $\mathrm{z_L} = 39$, it has that $z^*_{C_6} > z_L$ with a non integer $\boldsymbol{x}^*_{C_6}$, therefore one has to proceed with the branching Step.

Selecting the variable x_2 we obtain the subregions S_7 and S_8 (with P_7 and P_8 the corresponding feasible regions of the linear relaxations). Insert in the list L the problems I_7 and I_8 and remove I_6.

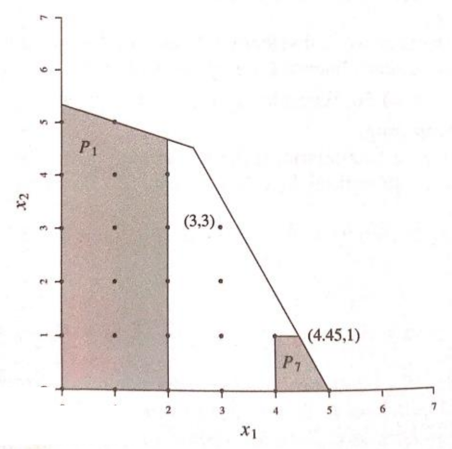
$$S_7 = \{\mathbf{x} \in S_6 : \ x_2 \leq 1\} \qquad S_8 = \{\mathbf{x} \in S_6 : \ x_2 \geq 2\}$$

Numerical Examples

Select now from the list L the problem I_8.

$$z^*_{I_8} = \max\ 8x_1 + 5x_2$$
$$s.t.\ 9x_1 + 5x_2 \leq 45$$
$$x_1 + 3x_2 \leq 16$$
$$x_1, x_2 \in S_8$$

The feasible region of the problems I_7 and I_8 and their continuous relaxations are depicted in the figure.



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For I_8 the feasible region is empty.

Numerical Examples

The relaxation C_8 is infeasible and therefore I_8 is removed from the list L. Then select problem I_7 .

The solution fo the linear relaxation is $\boldsymbol{x}^*_{C_7} = (4.44, 1)$ with optimal value $z^*_{C_7} = 40.56.$

Since $z_L = 39$, it has that $z^*_{C_7} > z_L$ with a non integer $\boldsymbol{x}^*_{C_7}$, therefore one has to proceed with the branching Step.

Selecting the variable x_1 we obtain the subregions S_9 and S_{10} (with P_9 and P_{10} the corresponding feasible regions of the linear relaxations). Insert in the list L the problems I_9 and I_{10} and remove I_8.

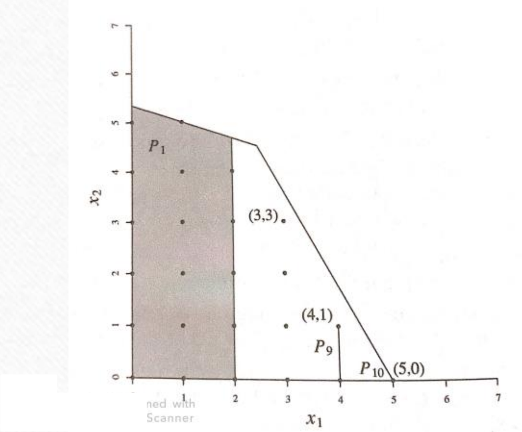
$$S_9 = \{\mathbf{x} \in S_8 : \, x_1 \leq 4\} \qquad S_{10} = \{\mathbf{x} \in S_8 : \, x_1 \geq 5\}$$

Numerical Examples

Select now from the list L the problem I_9.

$$z^*_{I_9} = \max\ 8x_1 + 5x_2$$
$$s.t.\ 9x_1 + 5x_2 \leq 45$$
$$x_1 + 3x_2 \leq 16$$
$$_1, x_2 \in S_9$$

The feasible region of the problems I_9 and I_{10} and their continuous relaxations are depicted in the figure.



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Numerical Examples

The solution for the linear relaxation is $\boldsymbol{x}^*_{C_9} = \boldsymbol{x}^*_{I_9} = (4, 1)$ with $z^*_{C_9} = z^*_{I_9} = 37$.

Since $\mathrm{z_L} = 39$, it has that $z^*_{C_9} < z_L$.

Problem I_9 is removed from the list L. Proceed with Step 2.

Select now from the list L the problem I_{10}.

Numerical Examples

The solution for the linear relaxation is $\boldsymbol{x}^*_{C_{10}} = \boldsymbol{x}^*_{I_{10}} = (5, 0)$ with $z^*_{C_{10}} = z^*_{I_{10}} = 40.$

Since $\mathrm{z_L} = 39$, it has that $z^*_{C_{10}} > z_L$ with an integer $\boldsymbol{x}^*_{C_{10}}$, therefore one has to proceed with the updating of the lower bound $\mathrm{z_L} = z^*_{I_{10}}$ and the solution $\boldsymbol{x}^*_{I_{10}}$ is stored as the best known solution (incumbent solution).

Problem I_{10} is removed from the list L. Proceed with Step 2.

Select now from the list L the problem I_1. This is the last element in the list L.

Numerical Examples

The solution for the linear relaxation is $\boldsymbol{x}^*_{C_1} = (2, 3.66)$ with optimal value $z^*_{C_2} = 39.34.$

Since $\mathrm{z_L} = 40$, it has that $z^*_{C_1} < z_L$.

Problem I_1 is removed from the list L. Proceed with Step 2.

The list L is empty and therefore the algorithm stops. The optimal solution is

$$z_I^* = 40$$
$$\mathbf{x}_I^* = (5, 0)$$

Numerical Examples

Here is depicted the tree for the branch and bound algorithm.