Cutting Plane Method

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Preliminary Definition

A mixed-integer linear program in standard form is formulated as

$$\begin{aligned} \max\ & \mathbf{c}'\mathbf{x} + \mathbf{d}'\mathbf{y} \\ s.t.\ & \mathbf{A}\mathbf{x} + \mathbf{E}\mathbf{y} = \mathbf{b} \\ & \mathbf{x} \in \mathcal{Z}_+^n,\ \mathbf{y} \geq \mathbf{0} \end{aligned}$$

Note that if $\boldsymbol{x} \in \mathbb{Z}^n$ we can reformulate the problem by introducing $\boldsymbol{x}^+, \boldsymbol{x}^- \in \mathbb{Z}_+^n$ such that $\boldsymbol{x} = \boldsymbol{x}^+ - \boldsymbol{x}^-$.

with $\mathbf{y} \in \mathcal{R}^q$, $\mathbf{A} \in \mathcal{R}^{m \times n}$ and $\mathbf{E} \in \mathcal{R}^{m \times q}$

Preliminary Definition

- If all the variables are integer ($q = 0$) we have a *pure integer linear programming problem.*

$$\begin{aligned} \max\ & \mathbf{c}'\mathbf{x} + \mathbf{d}'\mathbf{y} \\ s.t.\ & \mathbf{A}\mathbf{x} + \mathbf{E}\mathbf{y} = \mathbf{b} \\ & \mathbf{x} \in \mathcal{Z}_+^n,\ \mathbf{y} \geq \mathbf{0} \end{aligned}$$

- If $\boldsymbol{x} \in \{0,1\}^n$ we speak of *binary optimization problem.*

Linear Continuous Relaxation

Let consider a *pure integer linear programming problem*

$$\begin{aligned} z_I^* = \max\ & \mathbf{c}'\mathbf{x} \\ s.t.\ & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \in \mathcal{Z}_+^n \end{aligned}$$

The feasible reagion of the problem is defined as

$$S = \{\mathbf{x} \in \mathcal{R}^n : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \in \mathcal{Z}_+^n\}$$

Linear Continuous Relaxation

Its linear countinuos relaxation is obtained by removing the integer condition ($\boldsymbol{x} \in \mathbb{Z}_+^n$) for all the integer variables.

$$
\begin{aligned}
z_I^* &= \max\ \mathbf{c}'\mathbf{x} \\
s.t.\ \mathbf{A}\mathbf{x} &= \mathbf{b} \\
\mathbf{x} &\in \mathcal{Z}_+^n
\end{aligned}
\quad \Longrightarrow \quad
\begin{aligned}
z_C^* &= \max\ \mathbf{c}'\mathbf{x} \\
s.t.\ \mathbf{A}\mathbf{x} &= \mathbf{b} \\
\mathbf{x} &\geq 0
\end{aligned}
$$

Linear Continuous Relaxation

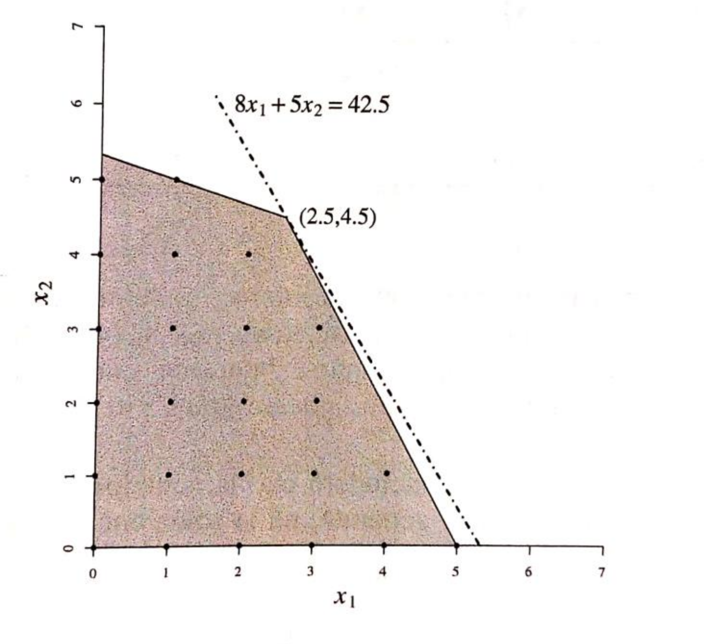
Theorem 1: given a linear integer optimization in the form of maximization and its linear continuous relaxation it holds $z_I^* \leq z_C^*$.

Corollary 1: If the solution of the linear countinuos relaxation $\boldsymbol{x}_C^* \in \mathbb{Z}_+^n$, then $\boldsymbol{x}_C^*$ is optimal also for the integer optimization problem.

Numerical Example

Let consider the pure integer optimization problem

$$\max\ 8x_1 + 5x_2$$
$$s.t.\ 9x_1 + 5x_2 \leq 45$$
$$x_1 + 3x_2 \leq 16$$
$$x_1, x_2 \in \mathcal{Z}_+^n$$



$$\mathbf{x}_C^* = (2.5,\ 4.5) \rightarrow z_C^* = 42.5$$
$$\mathbf{x}_I^* = (5,\ 0) \rightarrow z_I^* = 40$$

The optimal value of the linear relaxation is not a good approximation for the original integer problem.

Alternative Formulations

Definition 2: a polyhedron $P = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{Ax} \leq \boldsymbol{b}\}$ is a linear formulation of an integer optimization problem with feasible region S if $S = P \cap \mathbb{Z}_+^n$.

Definition 3: given two equivalent linear formulations P_1 and P_2 of an integer optimization, we say that P_1 is more stringent, and therefore better, if $P_1 \subset P_2$.

Alternative Formulations

$$\max\ 8x_1 + 5x_2$$
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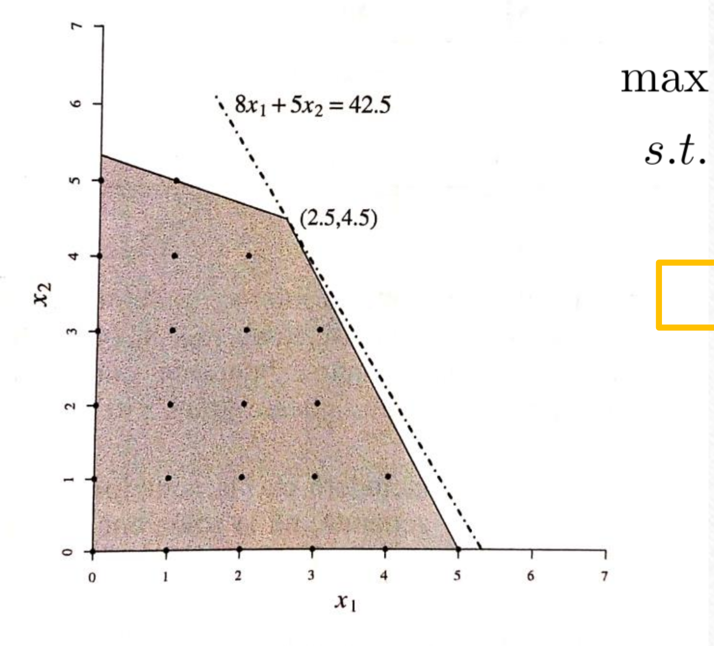
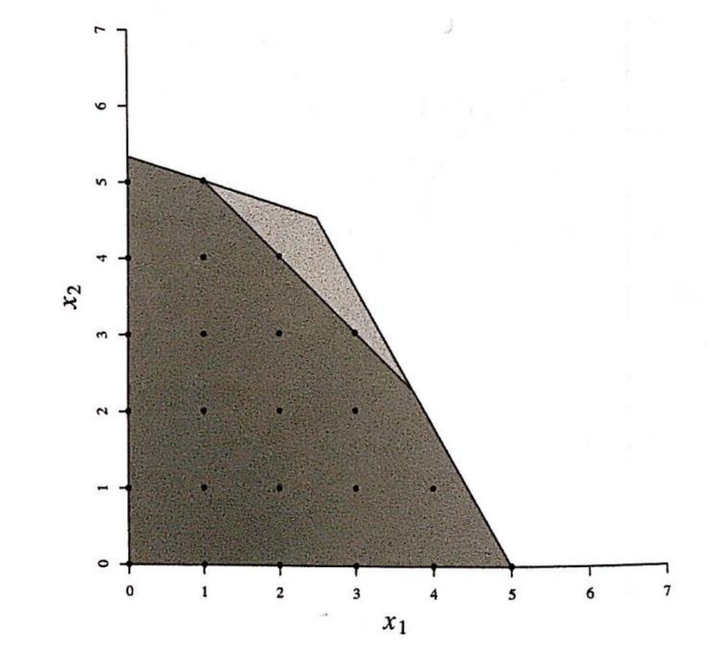


C. Vercellis, Ottimizzazione. Teoria, Metodi, applicazioni, 2008

$$\max\ 8x_1 + 5x_2$$
$$s.t.\ 9x_1 + 5x_2 \leq 45$$
$$x_1 + 3x_2 \leq 16$$
$$x_1 + x_2 \leq 6$$
$$x_1, x_2 \in \mathcal{Z}_+^n$$



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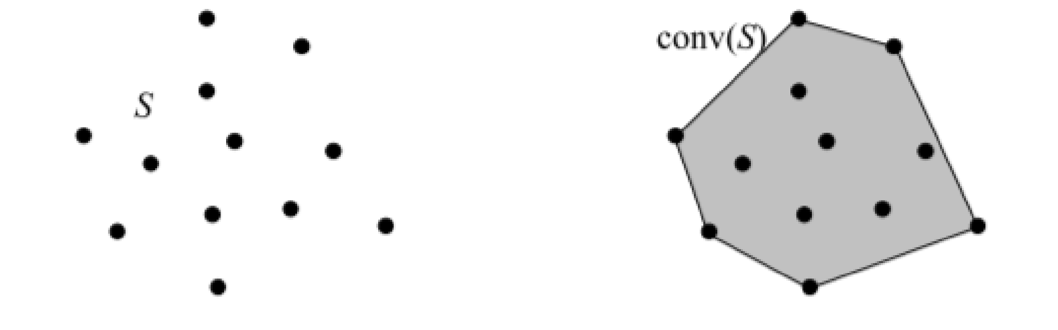
Convex Hull

Definition 4: given a set $X \subseteq \mathbb{R}^n$, we define the *convex hull* of X (denoted by $conv(X)$) as the smallest convex set in $\mathbb{R}^n$ which contains X.



L. De Giovanni, M. Di Summa, G. Zambelli, Solution Methods for Integer Linear Programming.

Ideal Formulation

Definition 5: a linear formulation with feasible region P of an integer optimization problem with feasible region S is said to be *ideal* if P is the convex hull of S, i.e. $P = conv(S)$.

The ideal formulation is the most stringent linear formulation of an integer problem.

In case of an ideal formulation one has that

$$\max\{\mathbf{c}'\mathbf{x} : \mathbf{x} \in S\} = \max\{\mathbf{c}'\mathbf{x} : \mathbf{x} \in conv(S)\}$$

Ideal Formulation

$$\begin{aligned} \max\ & 8x_1 + 5x_2 \\ s.t.\ & 9x_1 + 5x_2 \leq 45 \\ & x_1 + 3x_2 \leq 16 \\ & x_1 + x_2 \leq 6 \\ & x_1, x_2 \in \mathcal{Z}_+^n \end{aligned}$$

$$\begin{aligned} \max\ & 8x_1 + 5x_2 \\ s.t.\ & x_2 \leq 5 \\ & x_1 + x_2 \leq 6 \\ & 3x_1 + 2x_2 \leq 15 \\ & x_1, x_2 \in \mathcal{Z}_+^n \end{aligned}$$

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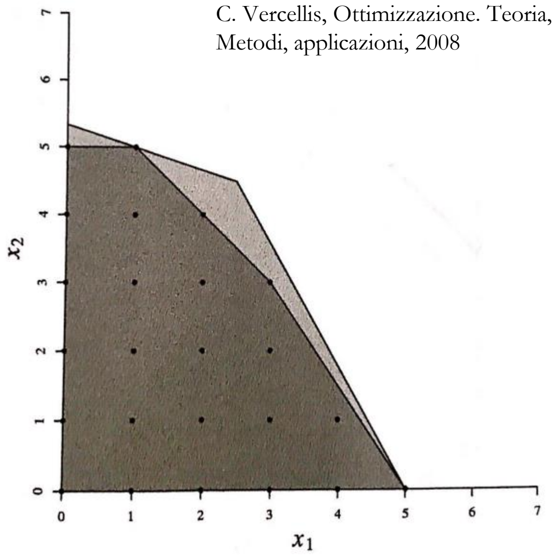


Unfortunately, only in few cases it is possible to determine the ideal formulation of a linear integer optimization problem (see *unimodularity*).

Cutting Plane Idea

Idea: iteratively solve a sequence of linear relaxations that approximate better and better the convex hull of the feasible region around the optimal solution.

At the k-th iteration:

- compute the optimal solution $\boldsymbol{x}^*_{C_k}$ of the linear relaxation
- if $\boldsymbol{x}^*_{C_k}$ is integer, then $\boldsymbol{x}^*_{C_k} = \boldsymbol{x}^*_{I_k}$ is the optimal solution of the integer problem
- otherwise, add to the optimization problem a new constraints which is violated by $\boldsymbol{x}^*_{C_k}$ but satisfied by all the feasible solutions of the original integer problem.

Cutting Plane Idea

Let consider I_k as the optimization problem at the *k*-th iteration and C_k the correspondent linear relaxation. The problem I_k is then given by

$$\begin{aligned} z^*_{I_k} = \max\ & \mathbf{c}'\mathbf{x} \\ s.t.\ & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{v}'_h\mathbf{x} = g_h,\ h = 1, \cdots, k \\ & \mathbf{x} \in \mathcal{Z}^n_+ \end{aligned}$$

where $\boldsymbol{v}'_h\boldsymbol{x} = g_h, h = 1, \ldots, k$ are the set of constraints added during the previous iterations.

Cutting Plane Idea

Definition 6: the constraint $\boldsymbol{v'}_{k+1}\, \boldsymbol{x} = g_{k+1}$ is said to be a valid cut for the problem I_k at the k-th iteration if

- the constraint is violated by the optimal solution $\boldsymbol{x}^*_{\boldsymbol{C_k}}$ of the linear continuos relaxation C_k
- the constraint is satisfied by all the feasible solution of I

Cutting Plane Idea

Let consider the following linear integer optimization problem I and its relaxation C

$$z_I^* = \max \mathbf{c}'\mathbf{x} \quad s.t.\ \mathbf{A}\mathbf{x} = \mathbf{b} \quad \mathbf{x} \in \mathcal{Z}_+^n$$

$$z_C^* = \max \mathbf{c}'\mathbf{x} \quad s.t.\ \mathbf{A}\mathbf{x} = \mathbf{b} \quad \mathbf{x} \geq 0$$

and consider $\boldsymbol{B}$ the optimal base for the linear relaxation C and its correspondent optimal solution $\boldsymbol{x}^* = (\boldsymbol{x_B}, \boldsymbol{x_D}) = (\boldsymbol{B^{-1}b}, \boldsymbol{0})$. If the solution is not integer it exists an index $\boldsymbol{t}$ for which $x_t^* \in \mathbb{R}_+ \backslash \mathbb{Z}_+$. Assume that $\boldsymbol{A} = [\boldsymbol{B}\ \boldsymbol{D}]$, with $\boldsymbol{D}$ associated to non basic variables.

Let finally define

$$y_{ij} = (\mathbf{B^{-1}D})_{ij}, \qquad \omega_i = (\mathbf{B^{-1}b})_i$$

Cutting Plane Idea

Definition 7: the *Gomory cut* is given by the constraint

$$\sum_{j \in \mathcal{D}} v_{tj} x_j \geq g_t$$

where $\mathcal{D} = \{m+1, m+2, \cdots, n\}$ is the set of index of non basic variables and

$$v_{tj} = y_{tj} - \lfloor y_{tj} \rfloor, \qquad g_t = \omega_t - \lfloor \omega_t \rfloor$$

Theorem 2: the Gomory cut is a valid cut.

Cutting Plane Algorithm

- **Initialization**: assign $k = 0$ to the iteration index and set $I_0 = I$, where I is the original problem.
- **Stopping criteria**: solve the linear relaxation C_k of the problem I_k. If the solution $\boldsymbol{x}^*_{C_k}$ is integer the algorithm stops since $\boldsymbol{x}^*_{C_k}$ is also solution of the problem I_k and therefore of I.
- **Cut generation**: generate a valid cut $\boldsymbol{v'}_{k+1}\boldsymbol{x} = g_{k+1}$ and add it to the problem I_k, obtaining the problem I_{k+1}. Finally, update $k = k + 1$.

Numerical Examples

Solve the following problem with Gomory cutting plan method.

$$\begin{aligned} z_I^* = \max\ & x_1 + 2x_2 \\ s.t.\ & -2x_1 + 2x_2 \leq 5 \\ & 6x_1 + 4x_2 \leq 25 \\ & x_1, x_2 \in \mathcal{Z}_+^n \end{aligned}$$

Consider two slack variables in order to transform the problem into standard form.

$$\begin{aligned} z_I^* = \max\ & x_1 + 2x_2 \\ s.t.\ & -2x_1 + 2x_2 + s_1 = 5 \\ & 6x_1 + 4x_2 + s_2 = 25 \\ & x_1, x_2 \in \mathcal{Z}_+^n,\ s_1, s_2 \geq 0 \end{aligned}$$

Numerical Examples

Consider two slack variables in order to transform the problem into standard form.

$$z^*_{C_0} = 9.5, \qquad x^*_{C_0} = (1.5,\, 4,\, 0,\, 0) \qquad \mathbf{B}^{-1}\mathbf{D} = \begin{bmatrix} -0.2 & 0.1 \\ 0.3 & 0.1 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} -2 & 2 \\ 6 & 4 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$v_{tj} = y_{tj} - \lfloor y_{tj} \rfloor, \qquad g_t = \omega_t - \lfloor \omega_t \rfloor$$

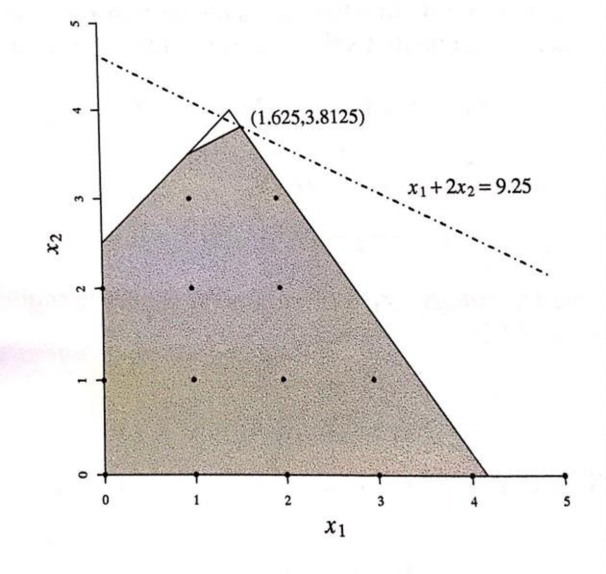
$$y_{ij} = (\mathbf{B}^{-1}\mathbf{D})_{ij}, \qquad \omega_i = (\mathbf{B}^{-1}\mathbf{b})_i$$

Gomory cut associated with x_1: $8s_1 + s_2 \geq 5$

Numerical Examples

The Gomory cut can be esxpressed as function of the original variable as $-x_1 + 2x_2 \leq 6$

and the problem I_1 becomes



$$
\begin{aligned}
z_I^* = \max\ & x_1 + 2x_2 \\
s.t.\ & -2x_1 + 2x_2 + s_1 = 5 \\
& 6x_1 + 4x_2 + s_2 = 25 \\
& -x_1 + 2x_2 + s_3 = 6 \\
& x_1, x_2 \in \mathcal{Z}_+^n,\ s_1, s_2, s_3 \geq 0
\end{aligned}
$$

Numerical Examples

The optimal solution of the relaxation problem C_1 is given by

$$z^*_{C_1} = 9.25, \qquad x^*_{C_1} = (\frac{13}{8}, \frac{61}{16}, \frac{5}{8}, 0, 0)$$

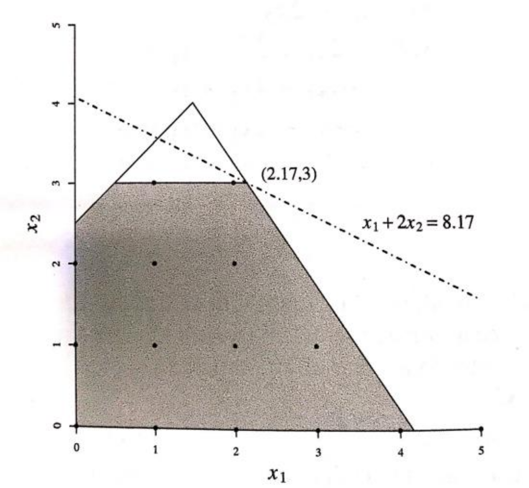
$$\mathbf{B} = \begin{bmatrix} -2 & 2 & 1 \\ 6 & 4 & 0 \\ -1 & 2 & 0 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{B}^{-1}\mathbf{D} = \begin{bmatrix} \frac{1}{8} & -\frac{1}{4} \\ \frac{1}{16} & +\frac{3}{8} \\ \frac{1}{8} & -\frac{5}{4} \end{bmatrix}$$

Gomory cut associated with x_2: $s_2 + 6s_3 \geq 13$

Numerical Examples

The Gomory cut can be esxpressed as function of the original variable as $x_2 \leq 3$

and the problem I_2 becomes



$$
\begin{aligned}
z_I^* = \max\ & x_1 + 2x_2 \\
s.t.\ & -2x_1 + 2x_2 + s_1 = 5 \\
& 6x_1 + 4x_2 + s_2 = 25 \\
& -x_1 + 2x_2 + s_3 = 6 \\
& x_2 + s_4 = 3 \\
& x_1, x_2 \in \mathcal{Z}_+^n,\ s_1, s_2, s_3 \geq 0
\end{aligned}
$$

Numerical Examples

The optimal solution of the relaxation problem C_2 is given by

$$z^*_{C_2} = 8.17, \qquad x^*_{C_2} = (\frac{13}{6}, 3, \frac{10}{3}, 0, \frac{13}{6}, 0)$$

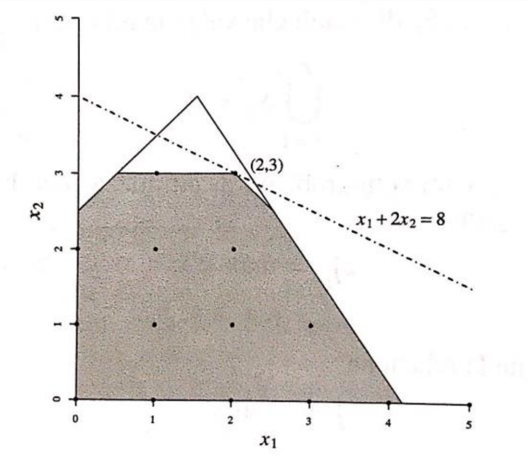
$$\mathbf{B} = \begin{bmatrix} -2 & 2 & 1 & 0 \\ 6 & 4 & 0 & 0 \\ -1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{B}^{-1}\mathbf{D} = \begin{bmatrix} \frac{1}{6} & -\frac{2}{3} \\ 0 & 1 \\ \frac{1}{3} & -\frac{10}{3} \\ \frac{1}{6} & -\frac{8}{3} \end{bmatrix}$$

Gomory cut associated with x_1: $\quad s_2 + 2s_4 \geq 1$

Numerical Examples

The Gomory cut can be esxpressed as function of the original variable as $x_1 + x_2 \leq 5$

and the problem I_3 becomes



$$
\begin{aligned}
z_I^* = \max\ & x_1 + 2x_2 \\
s.t.\ & -2x_1 + 2x_2 + s_1 = 5 \\
& 6x_1 + 4x_2 + s_2 = 25 \\
& -x_1 + 2x_2 + s_3 = 6 \\
& x_2 + s_4 = 3 \\
& x_1 + x_2 + s_5 = 5 \\
& x_1, x_2 \in \mathcal{Z}_+^n,\ s_1, s_2, s_3 \geq 0
\end{aligned}
$$

Numerical Examples

The optimal solution of the relaxation problem C_3 is given by

$$z^*_{C_3} = 8, \qquad x^*_{C_3} = (2,\ 3,\ 0,\ 0,\ 0,\ 0,\ 0)$$

The algorithm stops since the obtained solution is integer, and therefore

$$z^*_{I_3} = z^*_{C_3} = 8, \qquad x^*_{I_3} = x^*_{C_3} = (2,\ 3,\ 0,\ 0,\ 0,\ 0,\ 0)$$