Duality and Multiparametric Programming

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Duality

standard form problem (without equality constraints)

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \ i = 1, \ldots, m \end{array}$$

- optimal value $p^\star$, domain D
- called **primal problem** (in context of duality)

(for now) we **don't** assume convexity

Duality

Lagrangian $L : \mathbf{R}^{n+m} \to \mathbf{R}$

$$L(x, \lambda) = f_0(x) + \lambda_1 f_1(x) + \cdots + \lambda_m f_m(x)$$

- λ_i called *Lagrange multipliers* or *dual variables*
- objective is *augmented* with weighted sum of constraint functions

Lagrange dual function

(Lagrange) dual function $g : \mathbf{R}^m \to \mathbf{R} \cup \{-\infty\}$

$$
\begin{aligned}
g(\lambda) &= \inf_x L(x, \lambda) \\
&= \inf_x \left(f_0(x) + \lambda_1 f_1(x) + \cdots + \lambda_m f_m(x)\right)
\end{aligned}
$$

- minimum of augmented cost as function of weights
- can be $-\infty$ for some λ
- g is concave (even if f_i not convex!)

Not easy to understand
(we will check it for QPs and LPs)

Lagrange dual function: LP example

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x - b_i \leq 0, \; i = 1, \ldots, m \end{array}$$

$$\begin{aligned} L(x, \lambda) &= c^T x + \sum_{i=1}^{m} \lambda_i (a_i^T x - b_i) \\ &= -b^T \lambda + (A^T \lambda + c)^T x \end{aligned}$$

hence $g(\lambda) = \begin{cases} -b^T \lambda & \text{if } A^T \lambda + c = 0 \\ -\infty & \text{otherwise} \end{cases}$

Lower bound property

if $\lambda \succeq 0$ and x is primal feasible, then

$$g(\lambda) \le f_0(x)$$

proof: if $f_i(x) \le 0$ and $\lambda_i \ge 0$,

$$\begin{aligned} f_0(x) &\ge f_0(x) + \sum_i \lambda_i f_i(x) \\ &\ge \inf_z \left(f_0(z) + \sum_i \lambda_i f_i(z) \right) \\ &= g(\lambda) \end{aligned}$$

$f_0(x) - g(\lambda)$ is called the **duality gap** of (primal feasible) x and $\lambda \succeq 0$

minimize over primal feasible x to get, for any $\lambda \succeq 0$,

$$g(\lambda) \le p^\star$$

$\lambda \in \mathbf{R}^m$ is **dual feasible** if $\lambda \succeq 0$ and $g(\lambda) > -\infty$

dual feasible points yield lower bounds on optimal value!

Lagrangian dual problem

$$\begin{array}{ll} \text{maximize} & g(\lambda) \\ \text{subject to} & \lambda \succeq 0 \end{array}$$

- called **(Lagrange) dual problem** (associated with primal problem)
- always a convex problem, even if primal isn't!
- optimal value denoted $d^\star$
- we always have $d^\star \leq p^\star$ (called *weak duality*)
- $p^\star - d^\star$ is *optimal duality gap*

Strong duality

for convex problems, we (usually) have *strong duality:*

$$d^\star = p^\star$$

when strong duality holds, dual optimal $\lambda^\star$ serves as **certificate of optimality** for primal optimal point $x^\star$

Strong duality

many conditions or *constraint qualifications* guarantee strong duality for convex problems

Slater's condition: if primal problem is strictly feasible (and convex), *i.e.*, there exists $x \in \textbf{relint}\, D$ with

$$f_i(x) < 0, \; i = 1, \ldots, m$$

then we have $p^\star = d^\star$

Dual of linear program

(primal) LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \end{array}$$

- n variables, m inequality constraints

dual of LP is (after making implicit equality constraints explicit)

$$\begin{array}{ll} \text{maximize} & -b^T \lambda \\ \text{subject to} & A^T \lambda + c = 0 \\ & \lambda \succeq 0 \end{array}$$

- dual of LP is also an LP (indeed, in std LP format)
- m variables, n equality constraints, m nonnegativity contraints

for LP we have strong duality except in one (pathological) case: primal and dual *both* infeasible ($p^\star = +\infty$, $d^\star = -\infty$)

Dual of quadratic program

(primal) QP

$$\begin{array}{ll} \text{minimize} & x^T P x \\ \text{subject to} & Ax \preceq b \end{array}$$

we assume $P \succ 0$ for simplicity

Lagrangian is $L(x, \lambda) = x^T P x + \lambda^T (Ax - b)$

$\nabla_x L(x, \lambda) = 0$ yields $x = -(1/2)P^{-1}A^T\lambda$, hence dual function is

$$g(\lambda) = -(1/4)\lambda^T A P^{-1} A^T \lambda - b^T \lambda$$

- concave quadratic function
- all $\lambda \succeq 0$ are dual feasible

dual of QP is

$$\begin{array}{ll} \text{maximize} & -(1/4)\lambda^T A P^{-1} A^T \lambda - b^T \lambda \\ \text{subject to} & \lambda \succeq 0 \end{array}$$

. . . another QP

Duality in algorithms

many algorithms produce at iteration k

- a primal feasible $x^{(k)}$
- and a dual feasible $\lambda^{(k)}$

with $f_0(x^{(k)}) - g(\lambda^{(k)}) \to 0$ as $k \to \infty$

hence at iteration k we **know** $p^\star \in \left[g(\lambda^{(k)}), f_0(x^{(k)})\right]$

- useful for stopping criteria
- algorithms that use dual solution are often more efficient ($e.g.$, LP)

Stopping criteria

absolute error $= f_0(x^{(k)}) - p^\star \leq \epsilon$

stopping criterion:

$$\textbf{until} \quad \left(f_0(x^{(k)}) - g(\lambda^{(k)}) \leq \epsilon \right)$$

relative error $= \dfrac{f_0(x^{(k)}) - p^\star}{|p^\star|} \leq \epsilon$

stopping criterion:

$$\textbf{until} \quad \left(g(\lambda^{(k)}) > 0 \;\;\&\;\; \frac{f_0(x^{(k)}) - g(\lambda^{(k)})}{g(\lambda^{(k)})} \leq \epsilon \right)$$

$$\textbf{or} \quad \left(f_0(x^{(k)}) < 0 \;\;\&\;\; \frac{f_0(x^{(k)}) - g(\lambda^{(k)})}{-f_0(x^{(k)})} \leq \epsilon \right)$$

achieve **target value** ℓ or, prove ℓ is unachievable ($i.e.$, determine either $p^\star \leq \ell$ or $p^\star > \ell$)

stopping criterion:

$$\textbf{until} \quad \left(f_0(x^{(k)}) \leq \ell \;\textbf{ or }\; g(\lambda^{(k)}) > \ell \right)$$

Complementary slackness condition

suppose $x^\star$, $\lambda^\star$ are primal, dual feasible with zero duality gap (hence, they are primal, dual optimal)

$$\begin{aligned} f_0(x^\star) &= g(\lambda^\star) \\ &= \inf_x \left(f_0(x) + \sum_{i=1}^m \lambda_i^\star f_i(x) \right) \\ &\leq f_0(x^\star) + \sum_{i=1}^m \lambda_i^\star f_i(x^\star) \end{aligned}$$

hence we have $\sum_{i=1}^m \lambda_i^\star f_i(x^\star) = 0$, and so

$$\lambda_i^\star f_i(x^\star) = 0, \quad i = 1, \ldots, m$$

- ith constraint inactive at optimum $\Longrightarrow \lambda_i = 0$
- $\lambda_i^\star > 0$ at optimum $\Longrightarrow$ ith constraint active at optimum

KKT optimality conditions

suppose

- f_i are differentiable
- $x^\star$, $\lambda^\star$ are (primal, dual) optimal, with zero duality gap

by complementary slackness we have

$$f_0(x^\star) + \sum_i \lambda_i^\star f_i(x^\star) = \inf_x \left(f_0(x) + \sum_i \lambda_i^\star f_i(x) \right)$$

$$\Rightarrow \quad \boxed{\nabla f_0(x^\star) + \sum_i \lambda_i^\star \nabla f_i(x^\star) = 0}$$

i.e., $x^\star$ minimizes $L(x, \lambda^\star)$

KKT optimality conditions

if $x^\star$, $\lambda^\star$ are (primal, dual) optimal, with zero duality gap, they satisfy

$$\begin{aligned} f_i(x^\star) &\leq 0 \\ \lambda_i^\star &\geq 0 \\ \lambda_i^\star f_i(x^\star) &= 0 \\ \nabla f_0(x^\star) + \sum_i \lambda_i^\star \nabla f_i(x^\star) &= 0 \end{aligned}$$

the **Karush-Kuhn-Tucker** (KKT) optimality conditions

conversely, if the problem is convex and $x^\star$, $\lambda^\star$ satisfy KKT, then they are (primal, dual) optimal

Equality constraints

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

- optimal value $p^\star$
- again assume (for now) not necessarily convex

define **Lagrangian** $L : \mathbf{R}^{n+m+p} \to \mathbf{R}$ as

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$$

dual function is $g(\lambda, \nu) = \inf_x L(x, \lambda, \nu)$

(λ, ν) is dual feasible if $\lambda \succeq 0$ and $g(\lambda, \nu) > -\infty$

(no sign condition on ν)

Dual of LP in standard form

- Primal LP
$$\begin{aligned} \min_x \quad & c^\top x \\ \textit{subj. to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$
$$\begin{aligned} \min_x \quad & c^\top x \\ \textit{subj. to} \quad & b - Ax = 0 \\ & -x \leq 0 \end{aligned}$$
- Lagrangian
$$L(x, \lambda, \nu) = c^\top x - \lambda^\top x + \nu^\top (b - Ax)$$
$$g(\lambda, \nu) = \min_x L(x, \lambda, \nu) = \min_x (c^\top - \lambda^\top - \nu^\top A)x + \nu^\top b = \min_x (c - \lambda - A^\top \nu)^\top x + b^\top \nu$$
$$g(\lambda, \nu) = \begin{cases} b^\top \nu & \textit{if } (c - \lambda - A^\top \nu) = 0 \\ -\infty & \textit{otherwise} \end{cases}$$

Resulting dual LP
$$\begin{aligned} \max_{\lambda,\nu} \quad & b^\top \nu \\ \textit{subj. to} \quad & A^\top \nu + \lambda = c \\ & \lambda \geq 0 \end{aligned}$$

Properties

lower bound property: if x is primal feasible and (λ, ν) is dual feasible, then $g(\lambda, \nu) \le f_0(x)$, hence

$$g(\lambda, \nu) \le p^\star$$

dual problem: find best lower bound

$$\begin{array}{ll} \text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \succeq 0 \end{array}$$

(note ν unconstrained), optimal value $d^\star$

weak duality: $d^\star \le p^\star$ always

strong duality: if primal is convex then (usually) $d^\star = p^\star$

Slater condition: if primal is convex (*i.e.*, f_i convex, h_i affine) and strictly feasible, *i.e.*, there exists $x \in \textbf{relint}\, D$ s.t.

$$f_i(x) < 0, \quad h_i(x) = 0,$$

then $d^\star = p^\star$

KKT optimality conditions with equality constraints

assume f_i, h_i differentiable

if $x^\star$, $\lambda^\star$, $\nu^\star$ are optimal, with zero duality gap, then they satisfy KKT conditions

$$\begin{aligned} f_i(x^\star) \leq 0,\ h_i(x^\star) &= 0 \\ \lambda_i^\star &\geq 0 \\ \lambda_i^\star f_i(x^\star) &= 0 \\ \nabla f_0(x^\star) + \sum_i \lambda_i^\star \nabla f_i(x^\star) + \sum_i \nu_i^\star \nabla h_i(x^\star) &= 0 \end{aligned}$$

conversely, if they satisfy KKT and the problem is convex, then $x^\star$, $\lambda^\star$, $\nu^\star$ are optimal

KKT of LP in standard form

$$Ax = b$$
$$x \geq 0$$
$$A^\top \nu + \lambda = c$$
$$\lambda \geq 0$$
$$x^\top \lambda = 0$$

$$\lambda = \nu$$
$$s = \lambda$$

change of notation

(the second one is used in some scripts)

$$Ax = b$$
$$x \geq 0$$
$$A^\top \lambda + s = c$$
$$s \geq 0$$
$$x^\top s = 0$$

Duality problem: example

Consider the following LP

$$\begin{aligned} max\ 5x_1 + 12x_2 + 4x_3 \\ x_1 + 2x_2 + x_3 \leq 10 \\ 2x_1 - x_2 + 3x_3 = 8 \\ x_1, x_2, x_3 \geq 0 \end{aligned}$$

Find the dual problem (first rewrite the primal LP in standard form)

Parametric programming

- General formulation

$$f^*(x) = \inf_z \; f(z,x)$$
$$g(z,x) \leq 0$$

  - $\boldsymbol{z} \in \boldsymbol{Z} \subseteq \boldsymbol{R}^m$ is the vector of optimization variables
  - $\mathrm{x} \in \boldsymbol{X} \subseteq \boldsymbol{R}^n$ is the vector of parameters (i.e. initial state conditions)
    - When $\boldsymbol{x} \in \boldsymbol{R}^n$ with $n>1$, then we talk about multi-parametric programming

- **Parametric program:** solution for the full range of parameters **x**

- **We consider**

  1. **Multi-parametric LPs (mp-LPs)**
  2. **Multi-parametric QPs (mp-QPs)**

Multiparametric LP (mp-LP)

Formulation: given $c \in R^m, G \in R^{q\times m}, S \in R^{q\times n}, W \in R^q$

- Primal problem

$$J^*(x) = \min_z \quad J(z,x) = c^T z,$$
$$\text{subj. to} \quad Gz \leq W + Sx$$

Dual problem

$$\max_\pi \quad (W + Sx)^T \pi,$$
$$\text{subj. to} \quad G^T \pi = c,$$
$$\pi \leq 0$$

KKT Conditions

- $Gz - Sx - W \leq 0$
- $\pi \leq 0$
- $(Gz - Sx - W)^T \pi = 0$
- $G^T \pi = c$

Multiparametric LP (mp-LP)

Feasible set: $\boldsymbol{X}^* \subseteq \boldsymbol{X}$ is the set of all points for which exist a solution to the primal (P.P.) and the dual problem (D.P.).

Denoting with $I = \{1, \ldots, q\}$ the constraint index set, define

- Set of Active constraints $\mathcal{A}(x) = \{i \in \mathcal{I} \mid \forall z : J(z,x) = J^*(x) \Rightarrow G_i z - S_i x - W_i = 0\}$
- Set of Inactive constraints $\mathcal{N}(x) = \{i \in \mathcal{I} \mid \exists z : J(z,x) = J^*(x) \wedge G_i z - S_i x - W_i < 0\}$
- Critical region for a given set $\mathcal{A}^* \subseteq \mathcal{I}$

$$\mathcal{CR}_{\mathcal{A}^*} = \{x \in \mathcal{X} \mid \mathcal{A}(x) = \mathcal{A}^*\}$$

mp-LP: geometric approach

- exploration of the parameter space X using geometric methods
- requires an LP solver

Step 1: solve an LP: for an initial parameter vector $x_0 \in \mathcal{X}$:

1. Solve P.P. & D.P. $\Rightarrow z^*(x_0)$ and $\pi^*(x_0)$
2. Obtain $\mathcal{A}_0 = \mathcal{A}(x_0)$ and $\mathcal{N}_0 = \mathcal{N}(x_0)$ and matrices

$$\{G_{\mathcal{A}_0}, S_{\mathcal{A}_0}, W_{\mathcal{A}_0}\} = \{G_i, S_i, W_i \mid i \in \mathcal{A}(x_0)\}$$
$$\{G_{\mathcal{N}_0}, S_{\mathcal{N}_0}, W_{\mathcal{N}_0}\} = \{G_i, S_i, W_i \mid i \in \mathcal{N}(x_0)\}$$

mp-LP: geometric approach

- exploration of the parameter space X using geometric methods
- requires an LP solver

Step 2: determine : $\mathcal{CR}_{\mathcal{A}_0}$, $z_0^*(x)$ and $J_0^*(x)$

1. From the primal feasibility conditions

$$\begin{aligned} G_{\mathcal{A}_0} z_0^*(x) &= W_{\mathcal{A}_0} + S_{\mathcal{A}_0} x \\ G_{\mathcal{N}_0} z_0^*(x) &< W_{\mathcal{N}_0} + S_{\mathcal{N}_0} x \end{aligned}$$

2. Compute optimizer (as a function of x)

$$z_0^*(x) = G_{\mathcal{A}_0}^{-1} S_{\mathcal{A}_0} x + G_{\mathcal{A}_0}^{-1} W_{\mathcal{A}_0} = F_0 x + g_0$$

3. Critical region

$$\mathcal{CR}_{\mathcal{A}_0} = \left\{ x \mid (G_{\mathcal{N}_0} F_0 - S_{\mathcal{N}_0}) x < W_{\mathcal{N}_0} - G_{\mathcal{N}_0} g_0 \right\}$$

4. Compute the optimal value function using the D.P. (strong duality holds)

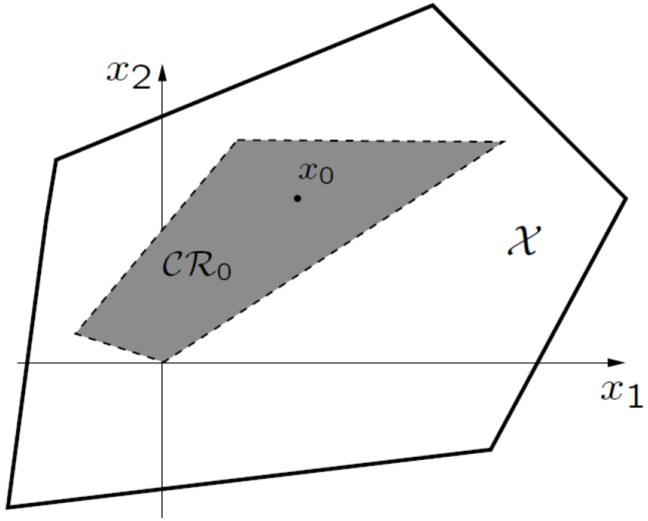
$$J_0^*(x) = (S_{\mathcal{A}_0} x + W_{\mathcal{A}_0})^T \pi_{\mathcal{A}_0}^*$$

mp-LP: geometric approach

- Critical region $\mathcal{CR}_{\mathcal{A}_0}$ ($\mathcal{CR}_0$ from now on) is defined by strict inequalities → an **open polyhedral set**
- $\mathcal{A} = \mathcal{A}(x)$ uniquely determines $\mathcal{CR}_{\mathcal{A}}$ → **the regions do not overlap!**
- the **optimizer** $z^*(x)$ is **affine** over $\mathcal{CR}_0$

Step 3: explore the rest of *X*

- *Replace* $\mathcal{CR}_0$ *by its closure* $\bar{\mathcal{CR}}_0$
- For $x \in \mathcal{X} \setminus \mathcal{CR}_0$ find optimality conditions and corresponding critical regions covering the entire feasible set X^*
- *Different exploration strategies*



mp-LP: geometric approach

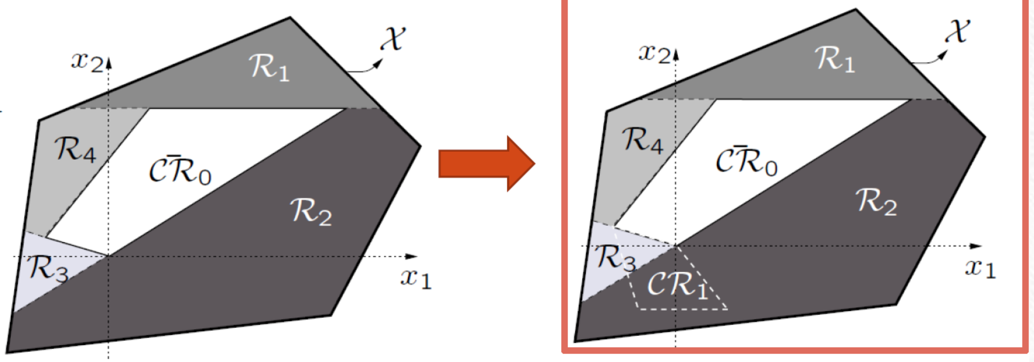
1st approach: reversing inequalities

- Note: regions *Ri* **are not** critical regions
- Proceed recursively: repeat the whole procedure for each Ri
- The entire set *X* is explored in finite number of iterations
- Problem: critical regions can be artificially divided among different Ri

$$\mathcal{CR}_0 = \{x \in \mathcal{X} \mid Hx \le K\}$$

$$\mathcal{R}_i = \{x \in \mathcal{X} \mid H_i x \ge K_i \wedge H_j x \le K_j,\ \forall j < i\}$$

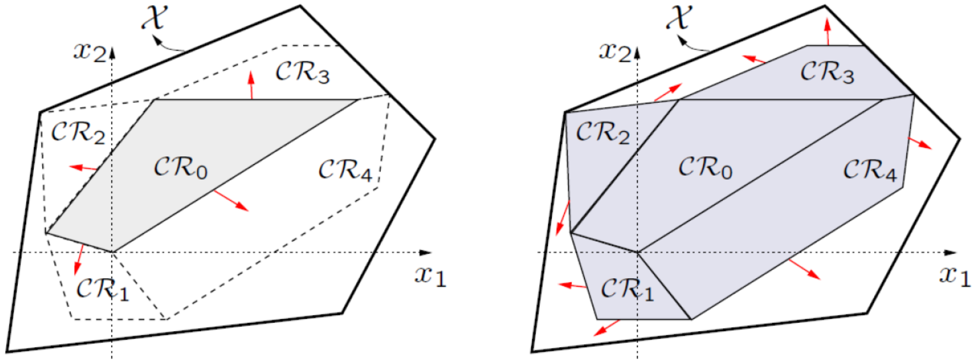
$$\mathcal{X} \setminus \mathcal{CR}_0 = \bigcup_i \mathcal{R}_i$$



mp-LP: geometric approach

2nd approach: crossing the facets

- For each of the facets of CR_0 a point outside the region but close to the facet is selected and the procedure is repeated
- Critical regions are computed "in one piece", no artificial splitting
- No formal proof that whole X is covered
  - In practice usually outperforms the strategy based on reversing inequalities



mp-LP: properties

1. Feasible set X* is closed and convex
2. If the optimal solution z* is unique for all x in X*, the optimizer function z(x) : $X^* \rightarrow R^m$ is
   - Continuous
   - Polyhedral piecewise affine (PPWA) over X*
   - Affine in each *CRi*

   If the solution is not unique, it is always possible to choose a continuous and PPWA optimizer function z*(x)
3. The value function J*(x): $X^* \rightarrow R$ is
   - Convex
   - PPWA over X*, affine in each *CRi*

mp-QP

Formulation: multi-parametric quadratic programs of the form *(Assume H>0)*

$$J^*(x) = \min_z \quad \left\{ J(z,x) = \tfrac{1}{2} z^T H z \right\}$$
$$\text{subj. to} \qquad Gz \leq W + Sx$$

where $z \in \mathcal{Z} \subseteq \mathbb{R}^m$, and $x \in \mathcal{X} \subseteq \mathbb{R}^n$, $G \in R^{q \times m}$.

KKT Conditions

$$Hz^* + G^T \lambda^* = 0,$$
$$\lambda_i^*(G_i z^* - S_i x - W_i) = 0, \quad i = 1, \ldots, q$$
$$\lambda^* \geq 0,$$
$$Gz^* \leq W + Sx$$

mp-QP: geometric algorithm

- exploration of the parameter space X using geometric methods
- requires an QP solver

Step 1: solve an QP: for an initial parameter vector $x_0 \in \mathcal{X}$:

1. Solve P.P. & D.P. $\Rightarrow$ $z^*(x_0)$ and $\lambda^*(x_0)$
2. Obtain $\mathcal{A}_0 = \mathcal{A}(x_0)$ and $\mathcal{N}_0 = \mathcal{N}(x_0)$ and matrices

$$\begin{aligned} \{G_{\mathcal{A}_0}, S_{\mathcal{A}_0}, W_{\mathcal{A}_0}\} &= \{G_i, S_i, W_i \mid i \in \mathcal{A}(x_0)\} \\ \{G_{\mathcal{N}_0}, S_{\mathcal{N}_0}, W_{\mathcal{N}_0}\} &= \{G_i, S_i, W_i \mid i \in \mathcal{N}(x_0)\} \end{aligned}$$

mp-QP: geometric algorithm

- exploration of the parameter space X using geometric methods
- requires an QP solver

Step 2: determine : $\mathcal{CR}_{\mathcal{A}_0}$, $z_0^*(x)$ and $J_0^*(x)$

1. From the primal feasibility conditions

$$\begin{aligned} G_{\mathcal{A}_0} z_0^*(x) &= W_{\mathcal{A}_0} + S_{\mathcal{A}_0} x \\ G_{\mathcal{N}_0} z_0^*(x) &< W_{\mathcal{N}_0} + S_{\mathcal{N}_0} x \end{aligned}$$

2. Since $z^* = -H^{-1}G^T\lambda^*$ and $\lambda_{\mathcal{A}}^*(x) = -(G_{\mathcal{A}}H^{-1}G_{\mathcal{A}}^T)^{-1}(W_{\mathcal{A}} + S_{\mathcal{A}}x)$

   the optimizer function is $z^*(x) = H^{-1}G_{\mathcal{A}}^T(G_{\mathcal{A}}H^{-1}G_{\mathcal{A}}^T)^{-1}(W_{\mathcal{A}}+S_{\mathcal{A}}x)$

3. Critical region

$$\mathcal{CR}_{\mathcal{A}} = \{x \mid Ax \le b\}$$

$$A = \begin{bmatrix} GH^{-1}G_{\mathcal{A}}^T(G_{\mathcal{A}}H^{-1}G_{\mathcal{A}}^T)^{-1}S_{\mathcal{A}} - S \\ (G_{\mathcal{A}}H^{-1}G_{\mathcal{A}}^T)^{-1}S_{\mathcal{A}} \end{bmatrix}$$

$$b = \begin{bmatrix} W + GH^{-1}G_{\mathcal{A}}^T(G_{\mathcal{A}}H^{-1}G_{\mathcal{A}}^T)^{-1}W_{\mathcal{A}} \\ (G_{\mathcal{A}}H^{-1}G_{\mathcal{A}}^T)^{-1}W_{\mathcal{A}} \end{bmatrix}$$

mp-QP: geometric algorithm

- Note that the function $z^*(x)$ is a **uniquely defined affine function** over the critical region CR_A
- Moreover, the critical region is a **polyhedral set** in the x-space

Step 3: explore the rest of *X (as before..)*

mp-QP: properties

1. Feasible set X* is closed and convex
2. The optimizer function z(x) : $X^* \rightarrow R^m$ is
   - Continuous
   - Polyhedral piecewise affine (PPWA) over X*
   - Affine in each *CRi*
3. The value function J*(x): X* → R is
   - Continuous
   - Convex
   - polyhedral piecewise quadratic (PPWQ) (piecewise quadratic over the critical regions CRi