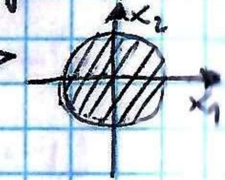
Exercises - Convexity

- $\min_{x_1 x_2} \; 3x_1 + 5x_2$ s.t. $(x_1+x_2)^2 \le 1$ Is a convex problem?
  - $\to$ cost is a linear function of $[x_1, x_2]^T$
  - $\to$ feasibility set is convex $\to$



$\Rightarrow$ The problem is convex

- $\min_{x_1 x_2} \; -x_1^2$ s.t. $(x_1+x_2)^2 \le 1$ isn't convex because the cost is concave

- $\max_{x_1 x_2} \; -x_1^2$ s.t. $(x_1+x_2)^2 \le 1$ is convex because $(\max -x_1^2) = (-\min x_1^2)$

- $\min_{x_1 x_2} \; x_1^2 + 3x_1 + 4x_1x_2 = \min_{x_1 x_2} [x_1 \; x_2]\begin{bmatrix}1 & 2\\ 2 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + [3 \; 0]\begin{bmatrix}x_1\\ x_2\end{bmatrix}$

$\hookrightarrow \det\begin{pmatrix}\lambda-1 & -2\\ -2 & \lambda\end{pmatrix} = (\lambda-1)\lambda - 4 = 0 \to \lambda^2 - \lambda - 4 = 0$

$\lambda_{1,2} = \frac{1 \pm \sqrt{17}}{2}$ $\to$ the matrix isn't semidefinite positive $\to$ not convex

Exercise - Duality

- $\min_{x_1 x_2} \; x_1^2 + 2x_1x_2 + 4x_2^2 + 6x_1$ $\to$ cost: $[x_1 \; x_2]\underbrace{\begin{bmatrix}1 & 1\\ 1 & 4\end{bmatrix}}_{P}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \underbrace{[6 \;\; 0]}_{c^T}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$

  s.t. $x_1 \le 3x_2 + 5$

  $x_1 - 3x_2 \le 5$

  $[1 \; -3]$ $\quad$ 5

First, check convexity: $\det\begin{pmatrix}\lambda-1 & -1\\ -1 & \lambda-4\end{pmatrix} =$

$= \lambda^2 - 5\lambda + 3 = 0$

$\Rightarrow \lambda_{1,2} = \frac{5 \pm \sqrt{13}}{2} > 0$ ok

$L(\lambda, x) = x^T P x + c^T x + \lambda^T (Ax - b)$

$g(\lambda) = \min_x L(\lambda, x) \longrightarrow \nabla_x L(\lambda, x) \overset{!}{=} 0 \to \underbrace{2Px + c + A^T\lambda}_{\mathbb{R}^{2\times 1}} = 0 \to x = \frac{-P^{-1}}{2}(c + A^T\lambda)$

$\Rightarrow g(\lambda) = \frac{1}{4}(P^{-1}(c+A^T\lambda))^T P (P^{-1}(c+A^T\lambda)) - \frac{1}{2}c^T(P^{-1}(c+A^T\lambda)) + \lambda^T(-\frac{1}{2}AP^{-1}(c+A^T\lambda) - b)$

Note: $P = P^T$

$= \frac{1}{4}(c+A^T\lambda)^T P^{-1} \cancel{P P^{-1}} (c+A^T\lambda) - \frac{1}{2}c^T P^{-1}(c+A^T\lambda) - \frac{1}{2}\lambda^T A P^{-1}(c+A^T\lambda) - \lambda^T b$

$= \frac{1}{4}(c+A^T\lambda)^T P^{-1}(c+A^T\lambda) - \frac{1}{2}(c+A^T\lambda)^T P^{-1}(c+A^T\lambda) - \lambda^T b$

$= -\frac{1}{4}(c+A^T\lambda)^T P^{-1}(c+A^T\lambda) - \lambda^T b \to$ split quadratic and linear part $\to$

Dual problem: $\left[\max_\lambda \; -\frac{1}{4}\lambda^T A P^{-1} A^T \lambda + \left(-\frac{1}{2}c^T P^{-1} A^T - b^T\right)\lambda\right] \left(-\frac{1}{4}c^T P^{-1} c\right)$ s.t. $\lambda \ge 0$