Dual of LP problems – Part 2

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Specialization of duality theory to LPs

Reference problems

$$\min_x c^T x \quad (P) \qquad\Bigg|\qquad \max_\lambda b^T\lambda \quad (D)$$

primal constraints | dual constraints

- x primal feasible and λ dual feasible $\Rightarrow c^T x \geq b^T \lambda$
  - ↳ (P) unbounded $\Rightarrow$ (D) infeasible, meaning that no λ verifies the dual constraints
  - ↳ (D) unbounded $\Rightarrow$ (P) infeasible

  Proof: By contradiction assume (D) unbounded and (P) feasible. $\forall K>0\ \exists \lambda : b^T\lambda > K$. Pick $K = c^T x$, where x is feasible for (P). One gets $b^T\lambda > c^T x$ that is a contradiction (violation of weak duality)

Rmk. • It can happen that both (P) and (D) are infeasible

• (P) is convex with affine constraints. (P) feasible $\Rightarrow$ constraints are qualified $\Rightarrow$ $J^* = D^*$ (strong duality)

Economical meaning of strong duality

• Product mix: DUAL gives to PRIMAL the same money PRIMAL would get from selling the optimal number of cases

• Diet problem: PRIMAL pays the same amount of money for buying individual vitamines from DUAL or boxes of additive 1 and 2

Optimal choices lead to an equilibrium of market prices

Relations between optimizers of (P) and (D)

Thm. Let (P) be in standard form. (P) has an optimal solution x^* if and only if (D) has an optimal solution λ^*. Moreover, if B is the optimal basis for (P), one has

$$(\lambda^*)^T = c_B^T B^{-1}$$

Thm (complementary slackness). Let (P) and (D) be related by the transformation table. Let x be primal feasible and λ be dual feasible. The following conditions are necessary and sufficient for optimality of x and λ

$$\lambda_i \left(\underbrace{a_i^T}_{\text{row } i \text{ of } A} x - b_i \right) = 0 \qquad i = 1, \dots, m \qquad (CS1)$$

$$x_j \left(c_j - \underbrace{A_j^T}_{\text{row } j \text{ of } A^T} \lambda \right) = 0 \qquad j = 1, \dots, n \qquad (CS2)$$

Proof From the table, it holds by construction $\lambda_i (a_i^T x - b_i) \leq 0$ and $x_j (c_j - A_j^T \lambda) \leq 0$.

Then,

$$0 \geq c^T x - \lambda^T b = \lambda^T A x - \lambda^T b - \lambda^T A x + c^T x = \underbrace{\sum_{i=1}^{m} \lambda_i (a_i^T x - b_i)}_{\lambda^T A x - \lambda^T b} + \underbrace{\sum_{j=1}^{n} x_j (c_j - A_j^T \lambda)}_{c^T x - \lambda^T A x}$$

↳ weak duality. Since in the table the primal is a "max" problem then $\lambda^T b \geq c^T x$

($\Rightarrow$) If x and λ are optimal, then $c^T x - \lambda^T b = 0$ (strong duality). Hence all summands are zero (they have the same sign)

($\Leftarrow$) Complementarity slackness means that all summands are zero. Hence $c^T x - \lambda^T b = 0$ and it follows that x and λ are optimal

Remarks

Meaning of (CS1)

Let x and λ be optimal solutions.
From (CS1) one has

- $a_i^T x < b_i$ (inactive primal constraint) $\Rightarrow \lambda_i = 0$
- $\lambda_i > 0 \Rightarrow a_i^T x = b_i$ (active primal constraint)

Relations between (cs 2) and simplex optimality conditions

Complementarity slackness

$$\lambda_i \left(a_i^T x - b_i \right) = 0 \qquad i = 1, \dots, m \qquad (cs\ 1)$$

$$x_j \left(c_j - A_j^T \lambda \right) = 0 \qquad j = 1, \dots, n \qquad (cs\ 2)$$

Let (P) be in standard form and B be an optimal basis. The vector $r^T = c^T - c_B^T B^{-1} A$ verifies

$$r_B^T = c_B^T - c_B^T B^{-1} B = 0 \qquad \text{(reduced costs for basic variables)}$$

$$r_F^T = c_F^T - c_B^T B^{-1} F \qquad \text{(reduced costs)}$$

The reduced cost associated to x_j is

$$r_j = c_j - c_B^T B^{-1} A_j = c_j - A_j^T \left(B^{-1}\right)^T c_B = c_j - A_j^T \lambda$$

Complementarity slackness

$$\lambda_i \left(a_i^T x - b_i \right) = 0 \quad i = 1, \dots, m \qquad (CS1)$$
$$x_j \left(c_j - A_j^T \lambda \right) = 0 \quad j = 1, \dots, n \qquad (CS2)$$

Then,

(i) $x_j > 0$ (x_j is basic) $\Rightarrow c_j - A_j^T \lambda = r_j = 0$

↳ From (CS2)

(ii) $c_j - A_j' \lambda = r_j < 0 \Rightarrow x_j = 0$

because (P) is a "max"

↳ From (CS2)

→ Same remarks if (P) is a min and $c_j - A_j' \lambda > 0$ for λ optimal

Ex. Product mix

$$\max \ 30x_1 + 20x_2$$
$$8x_1 + 4x_2 \le 640$$
$$4x_1 + 6x_2 \le 540$$
$$x_1 + x_2 \le 100$$
$$x_1, x_2 \ge 0$$

$$x^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} 60 \\ 40 \end{bmatrix}$$

Pbl: compute optimal multipliers λ^*

Since the problem is in generic form and we know primal optimizers, we use complementarity slackness conditions

From (CS1)

$$\lambda_1^*\left(8\cdot 60+4\cdot 40-640\right)=0$$

$\hookrightarrow \lambda_1^*\cdot 0=0 \longrightarrow$ no constraint on λ_1^*

$$\lambda_2^*\left(4\cdot 60+6\cdot 40-540\right)=0$$

$\hookrightarrow \lambda_2^*\cdot 60=0 \longrightarrow \lambda_2^*=0$

$$\lambda_3^*\left(60+40-100\right)=0$$

$\hookrightarrow \lambda_3^*\cdot 0=0 \longrightarrow$ no constraint on λ_3^*

Complementarity slackness

$\lambda_i\left(a_i^T x-b_i\right)=0 \quad i=1,\dots,m \quad$ (CS1)

$x_j\left(c_j-A_j^T\lambda\right)=0 \quad j=1,\dots,n \quad$ (CS2)

PRIMAL

$\max\ 30x_1+20x_2$

$8x_1+4x_2\le 640 \quad (\lambda_1)$

$4x_1+6x_2\le 540 \quad (\lambda_2)$

$x_1+x_2\le 100 \quad (\lambda_3)$

$x_1,x_2\ge 0$

From (CS2)

$$x_1^* \left(30 - 8\lambda_1^* - 4\lambda_2^* - \lambda_3^* \right) = 0$$

$$x_2^* \left(20 - 4\lambda_1^* - 6\lambda_2^* - \lambda_3^* \right) = 0$$

$$\Big\downarrow \; x_1^*, x_2^* > 0 \text{ and } \lambda_2^* = 0$$

$$\begin{cases} 30 = 8\lambda_1^* + \lambda_3^* \\ 20 = 4\lambda_1^* + \lambda_3^* \end{cases}$$

↳ One gets $\lambda_1^* = \frac{5}{2}$ and $\lambda_3^* = 10$

Complementarity slackness

$\lambda_i \left(a_i^T x - b_i \right) = 0 \quad i = 1, \dots, m \quad$ (CS1)

$x_j \left(c_j - A_j^T \lambda \right) = 0 \quad j = 1, \dots, n \quad$ (CS2)

PRIMAL

$\max \; 30x_1 + 20x_2$

$8x_1 + 4x_2 \leq 640 \quad (\lambda_1)$

$4x_1 + 6x_2 \leq 540 \quad (\lambda_2)$

$x_1 + x_2 \leq 100 \quad (\lambda_3)$

$x_1, x_2 \geq 0$

Rmk. For using $(\lambda^*)^T = c_B^T B^{-1}$ one must put (P) in the standard form for getting the optimal basis B

@ Home: check that optimal primal and dual costs are the same.