Linear programming: sensitivity analysis

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Introduction

Sensitivity analysis is also known as "post-optimality" analysis because it has to be performed after an optimal solution has been computed.

How the optimizers change when

P1) the cost changes?

P2) the vector b changes?

Entries in the matrix A could change as well but this will not be covered in this class.

Ex. Product mix

$$\max \; 30x_1 + 20x_2$$
$$8x_1 + 4x_2 \le 640$$
$$4x_1 + 6x_2 \le 540$$
$$x_1 + x_2 \le 100$$
$$x_1, x_2 \ge 0$$

$$x^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} 60 \\ 40 \end{bmatrix}$$

(P1) $\longleftrightarrow$ robustness of the optimal production plan when revenues are uncertain

(P2) $\longleftrightarrow$ is it convenient to buy additional resources (i.e. man/hours in the production phases)?

Ex. of problem P1

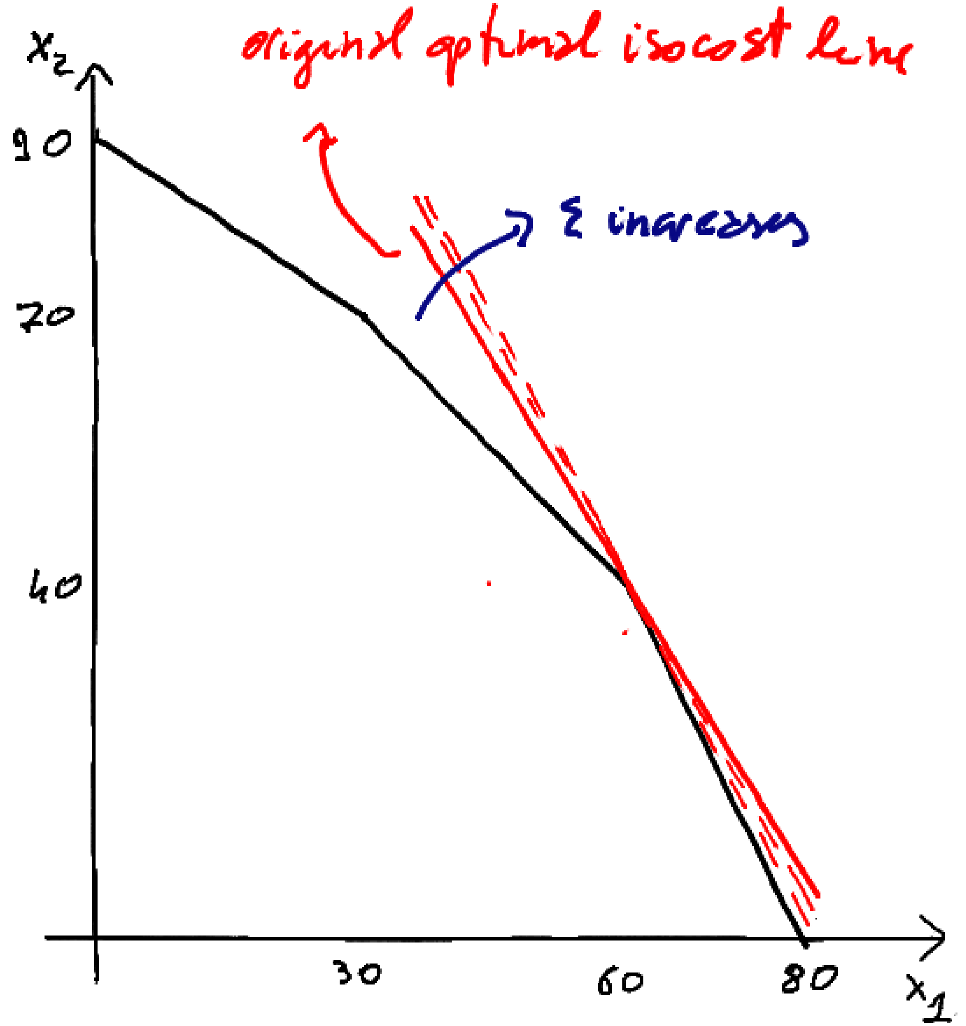
$c^T = [c_1 \; c_2] = [30 \;\; 20]$

How much c_1 can be increased without changing the optimal solution?

$$C_\alpha : (30+\varepsilon)\,x_1 + 20\,x_2 = \alpha$$

$$\hookrightarrow x_2 = \frac{\alpha}{20} - \frac{30+\varepsilon}{20}\,x_1$$

$\varepsilon > 0$ increases $\rightarrow C_\alpha$ rotates clockwise
$\rightarrow$ the optimal solution does not change until the optimal C_α intersects also the vertex $[80, 0]^T$

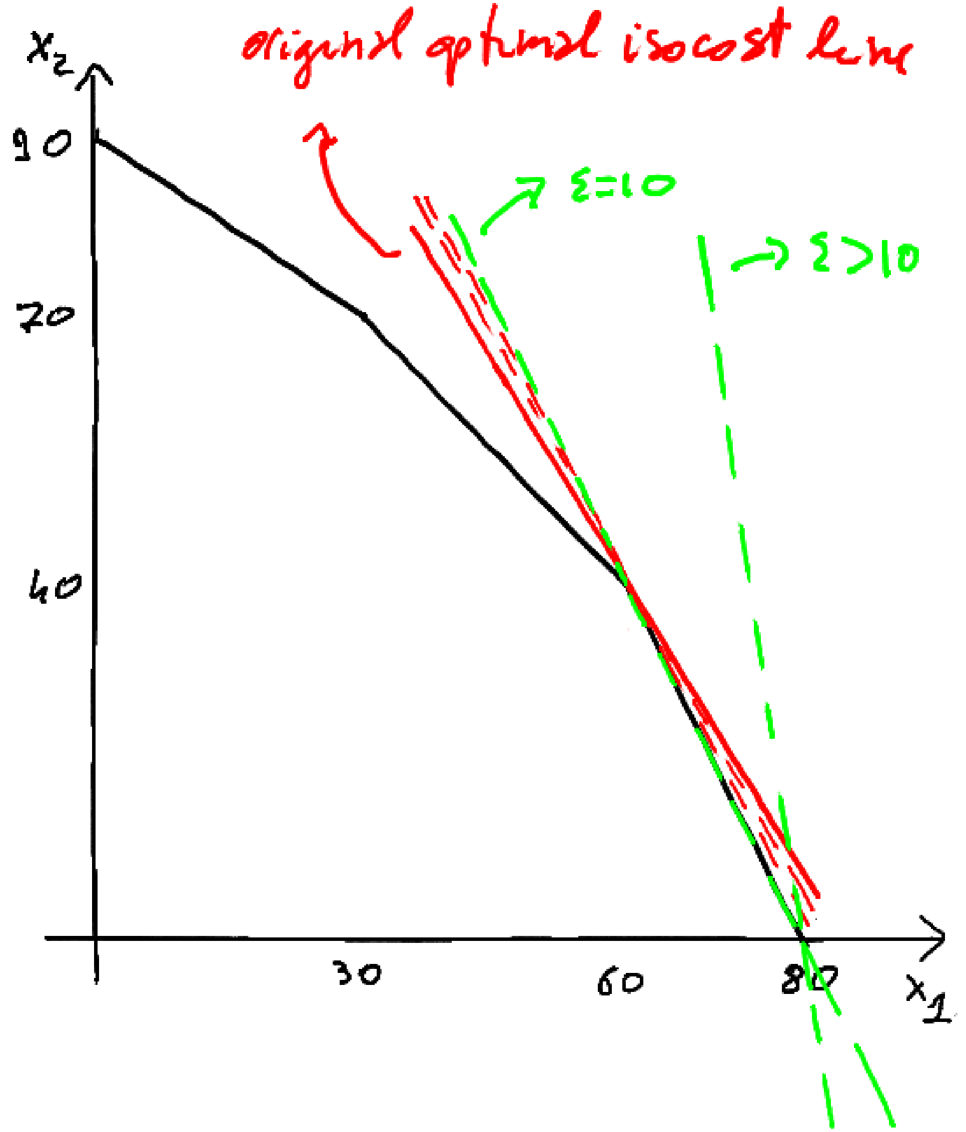


Compute the maximal ε

$$x^T = [60 \;\; 40] \rightarrow (30+\varepsilon)\,60 + 20 \cdot 40 = \alpha$$
$$x^T = [80, 0] \rightarrow (30+\varepsilon)\,80 = \alpha$$

$\varepsilon = 10$ and $\alpha = 3200$

- $\varepsilon = 10$: the optimal solution is not unique
- $\varepsilon > 10$: the optimal solution becomes $[80 \;\; 0]^T$

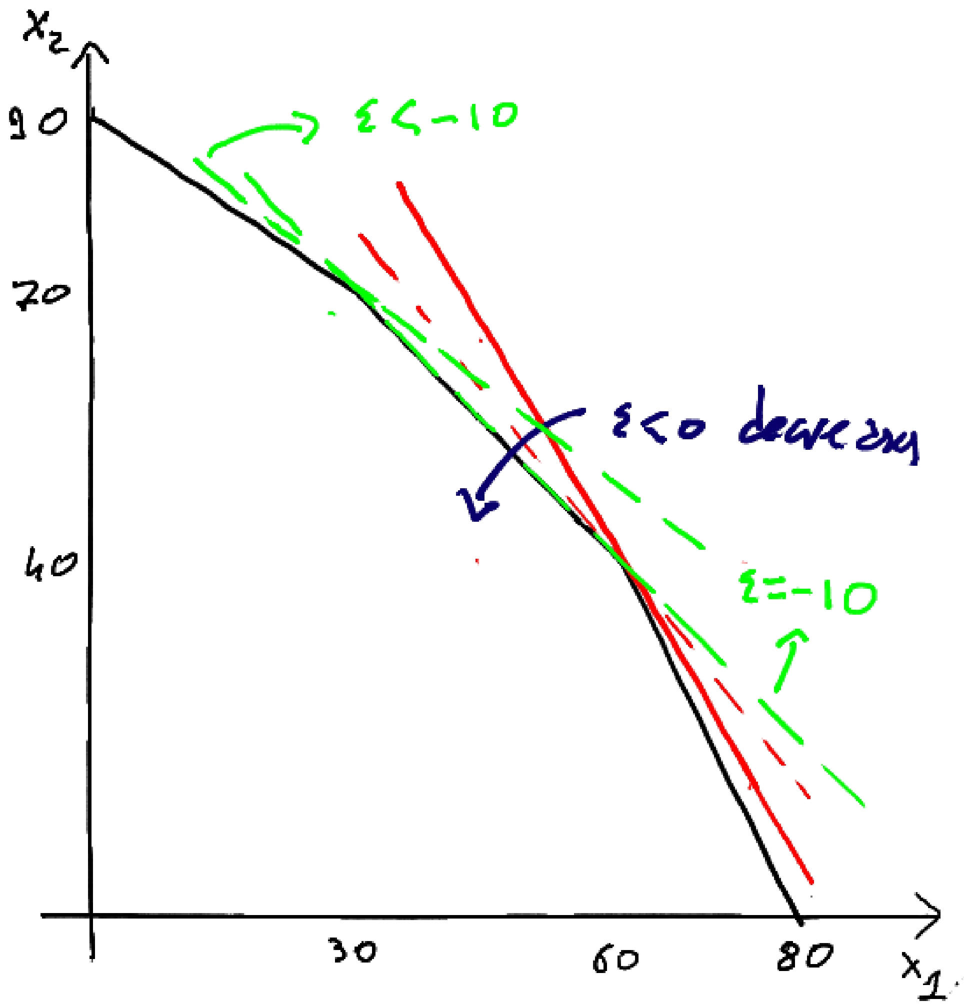


How much c_1 can be decreased without changing the optimal solution?

$\varepsilon < 0$ decreases $\to$ c_α rotates counterclockwise $\to$ the optimal solution does not change until c_α intersects $[30\ 70]^T$.

⋮ same computations as before

This happens for $\varepsilon = -10$



Conclusion: for $c_1 \in [20, 40]$ the optimal solution does not change

Ex. @ home: same analysis for $c_1 = 30$ and $c_2 = 20 + \varepsilon$

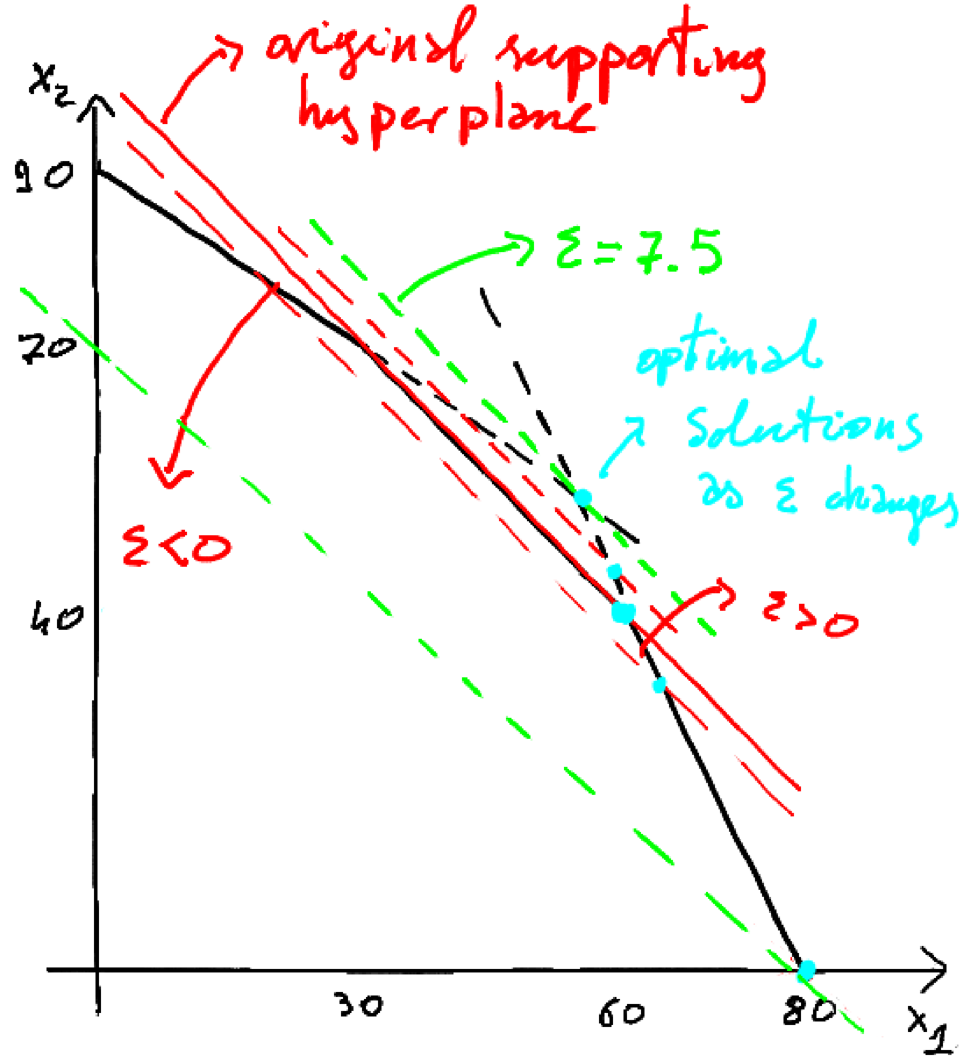
Ex. of problem P2

Change of resources in the quality control phase: the constraint $x_1 + x_2 \le 100$ gets replaced by

$$x_1 + x_2 \le 100 + \varepsilon$$

$$\downarrow$$

Supporting hyperplane: $x_2 = (100 + \varepsilon) - x_1$



The optimal solution changes for $\varepsilon \neq 0$ but it lies on the same supporting hyperplanes if ε does not vary too much

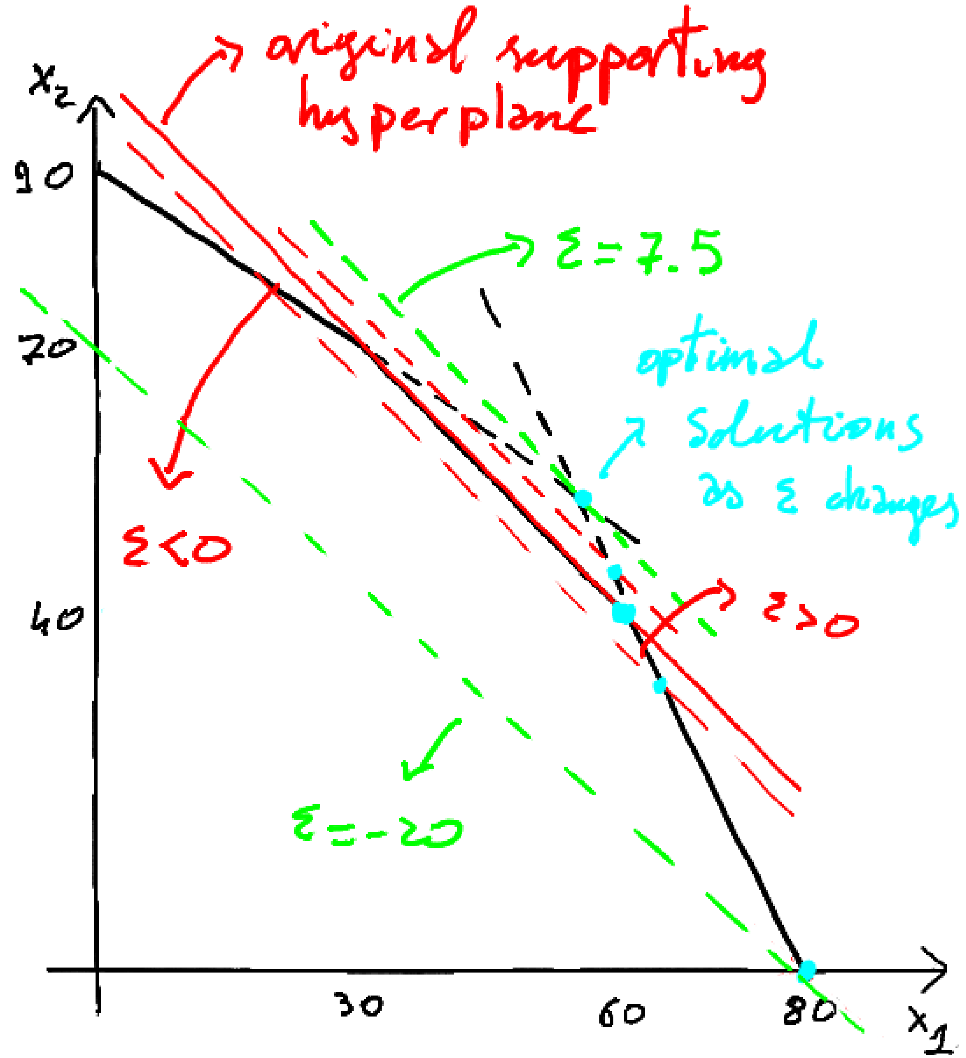
↳ The optimal basis does not change

$\varepsilon < 0$: the optimal basis does not change until $\varepsilon \in [-20, 0]$

↳ $\varepsilon = -20$: the supporting hyperplanes intersects $[80, 0]^T$

$\varepsilon > 0$: the optimal basis does not change until $\varepsilon \in [0, 7.5]$

↳ $\varepsilon = 7.5$: the constraint becomes redundant



Sensitivity analysis: general theory

Reference problem: standard LP

$$z^* = \max c^T x$$
$$Ax = b$$
$$x \geq 0$$

Optimal BFS given by x_B and $x_F = 0$. Assumption: it is not degenerate.

Optimality conditions

$$(O1): \quad x_B = B^{-1} b \geq 0 \qquad \text{(feasibility)}$$
$$(O2): \quad r_F^T = c_F^T - c_B^T B^{-1} F \leq 0 \qquad \text{(optimality)}$$

Sensitivity analysis: study when B is still optimal as

(a) only b changes
(b) only c_B changes
(c) only c_F changes

Case (a) : $b \longrightarrow b + \Delta b$

(O1) : $x_B = B^{-1}(b + \Delta b) \geq 0$

(O2) does not change

B does not change until Δb verifies

$$B^{-1} b \geq - B^{-1} \Delta b$$

admissible perturbations lie on a polyhedron

However the optimal solution changes and the optimal cost becomes

$$c_B^T B^{-1}(b + \Delta b) = (\lambda^*)^T (b + \Delta b) \qquad \lambda^* = \text{optimal multipliers}$$

Rmk. λ_i^* measures the sensitivity of the optimal cost to Δb_i

Case (b): $c_B \to c_B + \Delta c_B$

Let $\bar{r}_F^T = c_F^T - c_B^T B^{-1} F$ (reduced cost for $\Delta c_B = 0$)

(O1) does not change

(O2) $c_F^T - (c_B^T + \Delta c_B^T) B^{-1} F \leq 0$

$\hookrightarrow \underbrace{c_F^T - c_B^T B^{-1} F}_{\bar{r}_F^T} - \Delta c_B^T B^{-1} F \leq 0$

The optimal basis (and the optimal solution) does not change if Δc_B verifies

$$\Delta c_B^T B^{-1} F \geq \bar{r}_F^T$$

$\hookrightarrow$ admissible perturbations lie in a polyhedron

Case (c): $c_F \to c_F + \Delta c_F$

(O1) does not change

(O2): $c_F^T + \Delta c_F^T - c_B^T B^{-1} F \leq 0$

The optimal basis (and the optimal solution) does not change if Δc_F verifies

$$\Delta c_F \leq -\bar{r}_F$$

Rmk. For "min" problems the inequalities in cases (b) and (c) must be reversed

Ex.

$$\max \begin{bmatrix} 3 & 1 & 3 & 0 & 0 & 0 \end{bmatrix} x \qquad x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{bmatrix}^T$$

$$\begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$$

$$x \geq 0$$

The simplex algorithm produces the optimal tableau

$$\begin{array}{cc|cccccc}
 & & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\
 & -\frac{27}{5} & 0 & -\frac{7}{5} & 0 & -\frac{6}{5} & -\frac{3}{5} & 0 \\
\hline
x_1 & \frac{1}{5} & 1 & \frac{1}{5} & 0 & \frac{3}{5} & -\frac{1}{5} & 0 \\
x_3 & \frac{8}{5} & 0 & \frac{3}{5} & 1 & -\frac{1}{5} & \frac{2}{5} & 0 \\
x_6 & 4 & 0 & 1 & 0 & -1 & 0 & 1
\end{array}$$

Is B still optimal when

(i) $b_1 \to b_1 + 5$

(ii) $c_3 \to c_3 + 1$?

Useful quantities from the optimal tableau

$$
\begin{array}{c|c|cccccc}
 & & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\
 & -\frac{27}{5} & 0 & -\frac{7}{5} & 0 & -\frac{6}{5} & -\frac{3}{5} & 0 \\
\hline
x_1 & \frac{1}{5} & 1 & \frac{1}{5} & 0 & \frac{3}{5} & -\frac{1}{5} & 0 \\
x_3 & \frac{8}{5} & 0 & \frac{3}{5} & 1 & -\frac{1}{5} & \frac{2}{5} & 0 \\
x_6 & 4 & 0 & 1 & 0 & -1 & 0 & 1
\end{array}
$$

$$x_B = \begin{bmatrix} x_1 \\ x_3 \\ x_6 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 2 & 1 & 1 \end{bmatrix} \qquad x_B^* = B^{-1} b = \begin{bmatrix} 1/5 \\ 8/5 \\ 4 \end{bmatrix} \qquad x_F = \begin{bmatrix} x_2 \\ x_4 \\ x_5 \end{bmatrix}$$

$$\bar{r}_F = \begin{bmatrix} -7/5 \\ -6/5 \\ -3/5 \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} 3/5 & -1/5 & 0 \\ -1/5 & 2/5 & 0 \\ -1 & 0 & 1 \end{bmatrix} \qquad \bar{F} = B^{-1} F = \begin{bmatrix} 1/5 & 3/5 & -1/5 \\ 3/5 & -1/5 & 2/5 \\ 1 & -1 & 0 \end{bmatrix}$$

Problem (i): $b_1 \to b_1 + 5$

B optimal $\iff B^{-1}b \geq -B^{-1}\Delta b$

$$\Delta b = \begin{bmatrix} \Delta b_1 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} +\frac{1}{5} \\ \frac{8}{5} \\ 4 \end{bmatrix} \geq \begin{bmatrix} -\frac{3}{5}\Delta b_1 \\ \frac{1}{5}\Delta b_1 \\ \Delta b_1 \end{bmatrix} \longrightarrow \begin{cases} \Delta b_1 \geq -\frac{1}{3} \\ \Delta b_1 \leq 8 \\ \Delta b_1 \leq 4 \end{cases}$$

B does not change if $\Delta b_1 \in [-\frac{1}{3}, 4]$.

For $\Delta b_1 = 5$, B is no longer optimal

@ home: is B still optimal when $b_2 \to b_2 - 1$ or $b_3 \to b_3 - 1$?

Problem (ii): $c_3 \to c_3 + 1$

c_3 is an element of c_B

B is still optimal only if $\Delta c_B^T B^{-1} F \geq \bar{r}_F^T$

$$\begin{bmatrix} \Delta c_1 & \Delta c_3 & \Delta c_6 \end{bmatrix} \bar{F} \geq \begin{bmatrix} -\frac{7}{5} & -\frac{6}{5} & -\frac{3}{5} \end{bmatrix}$$

$$\left.\begin{aligned} \frac{1}{5}\Delta c_1 + \frac{3}{5}\Delta c_3 + \Delta c_6 &\geq -\frac{7}{5} \\ \frac{3}{5}\Delta c_1 - \frac{1}{5}\Delta c_3 - \Delta c_6 &\geq -\frac{6}{5} \\ -\frac{1}{5}\Delta c_1 + \frac{2}{5}\Delta c_3 &\geq -\frac{3}{5} \end{aligned}\right\} \xrightarrow{\Delta c_1 = \Delta c_6 = 0} \left\{\begin{aligned} \Delta c_3 &\geq -\frac{7}{5} \\ \Delta c_3 &\leq 6 \\ \Delta c_3 &\geq -\frac{3}{2} \end{aligned}\right.$$

B does not change if $\Delta c_3 \in [-\frac{3}{2}, 6]$

$\downarrow$

B is still optimal for $c_3 \to c_3 + 1$

Next: exercises !