Floyd-Warshall algorithm

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Shortest path problem

Problem (S3). Let $G = (V, E, c)$ be a directed network where all cycles have cost ≥ 0. Compute shortest paths from each vertex to all other vertices

Floyd-Warshall algorithm - two steps:

Step 1: init. $D \in \mathbb{R}^{n \times n}$ is the cost matrix ($n = |V|$). Set

$$d_{ij} = \begin{cases} c(i,j) & \text{if } (i,j) \in E \\ \infty & \text{otherwise} \end{cases}$$

When the algorithm stops, $d_{ij} < +\infty$ is the cost of a shortest path from i to j ($d_{ij} = +\infty$ means that j cannot be reached from i)

- $pred(i,j)$ is the predecessor of node j in the current path from node i. Set

$$pred(i,j) = i, \quad \forall i,j \in V$$

When the algorithm stops, $pred(i,j)$ is the predecessor of j in the computed shortest path from i to j

Step 2: main cycle. For $k = 1, \dots, n$

•) Triangular update (on node k). For $i = 1, \dots, n$ and $j = 1, \dots, n$

if

$$d_{ij} > d_{ik} + d_{kj}$$

then

$$d_{ij} = d_{ik} + d_{kj}$$
$$\text{pred}(i,j) = \text{pred}(k,j)$$

MEANING: it is convenient to go through k for reaching j

If $\exists\, i \in V : d_{ii} < 0$ then STOP: there is a cycle with cost < 0

Rmk. No need to verify a priori that all cycles must have cost ≥ 0!

Correctness of the algorithm

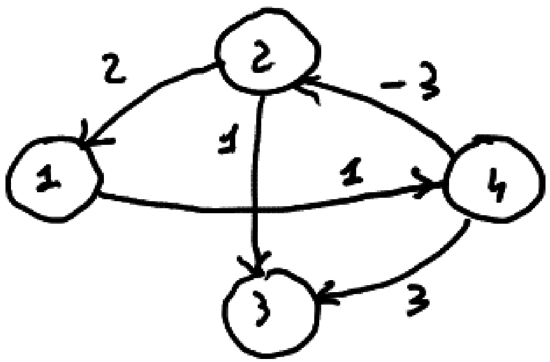
Thm. Let D be the matrix obtained after having completed a triangular operation on node k. Then, for all $i, j \in V$, the value d_{ij} is the cost of a shortest path from i to j in the subgraph $(\tilde{V}, E(\tilde{V}))$ where

$$\tilde{V} = \{1, \dots, k\} \cup \{i, j\}$$

Rmk. When $k = n$, d_{ij} is the cost of a shortest path from i to j

Complexity: there is an $O(n^3)$ implementation of the algorithm

Example



Compute a shortest path between each pair of nodes

Init.

$$D = \begin{bmatrix} \infty & \infty & \infty & 1 \\ 2 & \infty & 1 & \infty \\ \infty & \infty & \infty & \infty \\ \infty & -3 & 3 & \infty \end{bmatrix} \quad P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \end{bmatrix}$$

$p_{ij} = pred(i,j)$

$$D = \begin{bmatrix} \infty & \infty & \infty & 1 \\ 2 & \infty & 1 & \infty \\ \infty & \infty & \infty & \infty \\ \infty & -3 & 3 & \infty \end{bmatrix} \quad P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \end{bmatrix}$$

Triangular update on node $k=1$

i	j	$d_{ij} > d_{ik} + d_{kj}$?	updates
1	1	$\infty > \infty + \dots$	NO
1	2	$\infty > \infty + \dots$	NO
1	3	$\infty > \infty + \dots$	NO
1	4	$1 > \infty + \dots$	NO

i	j	$d_{ij} > d_{ik} + d_{kj}$?	updates
2	1	$2 > 2 + \infty$	NO
2	2	$\infty > 2 + \infty$	NO
2	3	$1 > 2 + \infty$	NO
2	4	$\infty > 2 + 1$	$d_{24} = 3$, pred(2,4) = pred(1,4) = 1
3	1	$\infty > \infty + \dots$	NO
3	2	$\infty > \infty + \dots$	NO
3	3	$\infty > \infty + \dots$	NO
3	4	$\infty > \infty + \dots$	NO
4	1	$\infty > \infty + \dots$	NO
4	2	$-3 > \infty + \dots$	NO
4	3	$3 > \infty + \dots$	NO
4	4	$\infty > \infty + \dots$	NO

$$D = \begin{bmatrix} \infty & \infty & \infty & 1 \\ 2 & \infty & 1 & 3 \\ \infty & \infty & \infty & \infty \\ \infty & -3 & 3 & \infty \end{bmatrix} \quad P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \end{bmatrix}$$

Triangular update on node $k=2$

i	j	$d_{ij} > d_{ik} + d_{kj}$?	updates
1	1	$\infty > \infty + \dots$	NO
1	2	$\infty > \infty + \dots$	NO
1	3	$\infty > \infty + \dots$	NO
1	4	$1 > \infty + \dots$	NO

i	j	$d_{ij} > d_{ik} + d_{kj}$?	updates
2	1	$2 > \infty + \dots$	NO
2	2	$\infty > \infty + \dots$	NO
2	3	$1 > \infty + \dots$	NO
2	4	$3 > \infty + \dots$	NO
3	1	$\infty > \infty + \dots$	NO
3	2	$\infty > \infty + \dots$	NO
3	3	$\infty > \infty + \dots$	NO
3	4	$\infty > \infty + \dots$	NO
4	1	$\infty > -3 + 2$	$d_{41} = -1$, pred $(4,1)$ = pred $(2,1)$ = 2
4	2	$-3 > -3 + \infty$	NO
4	3	$3 > -3 + 1$	$d_{43} = -2$, pred $(4,3)$ = pred $(2,3)$ = 2
4	4	$\infty > -3 + 3$	$d_{44} = 0$, pred $(4,4)$ = pred $(2,4)$ = 1

$$D = \begin{bmatrix} \infty & \infty & \infty & 1 \\ 2 & \infty & 1 & 3 \\ \infty & \infty & \infty & \infty \\ -1 & -3 & -2 & 0 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 \\ 3 & 3 & 3 & 3 \\ 2 & 4 & 2 & 1 \end{bmatrix}$$

Triangular update on node $k=3$: no change (check @ home)

Triangular update on node $k=4$ gives

$$D = \begin{bmatrix} 0 & -2 & -1 & 1 \\ 2 & 0 & 1 & 3 \\ \infty & \infty & \infty & \infty \\ -1 & -3 & -2 & 0 \end{bmatrix} \quad P = \begin{bmatrix} 2 & 4 & 2 & 1 \\ 2 & 4 & 2 & 1 \\ 3 & 3 & 3 & 3 \\ 2 & 4 & 2 & 1 \end{bmatrix}$$

$\text{pred}(1,1) = \text{pred}(4,1) = 2$

$\text{pred}(1,2) = \text{pred}(4,2) = 4$

$\text{pred}(1,3) = \text{pred}(4,3) = 2$

$\text{pred}(2,2) = \text{pred}(4,2) = 4$

Reading final results

$$D = \begin{bmatrix} 0 & -2 & -1 & 1 \\ 2 & 0 & 1 & 3 \\ \infty & \infty & \infty & \infty \\ -1 & -3 & -2 & 0 \end{bmatrix} \quad P = \begin{bmatrix} 2 & 4 & 2 & 1 \\ 2 & 4 & 2 & 1 \\ 3 & 3 & 3 & 3 \\ 2 & 4 & 2 & 1 \end{bmatrix}$$

Example: a shortest path from 1 to 3 is given by

$v_0 = 3$ ← end node

$v_1 = \text{pred}(1, v_0) = 2$

$v_2 = \text{pred}(1, v_1) = 4$

$v_3 = \text{pred}(1, v_2) = 1$

build the path backwards

Then, the path is 1 4 2 3 and its cost is $d_{13} = -1$

Reading final results

$$D = \begin{bmatrix} 0 & -2 & -1 & 1 \\ 2 & 0 & 1 & 3 \\ \infty & \infty & \infty & \infty \\ -1 & -3 & -2 & 0 \end{bmatrix} \quad P = \begin{bmatrix} 2 & 4 & 2 & 1 \\ 2 & 4 & 2 & 1 \\ 3 & 3 & 3 & 3 \\ 2 & 4 & 2 & 1 \end{bmatrix}$$

No negative element on the diagonal of $D \Leftrightarrow$ no cycle with cost < 0 in the graph.

Next slide: template for your exercises!

$D = \begin{bmatrix} \quad \end{bmatrix}$ $P = \begin{bmatrix} \quad \end{bmatrix}$

i	j	$d_{ij} > d_{ik} + d_{kj}$?	updates

i	j	$d_{ij} > d_{ik} + d_{kj}$?	updates