Linear programming: introduction and examples

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Industrial Automation

Linear Programming (LP)

Widely used optimization technique in management science

- optimal allocation of limited resources for maximizing revenues or minimizing costs

Basic problem

$$\min_{\substack{g_i(x)\leq 0\\ i=1,2,\ldots,m}} f(x), \quad x \in \mathbb{R}^n \tag{1}$$

A Linear Programming (LP) problem is (1) with

- $f(x) = c^{\mathrm{T}}x$ (linear cost)
- $g_i(x) = a_i^{\mathrm{T}}x - b_i$ (affine constraints)

An LP problem is a convex optimization problem

Linear Programming (LP)

Canonical form

An LP problem is in *canonical form* if it is written as

$$\min_{\substack{a_i^{\mathrm{T}} x \leq b_i,\ i=1,2,\ldots,m \\ x_j \geq 0,\ j=1,2,\ldots,n}} c^{\mathrm{T}} x$$

or

$$\max_{\substack{a_i^{\mathrm{T}} x \leq b_i,\ i=1,2,\ldots,m \\ x_j \geq 0,\ j=1,2,\ldots,n}} c^{\mathrm{T}} x$$

"$\leq$" constraints and positivity constraints on all variables

PL - matrix notation

Vector inequalities

$$x \leq 0 \text{ means } \begin{cases} x_1 \leq 0 \\ x_2 \leq 0 \\ \dots \\ x_n \leq 0 \end{cases}$$

Constraints

$$\begin{cases} a_1^{\mathrm{T}} x \leq b_1 \\ a_2^{\mathrm{T}} x \leq b_2 \\ \dots \\ a_m^{\mathrm{T}} x \leq b_m \end{cases} \Leftrightarrow Ax \leq b, \quad A = \begin{bmatrix} a_1^{\mathrm{T}} \\ a_2^{\mathrm{T}} \\ \vdots \\ a_m^{\mathrm{T}} \end{bmatrix}, \ b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

PL - matrix notation

LP problem in generic form

$$\min_{Ax \leq b} c^{\mathrm{T}} x$$

LP problem in canonical form (LP-C)

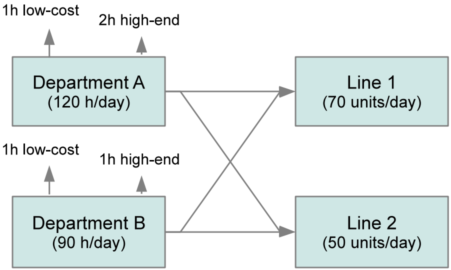
$$\min_{\substack{Ax \leq b \\ x \geq 0}} c^{\mathrm{T}} x$$

LP and management science

Typical decision problems in industry

- Product mix
- Diet problems
- Blending problems
- Transport problems
- Product mix with resource allocation
- Multiperiod production planning
- Portfolio optimization
- ...

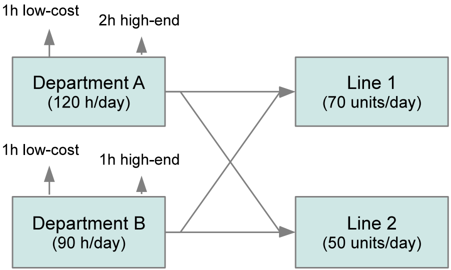
Product mix



A company manifactures two radio models (low-cost and high-end) and produces two components

- Department A: antennas (max. 120h hours of production per day)
  - 1h of work for a low-cost antenna
  - 2h of work for a high-end antenna
- Department B: case (max. 90h hours of production per day)
  - 1h of work for a low-cost case
  - 1h of work for a high-end case

Product mix



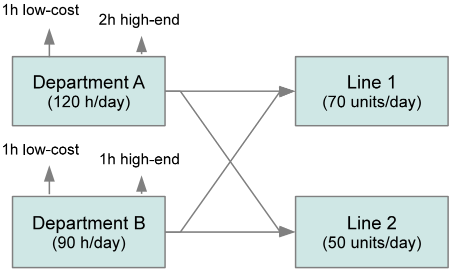
The company has two assembly lines (1 radio=1 antenna + 1 case)

- Line 1: production of low-cost models. No more than 70 units/day
- Line 2: production of high-end models. No more than 50 units/day

Profits: 20 Euros for a low-cost radio and 30 Euros for a high-end radio.

Assuming the company will sell all radios, which is the optimal number of units, for each model, that must be produced daily for maximizing the revenue?

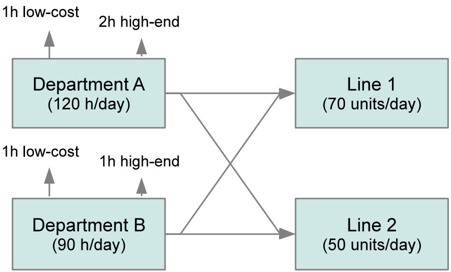
Product mix



Choice of variables

Usually this is the most difficult step in representing decision problems as optimization problems !
Guideline: cost and constraints must be a function of optimization variables only.

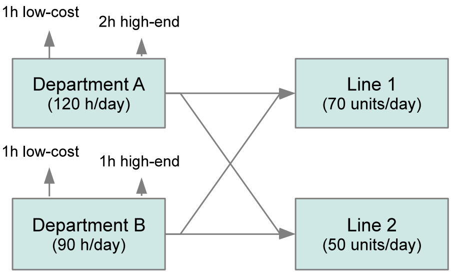
Product mix



Choice of variables - product mix

- x_1: number of produced low-cost radios per day
- x_2: number of produced high-end radios per day

Product mix



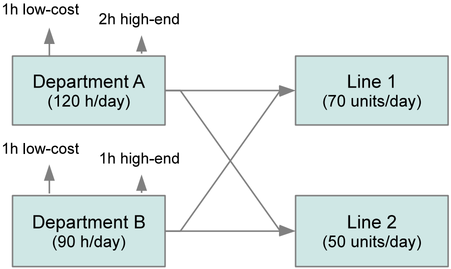
Choice of variables - product mix

- x_1: number of produced low-cost radios per day
- x_2: number of produced high-end radios per day

Cost and type of problem

Cost: $20x_1 + 30x_2$, to be *maximized*

Product mix

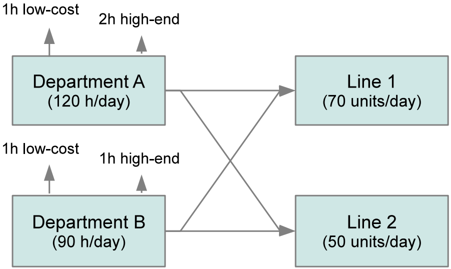


Constraints

Capacity constraints of assembly lines

- $x_1 \leq 70$ (line 1)
- $x_2 \leq 50$ (line 2)

Product mix



Constraints

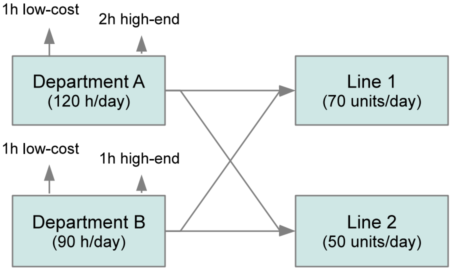
Capacity constraints of assembly lines

- $x_1 \leq 70$ (line 1)
- $x_2 \leq 50$ (line 2)

Capacity constraints of production departments

- $x_1 + 2x_2 \leq 120$ (department 1)
- $x_1 + x_2 \leq 90$ (department 2)

Product mix



Constraints

Capacity constraints of assembly lines

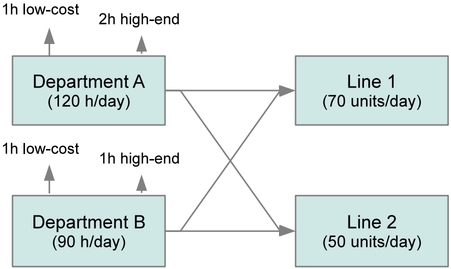
- $x_1 \leq 70$ (line 1)
- $x_2 \leq 50$ (line 2)

Capacity constraints of production departments

- $x_1 + 2x_2 \leq 120$ (department 1)
- $x_1 + x_2 \leq 90$ (department 2)

Positivity constraints: $x_1 \geq 0$, $x_2 \geq 0$

Product mix



LP problem

$$
\begin{aligned}
\max_{x_1,x_2} \quad & 20x_1 + 30x_2 \\
x_1 & \leq 70 \\
x_2 & \leq 50 \\
x_1 + 2x_2 & \leq 120 \\
x_1 + x_2 & \leq 90 \\
x_1 & \geq 0 \\
x_2 & \geq 0
\end{aligned}
$$

Product mix

LP problem

$$\begin{aligned} \max_{x_1,x_2} \quad & 20x_1 + 30x_2 \\ x_1 & \leq 70 \\ x_2 & \leq 50 \\ x_1 + 2x_2 & \leq 120 \\ x_1 + x_2 & \leq 90 \\ x_1 & \geq 0 \\ x_2 & \geq 0 \end{aligned}$$

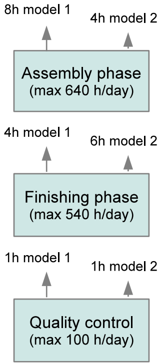
LP problem - matrix notation

$$\min_{\substack{Ax \leq b \\ x \geq 0}} c^{\mathrm{T}} x$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad c = \begin{bmatrix} 20 \\ 30 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 120 \\ 90 \\ 70 \\ 50 \end{bmatrix}$$

Product mix revised

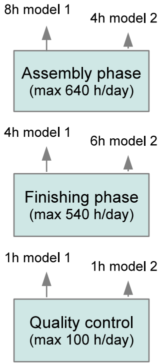


A company manifactures two models of cases for telephones (model 1 and model 2). The production cycle comprises three phases with bounded resources that must be allocated to the two products

- Each phase is modeled through the maximal availability of men-hours per day and men-hours required to process a single unit
- Profits: 30 Euros for a model 1 unit and 20 Euros for a model 2 unit

Assuming all cases will be sold, which is the optimal number of units, for each model, that must be produced daily for maximizing the revenue?

Product mix revised



Choice of variables

- x_1: n. of model 1 units
- x_2: n. of model 2 units

Cost and type of problem

Cost: $30x_1 + 20x_2$, to be *maximized*

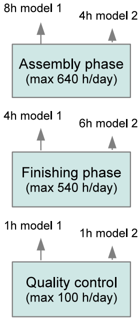
Constraints

Capacity constraints in each phase

- $8x_1 + 4x_2 \leq 640$ (assembly)
- $4x_1 + 6x_2 \leq 540$ (finishing)
- $x_1 + x_2 \leq 100$ (quality control)

Positivity constraints: $x_1 \geq 0$, $x_2 \geq 0$

Product mix revised



LP problem

$$\begin{aligned} \max_{x_1,x_2} \quad & 30x_1 + 20x_2 \\ 8x_1 + 4x_2 & \leq 640 \\ 4x_1 + 6x_2 & \leq 540 \\ x_1 + x_2 & \leq 100 \\ x_1 & \geq 0 \\ x_2 & \geq 0 \end{aligned}$$

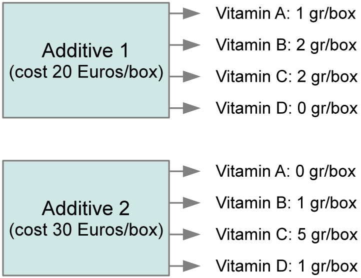
LP problem - matrix notation

$$\max_{\substack{Ax \leq b \\ x \geq 0}} c^{\mathrm{T}} x$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad c = \begin{bmatrix} 30 \\ 20 \end{bmatrix}$$

$$A = \begin{bmatrix} 8 & 4 \\ 4 & 6 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 640 \\ 540 \\ 100 \end{bmatrix}$$

Diet problem

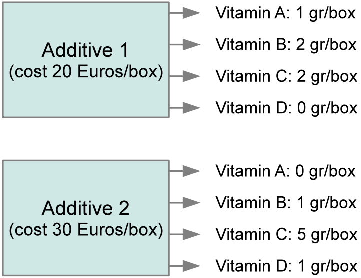


A company producing chicken food can use two types of additives with different vitamin contents. The product is obtained by mixing additives in a suitable quantity

- Vitamin contents and additive costs are given in the picture
- 1 box of chicken food must contain at least 2 gr. of vitamin A, at least 12 gr. of vitamin B, at least 36 gr. of vitamin C and at least 4 gr. of vitamin D

Which is the quantity of additives that allows the company to produce one box of chicken food while minimizing the costs ?

Diet problem



Choice of variables

- x_1: n. of boxes of Additive 1
- x_2: n. of boxes of Additive 2

Cost and type of problem

Cost: $20x_1 + 30x_2$, to be *minimized*

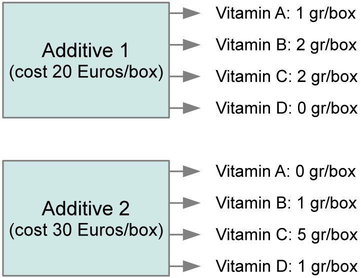
Constraints

Minimal quantities of vitamins

- $x_1 \geq 2$ (vitamin A)
- $2x_1 + x_2 \geq 12$ (vitamin B)
- $2x_1 + 5x_2 \geq 36$ (vitamin C)
- $x_2 \geq 4$ (vitamin D)

Positivity constraints: $x_1 \geq 0$, $x_2 \geq 0$

Diet problem



LP problem

$$
\begin{aligned}
\min_{x_1,x_2} \quad & 20x_1 + 30x_2 \\
x_1 & \geq 2 \\
2x_1 + x_2 & \geq 12 \\
2x_1 + 5x_2 & \geq 36 \\
x_2 & \geq 4 \\
x_1, x_2 & \geq 0
\end{aligned}
$$

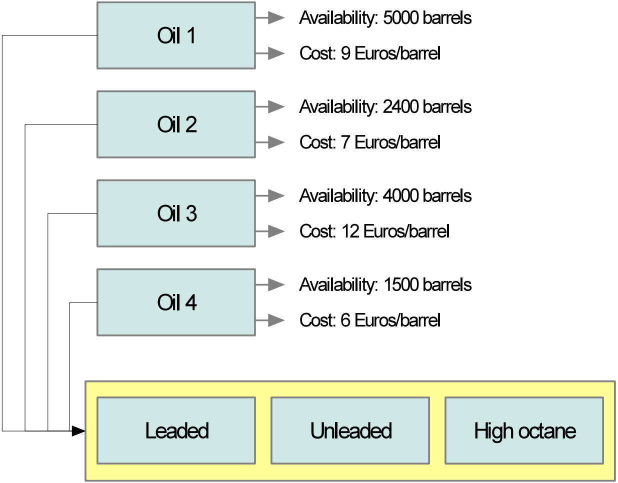
LP problem - matrix notation

$$
\min_{\substack{Ax \geq b \\ x \geq 0}} c^{\mathrm{T}} x
$$

$$
x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad c = \begin{bmatrix} 20 \\ 30 \end{bmatrix}
$$

$$
A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 2 & 5 \\ 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 12 \\ 36 \\ 4 \end{bmatrix}
$$

Blending problem



A company produces 3 types of gasoline (leaded, unleaded, high-octane) blending 4 types of oils in suitable proportions

Blending problem

Problem data

- The maximal number of available barrels and the buying costs are given in the picture
- Each gasoline type must fulfill the following composition requirements

GASOLINE	% of Oil 1	% of Oil 2	% of Oil 3	% of Oil 4
Leaded	at least 40%	no more than 20%	at least 30%	
Unleaded			at least 40%	
High octane	at least 10%	no more than 50%		

- Gasoline selling prices are: 12 Euros/barrel for leaded, 18 Euros/barrel for unleaded and 10 Euros/barrel for high-octane

Which are the optimal quantities of the 4 components that have to be bought in order to maximize the profits (profits = revenues - costs) ?

Blending problem

Choice of variables

x_{ij}: n. of barrels of oil i used for gasoline of type j, $i = 1, 2, 3, 4$, $j \in \{L, U, H\}$, $L=$ Leaded, $U=$ Unleaded, $H=$ High-octane

Cost and type of problem

Cost to be *maximized*

$$\underbrace{12\sum_{i=1}^{4} x_{iL} + 18\sum_{i=1}^{4} x_{iU} + 10\sum_{i=1}^{4} x_{iH}}_{\text{revenues}} -$$

$$\underbrace{-9\sum_{j\in\{U,L,H\}} x_{1j} - 7\sum_{j\in\{U,L,H\}} x_{2j} - 12\sum_{j\in\{U,L,H\}} x_{3j} - 6\sum_{j\in\{U,L,H\}} x_{4j}}_{\text{costs}}$$

Blending problem

Constraints

Maximal availability of barrels

- $x_{1L} + x_{1U} + x_{1H} \leq 5000$ (Oil 1)
- $x_{2L} + x_{2U} + x_{2H} \leq 2400$ (Oil 2)
- $x_{3L} + x_{3U} + x_{3H} \leq 4000$ (Oil 3)
- $x_{4L} + x_{4U} + x_{4H} \leq 1500$ (Oil 4)

Composition requirements

$\frac{x_{1L}}{x_{1L}+x_{2L}+x_{3L}+x_{4L}} \geq 0.4$ in affine form $0.6x_{1L} - 0.4x_{2L} - 0.4x_{3L} - 0.4x_{4L} \geq 0$

$\frac{x_{2L}}{x_{1L}+x_{2L}+x_{3L}+x_{4L}} \leq 0.2$ in affine form $-0.2x_{1L} + 0.8x_{2L} - 0.2x_{3L} - 0.2x_{4L} \leq 0$

$\frac{x_{3L}}{x_{1L}+x_{2L}+x_{3L}+x_{4L}} \geq 0.3$ in affine form $-0.3x_{1L} - 0.3x_{2L} + 0.7x_{3L} - 0.3x_{4L} \geq 0$

$\frac{x_{3U}}{x_{1U}+x_{2U}+x_{3U}+x_{4U}} \geq 0.4$ in affine form $-0.4x_{1U} - 0.4x_{2U} + 0.6x_{3U} - 0.4x_{4U} \geq 0$

$\frac{x_{1H}}{x_{1H}+x_{2H}+x_{3H}+x_{4H}} \geq 0.1$ in affine form $0.9x_{1H} - 0.1x_{2H} - 0.1x_{3H} - 0.1x_{4H} \geq 0$

$\frac{x_{2H}}{x_{1H}+x_{2H}+x_{3H}+x_{4H}} \leq 0.5$ in affine form $-0.5x_{1H} + 0.5x_{2H} - 0.5x_{3H} - 0.5x_{4H} \leq 0$

Positivity constraints: $x_{ij} \geq 0,\ i = 1, 2, 3, 4,\ j \in \{U, L, H\}$

Blending problem

LP problem - matrix notation

$$\max_{\substack{Ax \leq b \\ x \geq 0}} c^{\mathrm{T}} x$$

with

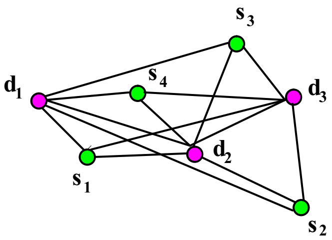
$$x = \begin{bmatrix} x_{1L} & x_{2L} & x_{3L} & x_{4L} & x_{1U} & x_{2U} & x_{3U} & x_{4U} & x_{1H} & x_{2H} & x_{3H} & x_{4H} \end{bmatrix}^T$$

$$c = \begin{bmatrix} 3 & 5 & 0 & 6 & 9 & 11 & 6 & 12 & 1 & 3 & -2 & 4 \end{bmatrix}^T$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ -0.6 & 0.4 & 0.4 & 0.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.2 & 0.8 & -0.2 & -0.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.3 & 0.3 & -0.7 & 0.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.4 & 0.4 & -0.6 & 0.4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.9 & 0.1 & 0.1 & 0.1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.5 & 0.5 & -0.5 & -0.5 \end{bmatrix}$$

$$b = \begin{bmatrix} 5000 & 2400 & 4000 & 1500 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

Transport problem

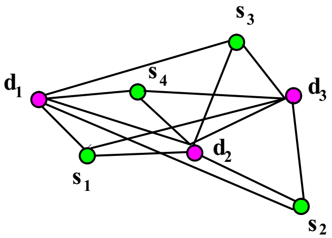


A company owns n fuel storage points (i.e. n supply points) and it must supply m gas stations (i.e. m demand points). Every gas station can be reached from all storage points.

Data and requirements

- Every storage point has a maximal availability of $a_i \geq 0$ $i = 1, \ldots, n$ liters of fuel
- Every gas station specifies a demand of $b_j \geq 0$, $j = 1, \ldots, m$ liters of fuel
- $t_{ij} \geq 0$ is the cost for shipping 1 liter of fuel from storage point i to gas station j

Transport problem



Which are the optimal quantities of fuel to be shipped from each storage point to each gas station in order to minimize transport costs while fulfilling exactly the demands ?

Transport problem

Choice of variables

x_{ij}, $i = 1, \dots, n$, $j = 1, \dots, m$, n. of liters of fuel shipped from storage point i to gas station j

Cost and type of problem

Cost to be *minimized*

$$\sum_{i=1}^{n} \sum_{j=1}^{m} t_{ij} x_{ij}$$

Constraints

Maximal availability of fuel at storage points

- $\sum_{j=1}^{m} x_{ij} \leq a_i$, $i = 1, \dots, n$

Demand constraints

- $\sum_{i=1}^{n} x_{ij} = b_j$, $j = 1, \dots, m$

Positivity constraints: $x_{ij} \geq 0$, $i = 1, \dots, n$, $j = 1, \dots, m$

Transport problem

LP problem

$$\min \quad \sum_{i=1}^{n} \sum_{j=1}^{n} t_{ij} x_{ij}$$

$$\begin{aligned} \sum_{j=1}^{m} x_{ij} &\leq a_i,\ i = 1, \dots, n \\ \sum_{i=1}^{n} x_{ij} &= b_j,\ j = 1, \dots, m \qquad \textbf{(A)} \\ x_{ij} &\geq 0,\ i = 1, \dots, n,\ j = 1, \dots, m \end{aligned}$$

Remarks

- Constraints (A) can be replaced by $\sum_{i=1}^{n} x_{ij} \geq b_j,\ j = 1, \dots, m$. Why ?

Transport problem

LP problem

$$\min \quad \sum_{i=1}^{n} \sum_{j=1}^{n} t_{ij} x_{ij}$$

$$\begin{aligned} \sum_{j=1}^{m} x_{ij} &\leq a_i, \ \ i = 1, \dots, n \\ \sum_{i=1}^{n} x_{ij} &= b_j, \ \ j = 1, \dots, m \qquad \textbf{(A)} \\ x_{ij} &\geq 0, \ \ i = 1, \dots, n, \ \ j = 1, \dots, m \end{aligned}$$

Remarks

- Constraints (A) can be replaced by $\sum_{i=1}^{n} x_{ij} \geq b_j, \ j = 1, \dots, m$. Why ?
- The LP problem is always feasible ?

Transport problem

LP problem

$$\min \quad \sum_{i=1}^{n} \sum_{j=1}^{n} t_{ij} x_{ij}$$

$$\begin{aligned} \sum_{j=1}^{m} x_{ij} &\leq a_i, \ i = 1, \dots, n \\ \sum_{i=1}^{n} x_{ij} &= b_j, \ j = 1, \dots, m \qquad \textbf{(A)} \\ x_{ij} &\geq 0, \ i = 1, \dots, n, \ j = 1, \dots, m \end{aligned}$$

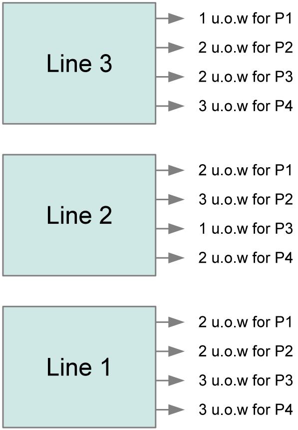
Remarks

- Constraints (A) can be replaced by $\sum_{i=1}^{n} x_{ij} \geq b_j, \ j = 1, \dots, m$. Why ?
- The LP problem is always feasible ? NO !
  - A necessary and sufficient condition for feasibility is

$$\sum_{i=1}^{n} a_i \geq \sum_{j=1}^{m} b_j$$

Product mix with resource allocation

In standard product mix problems, capacity constraints are fixed. We now assume to have resources that can be allocated to different production phases



A food company produces 4 products P_i, $i = 1, 2, 3, 4$. Each product must go through 3 production lines L_j, $j = 1, 2, 3$. Each line is modeled through the units of work (u.o.w.) required to process a single unit of product

Product mix with resource allocation

Requirements

- At least 1000 units of P_2 and no more than 500 units of P_1 must be produced
- U.o.w are men-hours that fall in 7 categories T_i, $i = 1, \ldots, 7$ according to the versatility of workers

Category	**Destination line**	**Max. availability**
T_1	Line 1	12000
T_2	Line 2	7000
T_3	Line 3	9000
T_4	Lines 1 and 2	4000
T_5	Lines 1 and 3	3000
T_6	Lines 2 and 3	3000
T_7	Lines 1,2 and 3	2000

- Profits per unit are 10 Euros for P_1, 12 Euros for P_2, 13 Euros for P_3 and 14 Euros for P_4

Assuming all products will be sold, which is a joint production and resource allocation plan that maximizes the profits ?

Product mix with resource allocation

Choice of variables

Two categories of variables are needed

- P_i, $i = 1, \ldots, 4$: units of product i
- T_4L_1, T_4L_2, T_5L_1, T_5L_3, T_6L_2, T_6L_3, T_7L_1, T_7L_2, T_7L_3. For instance T_4L_1 is the number of men-hours of category T_4 allocated to line 1.

Cost and type of problem

Cost to be *maximized*

$$10P_1 + 12P_2 + 13P_3 + 14P_4$$

Product mix with resource allocation

Constraints

Capacity constraints of production lines

- $2P_1 + 2P_2 + 3P_3 + 3P_4 \leq 12000 + T_4L_1 + T_5L_1 + T_7L_1$ (line 1)
- $2P_1 + 3P_2 + P_3 + 2P_4 \leq 7000 + T_4L_2 + T_6L_2 + T_7L_2$ (line 2)
- $P_1 + 2P_2 + 2P_3 + 3P_4 \leq 9000 + T_5L_3 + T_6L_3 + T_7L_3$ (line 3)

Bounds on the number of products

- $P_2 \geq 1000$
- $P_4 \leq 500$

Maximal availability of u.o.w. for versatile workers

- $T_4L_1 + T_4L_2 \leq 4000$
- $T_5L_1 + T_5L_3 \leq 3000$
- $T_6L_2 + T_6L_3 \leq 3000$
- $T_7L_1 + T_7L_2 + T_7L_3 \leq 2000$

Positivity contraints:

$P_1,\ P_2,\ P_3,\ P_4,\ T_4L_1,\ T_4L_2,\ T_5L_1,\ T_5L_3,\ T_6L_2,\ T_6L_3,\ T_7L_1,\ T_7L_2,\ T_7L_3 \geq 0$

Product mix with resource allocation

LP problem

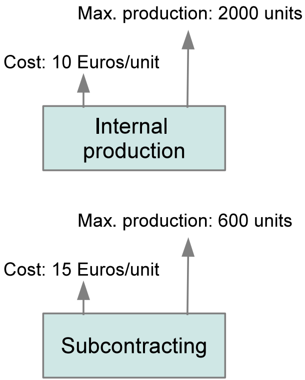
$$
\begin{aligned}
\max \quad & 10P_1 + 12P_2 + 13P_3 + 14P_4 \\
2P_1 + 2P_2 + 3P_3 + 3P_4 &\leq 12000 + T_4L_1 + T_5L_1 + T_7L_1 \\
2P_1 + 3P_2 + P_3 + 2P_4 &\leq 7000 + T_4L_2 + T_6L_2 + T_7L_2 \\
P_1 + 2P_2 + 2P_3 + 3P_4 &\leq 9000 + T_5L_3 + T_6L_3 + T_7L_3 \\
P_2 &\geq 1000 \\
P_4 &\leq 500 \\
T_4L_1 + T_4L_2 &\leq 4000 \\
T_5L_1 + T_5L_3 &\leq 3000 \\
T_6L_2 + T_6L_3 &\leq 3000 \\
T_7L_1 + T_7L_2 + T_7L_3 &\leq 2000 \\
P_1,\ P_2,\ P_3,\ P_4,\ T_4L_1,\ T_4L_2,\ T_5L_1,\ T_5L_3 &\geq 0 \\
T_6L_2,\ T_6L_3,\ T_7L_1,\ T_7L_2,\ T_7L_3 &\geq 0
\end{aligned}
$$

Exercise

Write the LP problem in matrix notation

Multiperiod production planning

A company must determine how many washing machines should be produced during each of the next 5 months. Washing machines production can be internal or subcontracted to another company. Production capacities and costs are given in the figure



Multiperiod production planning

Data and Requirements

- Inventory cost: 2 Euros/unit for a whole month
- Inventory at the beginning of the first month: 300 units
- Inventory at the end of the fifth month: 300 units
- Demand during each of the next five months (the company must meet demands on time)

Month	Demand
1	1200
2	2100
3	2400
4	3000
5	4000

How many washing machines must be produced (internally and by the subcontractor), each month, in order to meet demands on time and minimize the costs ?

Multiperiod production planning

Choice of variables

Three categories of variables are needed

- P_i, $i = 1, \ldots, 5$: units produced internally during month i
- S_i, $i = 1, \ldots, 5$: units produced by the subcontrator during month i
- I_i, $i = 1, \ldots, 4$: inventory at the end of month i

Cost and type of problem

Cost to be *minimized*

$$10 \sum_{i=1}^{5} P_i + 15 \sum_{i=1}^{5} S_i + 2 \sum_{i=1}^{4} I_i$$

Multiperiod production planning

Constraints

Maximal production

- $P_i \leq 2000,\ i = 1, \dots, 5$ (internal production)
- $S_i \leq 600,\ i = 1, \dots, 5$ (subcontracting)

Balance constraints: for month i,

$$\text{Inventory at the end of month } i = \text{Inventory at the end of month } i-1 + \\ + \text{month } i \text{ production} - \text{month } i \text{ demand}$$

- $I_1 = 300 + P_1 + S_1 - 1200$
- $I_2 = +I_1 + P_2 + S_2 - 2100$
- $I_3 = +I_2 + P_2 + S_2 - 2400$
- $I_4 = +I_3 + P_3 + S_3 - 3000$
- $300 = +I_4 + P_4 + S_4 - 4000$

Positivity contraints: $P_i, S_i \geq 0,\ i = 1 \dots, 5,\ I_j \geq 0,\ j = 1 \dots, 4$

Multiperiod production planning

LP problem

$$
\begin{array}{rl}
\min & 10\sum_{i=1}^{5} P_i + 15\sum_{i=1}^{5} S_i + 2\sum_{i=1}^{4} I_i \\
P_i & \leq 2000,\ \ i = 1,\ldots,5 \\
S_i & \leq 600,\ \ i = 1,\ldots,5 \\
I_1 - P_1 - S_1 & = 300 - 1200 \\
I_2 - I_1 - P_2 - S_2 & = -2100 \\
I_3 - I_2 - P_2 - S_2 & = -2400 \\
I_4 - I_3 - P_3 - S_3 & = -3000 \\
-I_4 - P_4 - S_4 & = -4000 - 300 \\
P_i, S_i & \geq 0,\ \ i = 1\ldots,5 \\
I_j & \geq 0,\ \ j = 1\ldots,4
\end{array}
$$

Portfolio optimization

A manager must decide how to invest 500000 Euros in different financial products. His goal is to maximize earnings while avoiding high risk exposure

Financial products and expected return on investment

Financial product	**Market**	**Return %**
T_1	Cars - Germany	10.3
T_2	Cars - Japan	10.1
T_3	Computers - USA	11.8
T_4	Computers - USA	11.4
T_5	Household appliances - Europe	12.7
T_6	Household appliances - Asia	12.2
T_7	Insurance - Germany	9.5
T_8	Insurance - USA	9.9
T_9	BOT	3.6
T_{10}	CCT	4.2

Portfolio optimization

Investment requirements

1. No more than 150000 Euros in the car options
2. No more than 150000 Euros in the computer options
3. No more than 100000 Euros in the appliance options
4. At least 100000 Euros in the insurance options
5. At least 125000 Euros in BOT or CCT
6. At least 40% of the money invested in CCT must be invested in BOT
7. No more than 250000 Euros must be invested in German options
8. No more than 200000 Euros must be invested in USA options

Portfolio optimization

Choice of variables

T_i, $i = 1, \ldots, 10$: thousands of Euros invested in the financial product i

Cost and type of problem

Cost to be *maximized*

$$0.103\,T_1 + 0.101\,T_2 + 0.118\,T_3 + 0.114\,T_4 + 0.127\,T_5 + 0.122\,T_6 + 0.095\,T_7 + \\ +0.099\,T_8 + 0.036\,T_9 + 0.042\,T_{10}$$

Portfolio optimization

Constraints

Total investment: $\sum_{i=1}^{10} T_i = 500$

Investment requirements:

1. $T_1 + T_2 \leq 150$
2. $T_3 + T_4 \leq 150$
3. $T_5 + T_6 \leq 100$
4. $T_7 + T_8 \geq 100$
5. $T_9 + T_{10} \geq 125$
6. $T_9 - 0.4\,T_{10} \geq 0$
7. $T_1 + T_7 \leq 250$
8. $T_3 + T_4 + T_8 \leq 200$

Positivity constraints: $T_i, \geq 0,\ i = 1 \ldots, 10$