Vertices of a polyhedron

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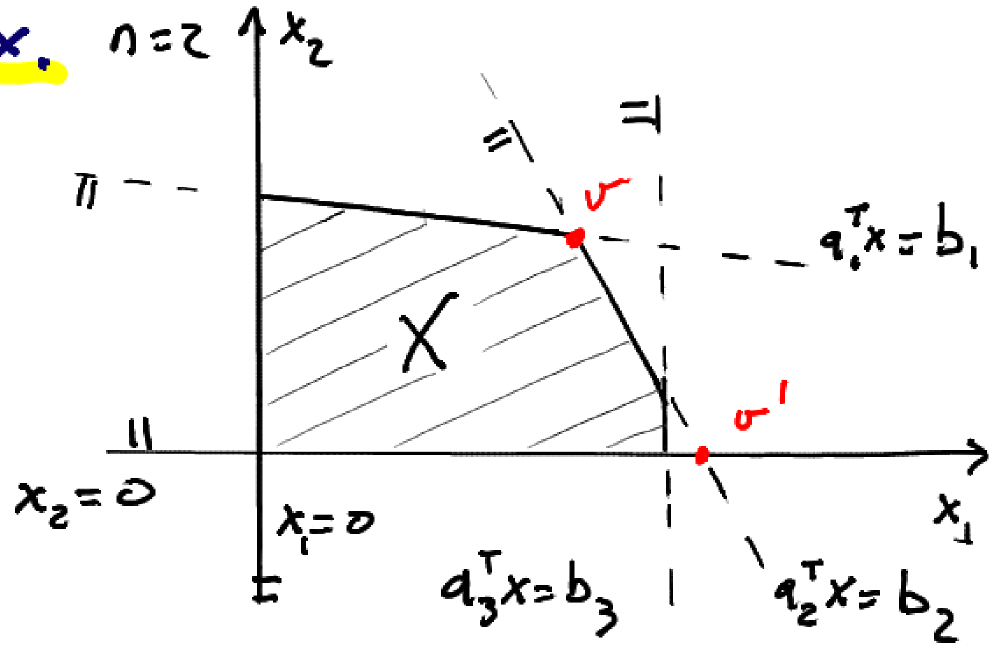
Basic solutions

Algebraic characterization of the vertices of a polyhedron $X \subseteq \mathbb{R}^n$

v is a vertex $\iff$ v is the intersection of n supporting hyperplanes
and $v \in X$

Ex.



- v verifies $\begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix} v = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ AND $v \in X$

- $\begin{cases} a_1^T x = b_1 \\ x_2 = 0 \end{cases}$ gives $x = v'$ but

$v' \notin X \Rightarrow v'$ is NOT a vertex

Def. The polyhedron $X \subseteq \mathbb{R}^n$ is in the standard form if

$X = \{x \in \mathbb{R}^n : Ax = b,\ x \geq 0\}$ where $A \in \mathbb{R}^{m \times n}$, $m < n$ and $\mathrm{rank}(A) = m$

Rmk. •) If $m = n$ and $\mathrm{rank}(A) = m$, then $\exists!$ solution x^* to $Ax = b$

↳ $X = \{x^*\}$ → the interesting case is $m < n$

•) If $m \leq n$ and $\mathrm{rank}(A)$ is not maximal, it is possible to remove suitable rows from A and b so as to obtain $\tilde{A}$ and $\tilde{b}$ with $\mathrm{rank}(\tilde{A})$ maximal and $X = \{x \in \mathbb{R}^n : \tilde{A}x = \tilde{b},\ x \geq 0\}$

Conclusion: every polyhedron can be written in standard form

Vertices of polyhedra in standard form

Supporting hyperplanes: $\left.\begin{array}{ll} a_{i,\cdot}\, x = b_i & i = 1, \dots, m \\ x_j = 0 & j = 1, \dots, n \end{array}\right\} n+m$

v is a vertex of X if and only if there is $B \subseteq \{1, \dots, n\}$ containing m elements such that v is the unique solution to the system of inequalities

$$Ax = b \qquad (1)$$

$$x_j = 0 \quad \forall j \notin B \qquad (2)$$

$$x \geq 0 \qquad (3)$$

Rmk. (1)-(2): intersection of n supporting hyperplanes
(1)-(3): $x \in X$

B-F notation for the std system $Ax=b$

B: matrix collecting the m columns of A indexed by $\mathcal{B}$

Recall: $Ax = Bx_B + Fx_F$

Then, x is a vertex if and only if $\exists B$ such that the inequalities

$$x_F = 0 \qquad (1') \qquad \text{[from (2)]}$$

$$Bx_B + Fx_F = b \longrightarrow Bx_B = b \qquad (2') \qquad \text{[from (1)]}$$

$$x_B \geq 0 \qquad (3') \qquad \text{[from (3)]}$$

have only one solution.

Unique solution $\Leftrightarrow$ $Bx_B = b$ has only one solution $\bar{x}_B$ $\Leftrightarrow$ $\det(B) \neq 0$

Theorem of vertices

Jargon for fat systems.

B such that $\det(B) \neq 0$: basis

x_B, x_F : BVs, NBVs

$\bar{x}$ verifying $\bar{x}_B = B^{-1}b$, $\bar{x}_F = 0$ is a BS

Def. A BS is *feasible* (BFS) if $\bar{x}_B = B^{-1}b \geq 0$. In this case, we also say that the basis is feasible.

A BS is *degenerate* (BDS) if some elements of $\bar{x}_B$ are zero. In this case, we also say that the basis is degenerate

Theorem. Let X be a nonempty polyhedron. Then $x \in X$ is a vertex *if and only if* there is a basis B such that x is a BFS

Example : product mix

Verify @ home that the LP in standard form is

$x = [x_1, x_2, s_1, s_2, s_3]$ $\quad$ s_i: slack variables

$\max \; c^T x$ $\quad$ $c^T = [30 \; 20 \; 0 \; 0 \; 0]$

$Ax = b$

$x \geq 0$

$$A = \begin{bmatrix} 8 & 4 & 1 & 0 & 0 \\ 4 & 6 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 640 \\ 540 \\ 100 \end{bmatrix}$$

X: feasible region

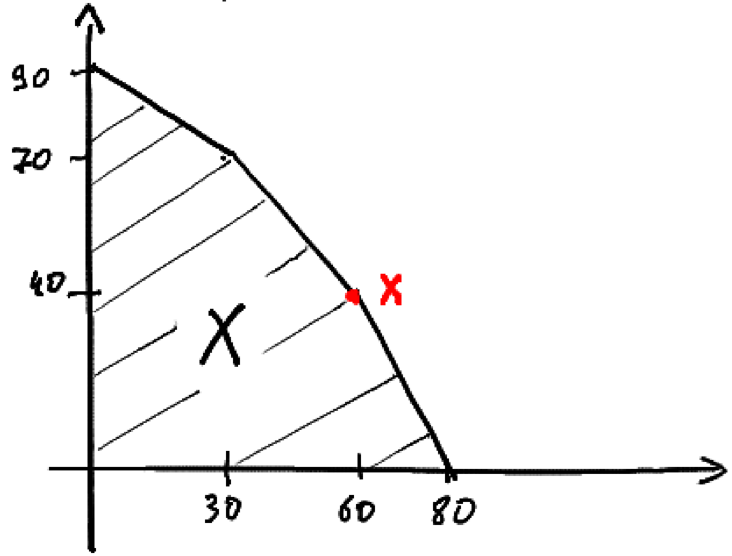
Pbl: compute a vertex of X

Pick $B = [A_1, A_2, A_4] = \begin{bmatrix} 8 & 4 & 0 \\ 4 & 6 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

- $\det(B) \neq 0 \Rightarrow B$ is a basis
- If $F = [A_3, A_5]$ then $x_B = [x_1 \ x_2 \ x_4]^T$ (BVs)

  $x_F = [x_3 \ x_5]^T$ (NBVs)

$\hookrightarrow$ BS $\quad \bar{x}_B = B^{-1} b = \begin{bmatrix} 60 \\ 40 \\ 60 \end{bmatrix}, \quad \bar{x}_F = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$



$\bar{x}_B \geq 0 \rightarrow \bar{x} = [60 \ 40 \ 0 \ 60 \ 0]^T$ is a BFS and then a vertex of X

At home. Consider $B = [A_1 \ A_4 \ A_5]$ and $B = [A_1 \ A_2 \ A_5]$. Do they define vertices?

Degeneracy

Ex. Constraints

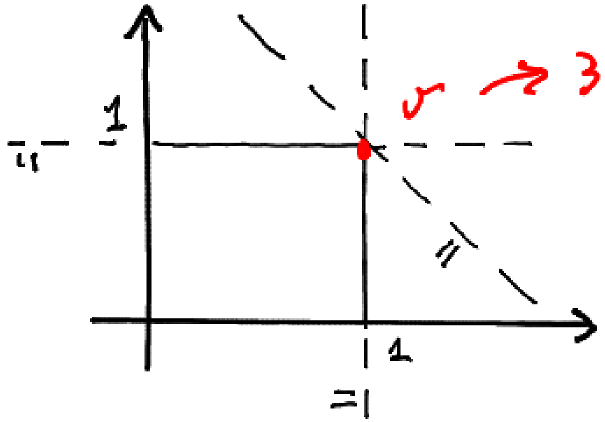
$x_1 \le 1$

$x_2 \le 1$

$x_1 + x_2 \le 2$

$x_1, x_2 \ge 0$

Feasible region X



$v \to$ 3 supporting hyperplanes through v

$\downarrow$

v is defined by any combination of two of them

X in standard form

$$
\begin{aligned}
x_1 + x_3 &= 1 \\
x_2 + x_4 &= 1 \\
x_1 + x_2 + x_5 &= 2 \\
x_1, \dots, x_5 &\geq 0
\end{aligned}
\qquad
A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}
\qquad
b = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}
$$

If $x_B = \begin{bmatrix} x_1 \\ x_2 \\ x_4 \end{bmatrix}$ then $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \longrightarrow B^{-1}b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

vertex v

$x_4 = 0$

If $x_B = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ then $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \longrightarrow B^{-1}b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

vertex v

$x_3 = 0$

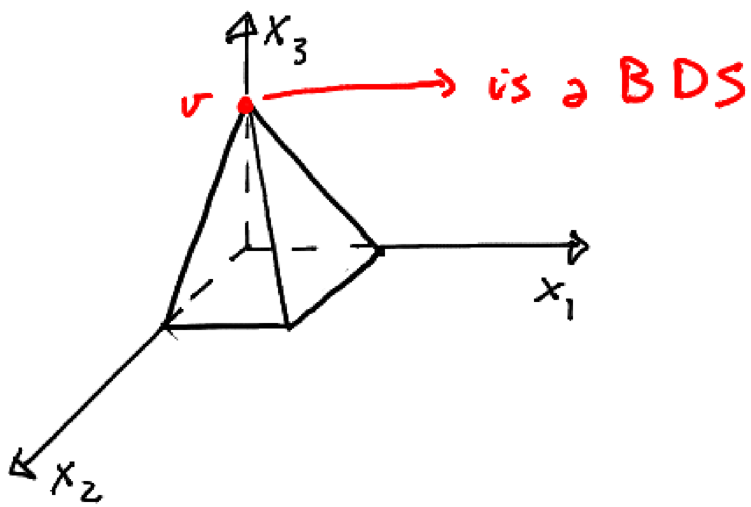
Both BDSs define the same vertex

Geometrical interpretation of BDS for $X \subseteq \mathbb{R}^n$

$q > n$ hyperplanes passing through the vertex v $\longrightarrow$ v is defined by any combination of n hyperplanes chosen among the q ones

↳ multiple BDSs describe the same vertex

Rmk. One can have a BDS and no redundant constraints



Naive algorithm for solving LPs

Fundamental theorem of linear programming (version 2). A feasible LP problem is either unbounded or there is a BFS that is optimal

Idea: compute all BFSs and choose the best one (vertex enumeration)

↳ can be computationally prohibitive because

- the largest number of matrices B is $\binom{n}{m} = \frac{n!}{m!\,(n-m)!}$ for n variables and m constraints

  ↳ possible ways of choosing m objects among n ones

- number of BFS $=$ $\underbrace{\text{number of vertices of } X}_{\text{can be very large! Smarter approaches are needed...}} \le \binom{n}{m}$