Duality in linear programming

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Dual of LP problems

Primal problem

$$\min c^T x$$
$$Ax = b$$
$$-x \leq 0$$

Notation: $f(x) = c^T x$, $h(x) = Ax - b$, $g(x) = -x$

Lagrangian: $L(x, \lambda, \mu) = c^T x + \lambda^T(-x) + \mu^T (Ax - b)$

Dual cost: $\ell(\lambda, \mu) = \min\limits_x \, (c^T - \lambda^T + \mu^T A)\, x - \mu^T b =$ constant

$$= \begin{cases} -\mu^T b & \text{if } c^T - \lambda^T + \mu^T A = 0 \\ -\infty & \text{otherwise} \end{cases}$$

Dual problem

$$D^* = \max_{\lambda \geq 0, \mu} \ell(\lambda, \mu) = \max_{\substack{\lambda \geq 0 \\ c^T - \lambda^T + \mu^T A = 0}} -\mu^T b =$$

↳ λ are slack variables

$$= \max_{c^T + \mu^T A \geq 0} -\mu^T b = \max_{-A^T \mu \leq c} -b^T \mu =$$

↳ set $\nu = -\mu$

$$= \max_{A^T \nu \leq c} b^T \nu$$

Rmk.
- The dual is an LP
-

Primal	Dual
min	max
c^T	b^T
A	A^T

Duals of LP problems in generic form

The dual can be found using the correspondences in the table

in the primal

Primal	Dual
variables $x \in \mathbb{R}^n$	variables: $\lambda \in \mathbb{R}^m$. m is the number of inequality constraints (except sign constraints)
$\max c^T x$	$\min b^T \lambda$
$a_i^T x \le b_i$	$\lambda_i \ge 0$
$a_i^T x = b_i$	λ_i sign free
[$a_i^T x \ge b_i$]	[$\lambda_i \le 0$]
$x_j \ge 0$	$A_j^T \lambda \ge c_j$
x_j sign free	$A_j^T \lambda = c_j$
[$x_j \le 0$]	[$A_j^T \lambda \le c_j$]

Sign constraints: $x_j \ge 0$, x_j sign free, [$x_j \le 0$]

A_j = j-th column of A

Example : dual of LPs in canonical form

- $$\min_{\substack{Ax\le b\\ x\ge 0}} c^T x = -\max_{\substack{Ax\le b\\ x\ge 0}} (-c^T)x = -\max_{\substack{(\lambda_1)\ a_1^T x\le b_1\\ \vdots\\ (\lambda_m)\ a_m^T x\le b_m\\ x_1\ge 0\\ \vdots\\ x_n\ge 0}} (-c^T)x \xrightarrow{\text{dual}} -\min_{\substack{\lambda_1\ge 0\\ \vdots\\ \lambda_m\ge 0\\ A_1^T\lambda\ge -c_1\\ \vdots\\ A_n^T\lambda\ge -c_n}} b^T\lambda =$$

$$A = \begin{bmatrix} a_1^T \\ \vdots \\ a_m^T \end{bmatrix}$$

$$A = [A_1, \dots, A_n]$$

$$= \max_{\substack{\nu\le 0\\ A^T\nu\le c}} b^T\nu \leftarrow \text{usual form found on textbooks}$$

set $\nu = -\lambda$

Example: dual of LPs in canonical form

- $$\begin{aligned} \max\ & c^T x \\ & Ax \le b \\ & x \ge 0 \end{aligned} \quad \xrightarrow[\text{(verify @ home)}]{\text{dual}} \quad \begin{aligned} \min\ & b^T \lambda \\ & \lambda \ge 0 \\ & A^T \lambda \ge c \end{aligned}$$

Duality in management science

• Dual of the product mix

The company PRIMAL plans the production of telephone cases solving the LP

$$\max\ 30x_1 + 20x_2$$

x_1, x_2: cases of type I and II

$$\left.\begin{aligned} 8x_1 + 4x_2 &\le 640 \\ 4x_1 + 6x_2 &\le 540 \\ x_1 + x_2 &\le 100 \end{aligned}\right\}\ \text{3 production phases}$$

$$x_1, x_2 \ge 0$$

→ men/hours in each production phase

The company DUAL produces the same cases and it wants to subcontract the production process to PRIMAL.

How much DUAL must pay each men/hour in each production phase so that PRIMAL accepts the subcontracting ?

RMK: pricing problem.

1) each man/hour given to dual is a gain loss for PRIMAL

2) Prices per man/hour allow DUAL to buy only a subset of PRIMAL resources

Let λ_1, λ_2 and λ_3 be the prices of a man/hour in phases 1, 2 and 3.

Assuming DUAL buys all available men/hours, DUAL minimizes the cost

$$640\,\lambda_1 + 540\,\lambda_2 + 100\,\lambda_3$$

under the constraints

1) $8 \cdot \lambda_1 + 4 \cdot \lambda_2 + 1 \cdot \lambda_3 \geq 30$

required hours in phases 1, 2 and 3 for producing a type-I case

↳ revenue for selling a type-I case

PRIMAL

$$\max\ 30\,x_1 + 20\,x_2$$
$$8\,x_1 + 4\,x_2 \leq 640$$
$$4\,x_1 + 6\,x_2 \leq 540$$
$$x_1 + x_2 \leq 100$$
$$x_1, x_2 \geq 0$$

2) $4\lambda_1 + 6\lambda_2 + \lambda_3 \geq 20$

3) $\lambda = [\lambda_1, \lambda_2, \lambda_3]^T \geq 0$

The LP that DUAL must solve is

$$\min\ 640\lambda_1 + 540\lambda_2 + 100\lambda_3$$
$$8\lambda_1 + 4\lambda_2 + \lambda_3 \geq 30$$
$$4\lambda_1 + 6\lambda_2 + \lambda_3 \geq 20$$
$$\lambda_1, \lambda_2, \lambda_3 \geq 0$$

that is the dual of the original problem (verify @ home)

Rmk. λ_1, λ_2 and λ_3 are called "shadow prices"

PRIMAL

$$\max\ 30x_1 + 20x_2$$
$$8x_1 + 4x_2 \leq 640$$
$$4x_1 + 6x_2 \leq 540$$
$$x_1 + x_2 \leq 100$$
$$x_1, x_2 \geq 0$$

Dual of the diet problem

The company PRIMAL produces chicken food solving the LP

$$\min \; 20x_1 + 30x_2$$

x_1, x_2 = boxes of additives 1 and 2

$$\left.\begin{array}{l} x_2 \geq 2 \\ 2x_1 + x_2 \geq 12 \\ 2x_1 + 5x_2 \geq 36 \\ x_2 \geq 4 \end{array}\right\}$$

required grams of vitamines A, B, C and D per box

$$x_1, x_2 \geq 0$$

The company DUAL produces the four vitamines. Which are the maximal selling prices DUAL can set so that PRIMAL will buy them instead of buying boxes of additives?

Let $\lambda_1, \dots, \lambda_4$ be the prices for each vitamine.

DUAL maximizes the cost (i.e. the revenue)

$$2\lambda_1 + 12\lambda_2 + 36\lambda_3 + 4\lambda_4$$

grams of vitamines in a box of chicken food

under the constraints

1) $0\cdot\lambda_1 + 2\cdot\lambda_2 + 2\cdot\lambda_3 + 0\cdot\lambda_4 \leq 20$

market price for a box of additive 1

Meaning: PRIMAL buys vitamines from DUAL only if it pays less than buying boxes of additive 1 and 2

PRIMAL

$$\min \; 20x_1 + 30x_2$$

$$x_2 \geq 2$$

$$2x_1 + x_2 \geq 12$$

$$2x_1 + 5x_2 \geq 36$$

$$x_2 \geq 4$$

$$x_1, x_2 \geq 0$$

2) $\lambda_1 + \lambda_2 + 5\lambda_3 + \lambda_4 \leq 30$

3) $\lambda_i \geq 0 \quad i = 1, \dots, 4$

The LP that DUAL must solve is

$$\max \; 2\lambda_1 + 12\lambda_2 + 36\lambda_3 + 4\lambda_4$$
$$2\lambda_2 + 2\lambda_3 \leq 20$$
$$\lambda_1 + \lambda_2 + 5\lambda_3 + \lambda_4 \leq 30$$
$$\lambda_1, \dots, \lambda_4 \geq 0$$

that is the dual of the original problem (verify @ home)

PRIMAL

$$\min \; 20x_1 + 30x_2$$
$$x_2 \geq 2$$
$$2x_1 + x_2 \geq 12$$
$$2x_1 + 5x_2 \geq 36$$
$$x_2 \geq 4$$
$$x_1, x_2 \geq 0$$