Mixed Integer Linear Programming

Part I

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A linear program..

Problem description

The ACME company has a project to build at least 900 smart washing machines. The production process of such devices can be conducted in three different ways : (i) manually, (ii) semi-automatically, and (iii) automatically. Each of the available approaches involves the allocation of different amounts of human resources. In particular, the manual production demands 1 minute of qualified work, 40 minutes of non-qualified work and 3 minutes of assemblage. If the semi-automatic solution would be chosen, 4 minutes of qualified work, 30 minutes of non-qualified work, and 2 minutes for the assemblage would be required. Finally, 8, 20 and 4 minutes respectively would be required for the automatic method. ACME has a pool of 4500 minutes of qualified work, 36000 minutes of non-qualified work and 2700 minutes of assembly. The production costs of a washing machine are 70 euros if produced manually, 80 euros if produced semi-automatically, and 85 euros if produced automatically. Each smart washing machine is sold at 130 euros. From a commercial point of view, the ACME company is interested in one of the following objective

Find the optimal number of washing machines to be produced in order to maximize the profit.

A linear program..

Problem formulation

The optimization variables of the problem are x_1, x_2 and x_3 which are used to represent the number of washing machines produced using the manual, semi-automatic, and automatic methods respectively.

If the company is interested in maximizing profit, then the following optimization problem needs to be solved

$$\max_{x_1,x_2,x_3} \; 60x_1 + 50x_2 + 45x_3$$

subject to

$$\begin{aligned} x_1 + x_2 + x_3 &\geq 900 \\ x_1 + 4x_2 + 8x_3 &\leq 4500 \\ 40x_1 + 30x_2 + 20x_3 &\leq 36000 \\ 3x_1 + 2x_2 + 4x_3 &\leq 2700 \\ x_1, x_2, x_3 &\geq 0 \end{aligned} \; .$$

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Does not take into account possible fixed costs related to the acquisition of new technologies (e.g. installation of semi-automatic and automatic assembling lines)

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How to incorporate logical statements in the optimization?

Basic logic concepts

- A basic concept used in propositional logic is the statement (a.k.a. atomic proposition)
- A statement is either true (T) or false (F).
- Compound propositions can be obtained by connecting statements with logical connectives

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$	$p \otimes q$
0	0	1	0	0	1	1	0
0	1	1	0	1	1	0	1
1	0	0	0	1	0	0	1
1	1	0	1	1	1	1	0

Equivalences

Statement	Equivalent Form	Observations
$\neg\neg p$	p	double negation
$p \otimes q$	$(\neg p \wedge q) \vee (p \wedge \neg q)$	exclusive OR (either p or q)
$\neg(p \vee q)$	$\neg p \wedge \neg q$	De Morgan's law
$\neg(p \wedge q)$	$\neg p \vee \neg q$	De Morgan's law
$p \wedge (q \vee r)$	$(p \wedge q) \vee (p \wedge r)$	distributive law
$p \vee (q \wedge r)$	$(p \vee q) \wedge (p \vee r)$	distributive law
$p \rightarrow q$	$\neg p \vee q$	implication (if p then q)
$p \leftrightarrow q$	$(p \rightarrow q) \wedge (q \rightarrow p)$	equivalence (p if and only if q)

Conjunctive Normal Form (CNF)

A CNF is a conjunction of clauses, where a clause is a disjunction of literals

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$$(A \vee B) \wedge (\neg B \vee C \vee \neg D) \wedge (D \vee \neg E)$$

conjunction of clauses

disjunction of literals

literals

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Procedure

- *Step 1*: remove implications
- *Step 2:* use De Morgan's law and the double negation to absorb all «not» into the atomic statements
- *Step 3:* use the distributive law to move the conjunctions out of the statements until each statement is a clause of pure disjunctions

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$$[(p_1 \wedge p_2) \Rightarrow (p_3 \wedge p_4)]$$

$$\neg(p_1 \wedge p_2) \vee (p_3 \wedge p_4)$$

$$\neg p_1 \vee \neg p_2 \vee (p_3 \wedge p_4)$$

$$(\neg p_1 \vee \neg p_2 \vee p_3) \wedge (\neg p_1 \vee \neg p_2 \vee p_4)$$

Translation of Logic Rules into Linear Integer Inequalitites

- Associate to each boolean variable p_i a binary integer variable δ_i

$$p_i \Leftrightarrow \{\delta_i = 1\}, \quad \neg p_i \Leftrightarrow \{\delta_i = 0\}$$

- Then, a logic proposition in CNF can be expressed in terms of linear integer inequalities, since

$\neg p_i$	$1 - \delta_i$
$p_i \vee p_j$	$\delta_i + \delta_j \geq 1$

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- Example: $(\neg p_1 \vee \neg p_2 \vee p_3) \wedge (\neg p_1 \vee \neg p_2 \vee p_4)$

- The above proposition is true iff

$$\begin{cases} \delta_1 + \delta_2 - \delta_3 & \leq & 1 \\ \delta_1 + \delta_2 - \delta_4 & \leq & 1 \\ \delta_{1,2,3} \in \{0,1\} & & \end{cases}$$

Combining logic rules and continuous data

- Consider the quantity $\sum_{j \in J} a_{kj} x_j - b_k$ which represents the k-th row of **Ax-b**
- Assume this admits both a lower and an upper bound $L_k \leq \sum_{j \in J} a_{kj} x_j - b_k \leq U_k$
- Then, it holds that:

1)

$$\delta_k = 1 \rightarrow \sum_j a_{kj} x_j - b_k \leq 0 \quad \equiv \quad \sum_j a_{kj} x_j - b_k \leq U_k(1 - \delta_k)$$

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2)

$$\delta_k = 1 \rightarrow \sum_j a_{kj} x_j - b_k \geq 0 \equiv \sum_j a_{kj} x_j - b_k \geq L_k (1 - \delta_k)$$

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Very small number
Machine precision
2.2204 e-16

3)

$$\delta_k = 1 \rightarrow \sum_j a_{kj} x_j - b_k < 0 \equiv \sum_j a_{kj} x_j - b_k \leq (U_k + \varepsilon)(1 - \delta_k) - \varepsilon$$

Combining logic rules and continuous data

- Consider the quantity $\sum_{j \in J} a_{kj} x_j - b_k$ which represents the k-th row of **Ax-b**
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- Then, it holds that:

4)

$$\delta_k = 1 \rightarrow \sum_j a_{kj} x_j - b_k > 0 \equiv \sum_j a_{kj} x_j - b_k \geq (L_k - \varepsilon)(1 - \delta_k) + \varepsilon$$

Combining logic rules and continuous data

- Since $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (\neg p \rightarrow \neg q)$ for the equivalence one gets

$\delta_k = 1 \leftrightarrow \sum_j a_{kj}x_j \leq b_k$	$\begin{cases} \sum_j a_{kj}x_j - b_k \leq U_k(1-\delta_k) \\ \sum_j a_{kj}x_j - b_k \geq (L_k - \varepsilon)\delta_k + \varepsilon \end{cases}$	$\begin{cases} L_k \leq 0 \\ U_k \geq +\varepsilon \end{cases}$
$\delta_k = 1 \leftrightarrow \sum_j a_{kj}x_j \geq b_k$	$\begin{cases} \sum_j a_{kj}x_j - b_k \geq L_k(1-\delta_k) \\ \sum_j a_{kj}x_j - b_k \leq (U_k + \varepsilon)\delta_k - \varepsilon \end{cases}$	$\begin{cases} L_k \leq -\varepsilon \\ U_k \geq 0 \end{cases}$
$\delta_k = 1 \leftrightarrow \sum_j a_{kj}x_j < b_k$	$\begin{cases} \sum_j a_{kj}x_j - b_k \leq (U_k + \varepsilon)(1-\delta_k) - \varepsilon \\ \sum_j a_{kj}x_j - b_k \geq L_k\delta_k \end{cases}$	$\begin{cases} L_k \leq -\varepsilon \\ U_k \geq 0 \end{cases}$
$\delta_k = 1 \leftrightarrow \sum_j a_{kj}x_j > b_k$	$\begin{cases} \sum_j a_{kj}x_j - b_k \geq (L_k - \varepsilon)(1-\delta_k) + \varepsilon \\ \sum_j a_{kj}x_j - b_k \leq U_k\delta_k \end{cases}$	$\begin{cases} L_k \leq 0 \\ U_k \geq +\varepsilon \end{cases}$

Thanks to all these properties, it is possible to translate the combination of logic rules and continuous information into **mixed integer linear inequalities**

Further properties

Bilinear terms

- product between binary variables

  $$\delta_3 = \delta_1\delta_2 \text{ is equivalent to } \begin{cases} -\delta_1 + \delta_3 \leq 0, \\ -\delta_2 + \delta_3 \leq 0, \\ \delta_1 + \delta_2 - \delta_3 \leq 1. \end{cases}$$

  - It relies on the introduction of a new binary variable

- product between a binary and a continous variable $\delta f(x)$, where $f: \mathbb{R}^n \mapsto \mathbb{R}$ and $\delta \in \{0, 1\}$
  - It relies on the introduction of a new continous variable $y \triangleq \delta f(x)$.
  - *The variable y can be expressed in terms of four inequalities*

$$\begin{aligned} &y \leq M\delta, \\ &y \geq m\delta, \\ &y \leq f(x) - m(1-\delta), \\ &y \geq f(x) - M(1-\delta). \end{aligned}$$

$$M \triangleq \max_{x \in \mathcal{X}} f(x),$$

$$m \triangleq \min_{x \in \mathcal{X}} f(x).$$

Back to the example!

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