Mixed Integer Linear Programming

Part II

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Back to the example!

Problem formulation

The optimization variables of the problem are x_1, x_2 and x_3 which are used to represent the number of washing machines produced using the manual, semi-automatic, and automatic methods respectively.

If the company is interested in maximizing profit, then the following optimization problem needs to be solved

$$\max_{x_1,x_2,x_3} \; 60x_1 + 50x_2 + 45x_3$$

subject to

$$\begin{aligned} x_1 + x_2 + x_3 &\geq 900 \\ x_1 + 4x_2 + 8x_3 &\leq 4500 \\ 40x_1 + 30x_2 + 20x_3 &\leq 36000 \; . \\ 3x_1 + 2x_2 + 4x_3 &\leq 2700 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

- if $x_2 > 0$ then installation of semi-automatic line is 1000€
- if $x_3 > 0$ then installation of the automatic line is 2000€

How to incorporate logical statements in the optimization?

Back to the example!

$$\max_{x_1,x_2,x_3} \; 60x_1 + 50x_2 + 45x_3$$

subject to

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- if $x_2 > 0$ then installation of semi-automatic line is 1000€
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Compute upper and lower bounds for x_2 and x_3

$$\begin{aligned} 4x_2 &\leq 4500 \\ 30x_2 &\leq 36000 \\ 2x_2 &\leq 2700 \end{aligned} \qquad 0 \leq x_2 \leq 1125$$

$$\begin{aligned} 8x_3 &\leq 4500 \\ 20x_3 &\leq 36000 \\ 4x_3 &\leq 2700 \end{aligned} \qquad 0 \leq x_3 \leq 562.5$$

Back to the example!

- if $x_2 > 0$ then installation of semi-automatic line is 1000€
- if $x_3 > 0$ then installation of the automatic line is 2000€

$$\delta_1 = 1 \Leftrightarrow x_2 > 0$$
$$\delta_2 = 1 \Leftrightarrow x_3 > 0$$

$$p \Leftrightarrow q \textit{ is equivalent to } p \Rightarrow q \wedge \neg p \Rightarrow \neg q$$

$$\delta_1 = 1 \Rightarrow x_2 > 0 \wedge \delta_1 = 0 \Rightarrow x_2 \leq 0$$
$$\delta_2 = 1 \Rightarrow x_3 > 0 \wedge \delta_2 = 0 \Rightarrow x_3 \leq 0$$

Back to the example!

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$$\delta_1 = 1 \Rightarrow x_2 > 0 \wedge \delta_1 = 0 \Rightarrow x_2 \leq 0$$
$$\delta_2 = 1 \Rightarrow x_3 > 0 \wedge \delta_2 = 0 \Rightarrow x_3 \leq 0$$

$$\delta_1 = 1 \Rightarrow x_2 > 0$$
$$\delta_1 = 1 \Rightarrow x_2 \geq \epsilon$$
$$x_2 \geq (0 - \epsilon)(1 - \delta_1) + \epsilon$$

$$\delta_1 = 0 \Rightarrow x_2 \leq 0$$
$$x_2 \leq 1125\delta_1$$

$$\delta_2 = 1 \Rightarrow x_3 > 0$$
$$\delta_2 = 1 \Rightarrow x_3 \geq \epsilon$$
$$x_3 \geq (0 - \epsilon)(1 - \delta_2) + \epsilon$$

$$\delta_2 = 0 \Rightarrow x_3 \leq 0$$
$$x_3 \leq 562.5\delta_2$$

Back to the example!

The optimization problem becomes

$$
\begin{aligned}
\max \quad & 60x_1 + 50x_2 + 45x_3 - 1000\delta_1 - 2000\delta_2 \\
& -x_1 - x_2 - x_3 \leq -900 \\
& x_1 + 4x_2 + 8x_3 \leq 4500 \\
& 40x_1 + 30x_2 + 20x_3 \leq 36000 \\
& 3x_1 + 2x_2 + 4x_3 \leq 2700 \\
& -x_2 + \epsilon\delta_1 \leq 0 \\
& x_2 - 1125\delta_1 \leq 0 \\
& -x_3 + \epsilon\delta_2 \leq 0 \\
& x_3 - 562.5\delta_2 \leq 0 \\
& x_1, x_2, x_3 \geq 0 \\
& \delta_1, \delta_2 \in \{0, 1\}
\end{aligned}
$$

Back to the example!

The optimization problem becomes

$$\begin{aligned}
\max \quad & 60x_1 + 50x_2 + 45x_3 - 1000\delta_1 - 2000\delta_2 \\
& -x_1 - x_2 - x_3 \leq -900 \\
& x_1 + 4x_2 + 8x_3 \leq 4500 \\
& 40x_1 + 30x_2 + 20x_3 \leq 36000 \\
& 3x_1 + 2x_2 + 4x_3 \leq 2700 \\
& -x_2 + \epsilon\delta_1 \leq 0 \\
& x_2 - 1125\delta_1 \leq 0 \\
& -x_3 + \epsilon\delta_2 \leq 0 \\
& x_3 - 562.5\delta_2 \leq 0 \\
& x_1, x_2, x_3 \geq 0 \\
& \delta_1, \delta_2 \in \{0, 1\}
\end{aligned}$$

Standard form

$$\begin{aligned}
\max \quad & 60x_1 + 50x_2 + 45x_3 - 1000\delta_1 - 2000\delta_2 \\
& -x_1 - x_2 - x_3 + s_1 = -900 \\
& x_1 + 4x_2 + 8x_3 + s_2 = 4500 \\
& 40x_1 + 30x_2 + 20x_3 + s_3 = 36000 \\
& 3x_1 + 2x_2 + 4x_3 + s_4 = 2700 \\
& -x_2 + \epsilon\delta_1 + s_5 = 0 \\
& x_2 - 1125\delta_1 + s_6 = 0 \\
& -x_3 + \epsilon\delta_2 + s_7 = 0 \\
& x_3 - 562.5\delta_2 + s_8 = 0 \\
& x_1, x_2, x_3, s_1, s_2, \ldots, s_8 \geq 0 \\
& \delta_1, \delta_2 \in \{0, 1\}
\end{aligned}$$

Back to the example!

The optimization problem becomes

$$
\begin{aligned}
\max \quad & 60x_1 + 50x_2 + 45x_3 - 1000\delta_1 - 2000\delta_2 \\
& -x_1 - x_2 - x_3 \leq -900 \\
& x_1 + 4x_2 + 8x_3 \leq 4500 \\
& 40x_1 + 30x_2 + 20x_3 \leq 36000 \\
& 3x_1 + 2x_2 + 4x_3 \leq 2700 \\
& -x_2 + \epsilon\delta_1 \leq 0 \\
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& x_1, x_2, x_3 \geq 0 \\
& \delta_1, \delta_2 \in \{0, 1\}
\end{aligned}
$$

Standard form with **constants on the rhs all positive**

$$
\begin{aligned}
\max \quad & 60x_1 + 50x_2 + 45x_3 - 1000\delta_1 - 2000\delta_2 \\
& x_1 + x_2 + x_3 - s_1 = 900 \\
& x_1 + 4x_2 + 8x_3 + s_2 = 4500 \\
& 40x_1 + 30x_2 + 20x_3 + s_3 = 36000 \\
& 3x_1 + 2x_2 + 4x_3 + s_4 = 2700 \\
& -x_2 + \epsilon\delta_1 + s_5 = 0 \\
& x_2 - 1125\delta_1 + s_6 = 0 \\
& -x_3 + \epsilon\delta_2 + s_7 = 0 \\
& x_3 - 562.5\delta_2 + s_8 = 0 \\
& x_1, x_2, x_3, s_1, s_2, \ldots, s_8 \geq 0 \\
& \delta_1, \delta_2 \in \{0, 1\}
\end{aligned}
$$

Final result

$x^\intercal = (x_1, x_2, x_3, s_1, s_2, \dots, s_8, \delta_1, \delta_2)$ $c^\intercal = (60, 50, 45, 0, 0, \dots, 0, -1000, -2000)$

$$A = \begin{pmatrix} 1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 8 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 40 & 30 & 20 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \epsilon & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1125 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \epsilon \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -562.5 \end{pmatrix}$$

$b^\intercal = (900, 4500, 36000, 2700, 0, 0, 0, 0)$

Final MILP

$$\begin{aligned} \max \quad & c^\intercal x \\ & Ax = b \\ & x \geq 0 \\ & x_{12}, x_{13} \in \{0, 1\} \end{aligned}$$

Final result

Final MILP

$$\begin{aligned} \max \quad & c^\intercal x \\ & Ax = b \\ & x \geq 0 \\ & x_{12}, x_{13} \in \{0, 1\} \end{aligned}$$

Note: a mixed integer linear program is not convex!

- how does the feasible set look like?
- integer linear programs (ILP) are also not convex!

Final result

Final MILP

$$\begin{aligned} \max \quad & c^\intercal x \\ & Ax = b \\ & x \geq 0 \\ & x_{12}, x_{13} \in \{0, 1\} \end{aligned}$$

Note: a mixed integer linear program is not convex!

- how does the feasible set look like?
- integer linear programs (ILP) are also not convex!

δ_1	δ_2
0	0
0	1
1	0
1	1

How do we solve it?

One way consists in enumerating all the possible combination of the binary variables and solve all the resulting LPs. The optimal solution is obtained by comparing the result of the LPs.

Combinatorial complexity! Any heuristic to speed up computation?