Properties of linear programming

Dipartimento di Ingegneria Industriale e dell'Informazione
Università degli Studi di Pavia

Industrial Automation

Linear Programming (LP)

Widely used optimization technique in management science

- optimal allocation of limited resources for maximizing revenues or minimizing costs

Basic problem

$$\min_{\substack{g_i(x)\leq 0\\ i=1,2,\ldots,m}} f(x), \quad x \in \mathbb{R}^n \tag{1}$$

A Linear Programming (LP) problem is (1) with

- $f(x) = c^{\mathrm{T}}x$ (linear cost)
- $g_i(x) = a_i^{\mathrm{T}}x - b_i$ (affine constraints)

An LP problem is a convex optimization problem

Linear Programming (LP)

Canonical form

An LP problem is in *canonical form* if it is written as

$$\min_{\substack{a_i^{\mathrm{T}} x \leq b_i,\ i=1,2,\ldots,m \\ x_j \geq 0,\ j=1,2,\ldots,n}} c^{\mathrm{T}} x$$

or

$$\max_{\substack{a_i^{\mathrm{T}} x \leq b_i,\ i=1,2,\ldots,m \\ x_j \geq 0,\ j=1,2,\ldots,n}} c^{\mathrm{T}} x$$

"$\leq$" constraints and positivity constraints on all variables

PL - matrix notation

Vector inequalities

$$x \leq 0 \text{ means } \begin{cases} x_1 \leq 0 \\ x_2 \leq 0 \\ \dots \\ x_n \leq 0 \end{cases}$$

Constraints

$$\begin{cases} a_1^{\mathrm{T}} x \leq b_1 \\ a_2^{\mathrm{T}} x \leq b_2 \\ \dots \\ a_m^{\mathrm{T}} x \leq b_m \end{cases} \Leftrightarrow \; Ax \leq b, \quad A = \begin{bmatrix} a_1^{\mathrm{T}} \\ a_2^{\mathrm{T}} \\ \vdots \\ a_m^{\mathrm{T}} \end{bmatrix}, \; b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

PL - matrix notation

LP problem in generic form

$$\min_{Ax \leq b} c^{\mathrm{T}} x$$

LP problem in canonical form (LP-C)

$$\min_{\substack{Ax \leq b \\ x \geq 0}} c^{\mathrm{T}} x$$

Outline

Outline

Representations of LP problems

LP in canonical form (LP-C)

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Inequality "$\leq$" constraints. Positivity constraints on all variables.

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LP in standard form (LP-S)

$$\min_{\substack{Ax = b \\ x \geq 0}} c^{\mathrm{T}} x$$

Equality constraints. Positivity constraints on all variables.

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Mixed constraints $\le$, $\ge$, $=$ and/or some variable is not constrained to be positive.

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Mixed constraints $\leq$, $\geq$, $=$ and/or some variable is not constrained to be positive.

The three forms are equivalent even if the conversion from one form to another one is possible only changing the number of variables and/or constraints.

Conversion between constraints

From $\leq$ to $=$

$$a_i^{\mathrm{T}} x \leq b_i \Leftrightarrow \exists s_i \in \mathbb{R} : \begin{cases} a_i^{\mathrm{T}} x + s_i = b_i \\ s_i \geq 0 \end{cases}$$

The additional variable s_i is called *slack variable*

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From $\geq$ to $=$

$$a_i^{\mathrm{T}} x \geq b_i \Leftrightarrow \exists s_i \in \mathbb{R} : \begin{cases} a_i^{\mathrm{T}} x - s_i = b_i \\ s_i \geq 0 \end{cases}$$

The additional variable s_i is called *excess variable*

Conversion between constraints

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In both cases, a single constraint is replaced by two constraints

Positivity constraints

Variables without sign constraints

$$x_i \in \mathbb{R} \Leftrightarrow \exists x_i^+, x_i^- \in \mathbb{R} : \begin{cases} x_i = x_i^+ - x_i^- \\ x_i^+ \geq 0 \\ x_i^- \geq 0 \end{cases}$$

x_i^+ and x_i^- are two new variables representing the positive and negative part of $x_i \in \mathbb{R}$, respectively

The variable x_i is replaced with $x_i^+ - x_i^-$ in the whole LP problem and constraints $x_i^+ \geq 0$, $x_i^- \geq 0$ are added

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Variables with sign constraints: from "≤ 0" to "≥ 0":

$$x_i \leq 0 \longrightarrow \xi_i \geq 0$$

with $\xi_i = -x_i$ that replaces x_i in the whole LP problem

Example 1

Write the following problem in standard form

$$\max_{x} \left\{ c^{\mathrm{T}} x : Ax = b \right\}$$

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- There is no positivity constraint: we introduce two vectors $x^{+} \in \mathbb{R}^{n}, x^{-} \in \mathbb{R}^{n}$ and substitute x with $x^{+} - x^{-}$. We get

$$\max_{x^{+}, x^{-}} \left\{ c^{\mathrm{T}}(x^{+} - x^{-}) : A(x^{+} - x^{-}) = b, x^{+} \geq 0, x^{-} \geq 0 \right\}.$$

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$$\max_{x^+, x^-} \left\{c^{\mathrm{T}}(x^+ - x^-) : A(x^+ - x^-) = b, x^+ \geq 0, x^- \geq 0\right\}.$$

- Defining $\xi = \begin{bmatrix} x^+ \\ x^- \end{bmatrix}$ the problem becomes

$$\max_\xi \left\{\begin{bmatrix} c^{\mathrm{T}} & -c^{\mathrm{T}} \end{bmatrix} \xi : \begin{bmatrix} A & -A \end{bmatrix} \xi = b, \xi \geq 0\right\}$$

In the conversion process the number of variables doubled

Example 2: conversion between canonical and standard forms

From canonical (LP-C) to standard (LP-S) form

$$\max_{\substack{Ax \leq b \\ x \geq 0}} c^{\mathrm{T}} x \quad \longrightarrow \quad \max_{\begin{bmatrix} x \\ s \end{bmatrix}} \left\{ \begin{bmatrix} c & 0 \end{bmatrix} \begin{bmatrix} x \\ s \end{bmatrix} : \begin{bmatrix} A & I \end{bmatrix} \begin{bmatrix} x \\ s \end{bmatrix} = b, \ \begin{bmatrix} x \\ s \end{bmatrix} \geq 0 \right\} \qquad (1)$$

We introduced the vector of slack variables $s \in \mathbb{R}^n$.

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From LP-S to LP-C

$$\max_{\substack{Ax = b \\ x \geq 0}} c^{\mathrm{T}} x \quad \longrightarrow \quad \max_{x} \left\{ c^{\mathrm{T}} x : \begin{bmatrix} A \\ -A \end{bmatrix} x \leq \begin{bmatrix} b \\ -b \end{bmatrix}, \; x \geq 0 \right\} \tag{2}$$

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From canonical (LP-C) to standard (LP-S) form

$$\max_{\substack{Ax\leq b\\ x\geq 0}} c^{\mathrm{T}}x \quad \longrightarrow \quad \max_{\begin{bmatrix}x\\ s\end{bmatrix}} \left\{ \begin{bmatrix} c & 0 \end{bmatrix} \begin{bmatrix}x\\ s\end{bmatrix} : \begin{bmatrix} A & I \end{bmatrix} \begin{bmatrix}x\\ s\end{bmatrix} = b,\ \begin{bmatrix}x\\ s\end{bmatrix} \geq 0 \right\} \qquad (1)$$

We introduced the vector of slack variables $s \in \mathbb{R}^n$.

From LP-S to LP-C

$$\max_{\substack{Ax=b\\ x\geq 0}} c^{\mathrm{T}}x \quad \longrightarrow \quad \max_{x} \left\{ c^{\mathrm{T}}x : \begin{bmatrix} A\\ -A \end{bmatrix} x \leq \begin{bmatrix} b\\ -b \end{bmatrix},\ x \geq 0 \right\} \qquad (2)$$

Meaning of equivalence between the two forms:

- In (1): x^* is optimal for LP-C $\Leftrightarrow$ $\exists s^* : \begin{bmatrix}x^*\\ s^*\end{bmatrix}$ is optimal for LP-S
- In (2): x^* is optimal for LP-S $\Leftrightarrow$ x^* is optimal for LP-C

Example 3

Write the following problem in canonical form

$$\min_{x_1,x_2,x_3} c_1x_1 + c_2x_2 + c_3x_3 \tag{3}$$

$$a_{11}x_1 + a_{12}x_2 \leq b_1 \tag{4}$$

$$a_{22}x_2 + a_{23}x_3 \geq b_2 \tag{5}$$

$$a_{31}x_1 + a_{32}x_3 = b_3 \tag{6}$$

$$x_1 \geq 0 \tag{7}$$

$$x_2 \leq 0 \tag{8}$$

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$$x_1 \geq 0 \tag{7}$$

$$x_2 \leq 0 \tag{8}$$

1. Positivity constraints on all variables:
   - replace x_2 with $\xi_2 = -x_2$
   - x_3 is not sign constrained: we set $x_3 = x_3^+ - x_3^-$ and add the constraints $x_3^+ \geq 0$ e $x_3^- \geq 0$

Example 3

The original problem is now

$$\min_{x_1,\xi_2,x_3^+,x_3^-} c_1x_1 - c_2\xi_2 + c_3x_3^+ - c_3x_3^- \tag{9}$$

$$a_{11}x_1 - a_{12}\xi_2 \leq b_1 \tag{10}$$

$$-a_{22}\xi_2 + a_{23}x_3^+ - a_{23}x_3^- \geq b_2 \tag{11}$$

$$a_{31}x_1 + a_{32}x_3^+ - a_{32}x_3^- = b_3 \tag{12}$$

$$x_1 \geq 0 \tag{13}$$

$$\xi_2 \geq 0 \tag{14}$$

$$x_3^+ \geq 0 \tag{15}$$

$$x_3^- \geq 0 \tag{16}$$

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$$\min_{x_1,\xi_2,x_3^+,x_3^-} c_1x_1 - c_2\xi_2 + c_3x_3^+ - c_3x_3^- \tag{9}$$

$$a_{11}x_1 - a_{12}\xi_2 \leq b_1 \tag{10}$$
$$-a_{22}\xi_2 + a_{23}x_3^+ - a_{23}x_3^- \geq b_2 \tag{11}$$
$$a_{31}x_1 + a_{32}x_3^+ - a_{32}x_3^- = b_3 \tag{12}$$
$$x_1 \geq 0 \tag{13}$$
$$\xi_2 \geq 0 \tag{14}$$
$$x_3^+ \geq 0 \tag{15}$$
$$x_3^- \geq 0 \tag{16}$$

2. Constraints "$\leq$":
   - we replace (11) with $a_{22}\xi_2 - a_{23}x_3^+ + a_{23}x_3^- \leq -b_2$
   - we replace (12) with $a_{31}x_1 + a_{32}x_3^+ - a_{32}x_3^- \leq b_3$ and $-a_{31}x_1 - a_{32}x_3^+ + a_{32}x_3^- \leq -b_3$

Example 3

The LP problem is now in canonical form

$$\min_{x_1,\xi_2,x_3^+,x_3^-} c_1x_1 - c_2\xi_2 + c_3x_3^+ - c_3x_3^- \tag{17}$$

$$a_{11}x_1 - a_{12}\xi_2 \leq b_1 \tag{18}$$

$$+a_{22}\xi_2 - a_{23}x_3^+ + a_{23}x_3^- \leq -b_2 \tag{19}$$

$$a_{31}x_1 + a_{32}x_3^+ - a_{32}x_3^- \leq b_3 \tag{20}$$

$$-a_{31}x_1 - a_{32}x_3^+ + a_{32}x_3^- \leq -b_3 \tag{21}$$

$$x_1 \geq 0 \tag{22}$$

$$\xi_2 \geq 0 \tag{23}$$

$$x_3^+ \geq 0 \tag{24}$$

$$x_3^- \geq 0 \tag{25}$$

Example 3 - matrix notation

We define $x = \begin{bmatrix} x_1 & \xi_2 & x_3^+ & x_3^- \end{bmatrix}^T$ and obtain

$$\min_{\substack{Ax \leq b \\ x \geq 0}} \begin{bmatrix} c_1 & -c_2 & c_3 & -c_3 \end{bmatrix} x \tag{26}$$

$$A = \begin{bmatrix} a_{11} & -a_{12} & 0 & 0 \\ 0 & a_{22} & -a_{23} & +a_{23} \\ a_{31} & 0 & a_{32} & -a_{32} \\ -a_{31} & 0 & -a_{32} & +a_{32} \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ -b_2 \\ b_3 \\ -b_3 \end{bmatrix}$$

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Meaning of equivalence between different forms

If $x^* = \begin{bmatrix} x_1^* & \xi_2^* & (x_3^+)^* & (x_3^-)^* \end{bmatrix}$ is an optimal solution to (26), then $\begin{bmatrix} \tilde{x}_1 & \tilde{x}_2 & \tilde{x}_3 \end{bmatrix}$ is an optimal solution to the original problem, where

$$\tilde{x}_1 = x_1^*$$
$$\tilde{x}_2 = -\xi_2^*$$
$$\tilde{x}_3 = (x_3^+)^* - (x_3^-)^*$$

Outline

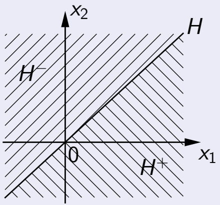
Convex geometry

Hyperplane

The set $H = \left\{x \in \mathbb{R}^n : a^{\mathrm{T}}x = b\right\}$ with $a \in \mathbb{R}^n, a \neq 0, b \in \mathbb{R}$ is called *hyperplane* in $\mathbb{R}^n$. The boundary of the closed half-spaces

$$H^- = \left\{x \in \mathbb{R}^n : a^{\mathrm{T}}x \leq b\right\}$$
$$H^+ = \left\{x \in \mathbb{R}^n : a^{\mathrm{T}}x \geq b\right\}$$

is the *supporting hyperplane H*



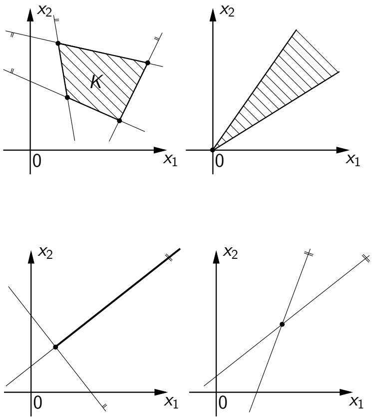
Convex geometry

Polyhedra and polytopes

A *polyhedron* in $\mathbb{R}^n$ is the intersections of a *finite and strictily positive* number of half-spaces in $\mathbb{R}^n$.

- If K is a polyhedron, $\exists A, b$ of suitable dimensions such that $K = \{x \in \mathbb{R}^n : Ax \leq b\}$.

- If K is bounded, it is called *polytope*.



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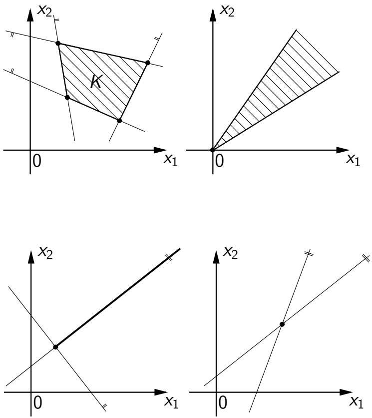

- A polytope is a closed and convex set

Convex geometry

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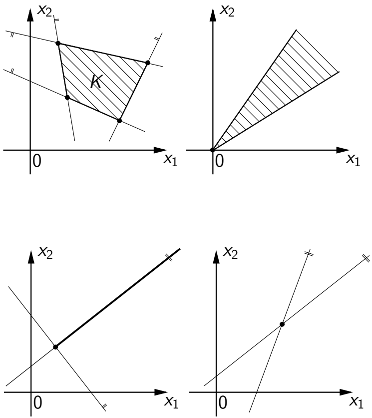

- A polytope is a closed and convex set
- The feasible region of an LP problem is a polyhedron

Convex geometry

Remarks

The pair (A, b) defining the polyhedron $K = \{x \in \mathbb{R}^n : Ax \leq b\}$ is not unique.

Convex geometry

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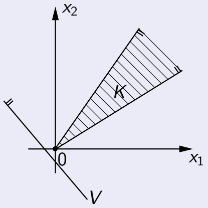
- $(\alpha A, \alpha b)$, $\alpha > 0$ defines K
- A constraint in $Ax \leq b$ is *redundant* if K does not change when removed. If redundant constraints are added to or removed form those defining K, one gets a new pair $(\tilde{A}, \tilde{b})$ that still defines K



Convex geometry

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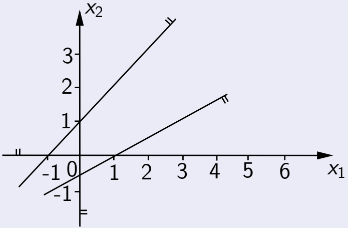

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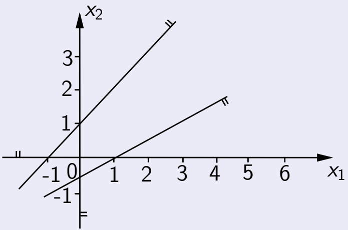

... it is also a polytope

Convex geometry

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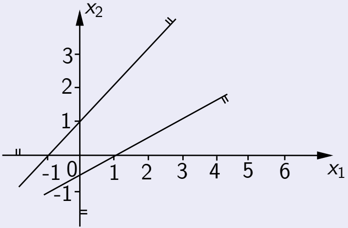


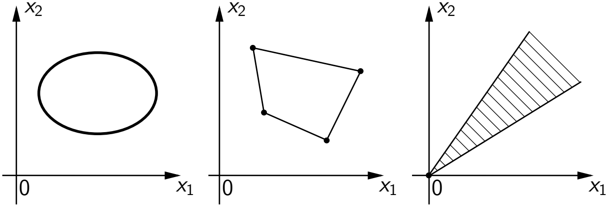
... it is also a polytope

- $\mathbb{R}^n$ is not a polyhedron

Convex geometry

Extreme points

Let $S \subseteq \mathbb{R}^n$ be a convex set. A point $z \in S$ is called *extreme point* if there are not two points $x, y \in S$ different from z, such that z belongs to the segment $\overline{xy}$.

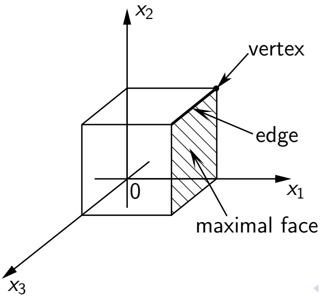


Convex geometry

Definition

Let $K \subset \mathbb{R}^n$ be a polyhedron. Then

- its extreme points are called *vertices*
- the intersection of K with one or more supporting hyperplanes is called *face*
- faces of dimension 1 are called *edges*. Faces of dimension $n-1$ are called *facets* or maximal faces.



Convex geometry

Theorem

A polyhedron has a finite number[a] of vertices.

[a] It can be zero.

Convex geometry

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Representations of a polytope

Definition

The point $z \in \mathbb{R}^n$ is a *convex combination* of k points $x_1, x_2, \ldots, x_k$ if $\exists \lambda_1, \lambda_2, \ldots, \lambda_k \geq 0$ verifying $\sum_{i=1}^{k} \lambda_i = 1$ and such that

$$z = \sum_{i=1}^{k} \lambda_k x_k \qquad (27)$$

A segment $\overline{xy}$ is the set of the convex combinations of x and y.

Convex geometry

Minkowski-Weyl theorem

Let P be a polytope. Then, a point $x \in P$ is a convex combination of the vertices of P

Convex geometry

Minkowski-Weyl theorem

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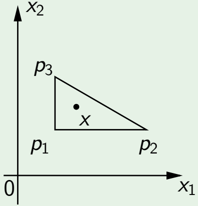
Example



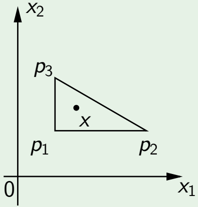
All points x of the triangle can be written as $x = \sum_{i=1}^{3} \lambda_i p_i$ for suitable $\lambda_i \geq 0$ such that $\lambda_1 + \lambda_2 + \lambda_3 = 1$

Convex geometry

Minkowski-Weyl theorem

Let P be a polytope. Then, a point $x \in P$ is a convex combination of the vertices of P

Example



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Remark

The theorem does not hold for generic polyhedra (think about a cone ...)

Outline

The graphical solution for two-variable LP problems

The feasible region and optimal solution of LP problems with only two variables $x = [x_1, x_2]^{\mathrm{T}}$ can be represented graphically.

Isocost lines

Given a level $\alpha \in \mathbb{R}$ the level surface of the cost is

$$C_\alpha \left[c^{\mathrm{T}} x\right] = \left\{x \in \mathbb{R}^2 : c^{\mathrm{T}} x = \alpha\right\}.$$

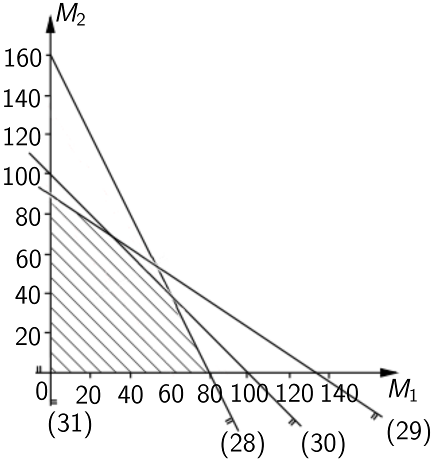
For different values of α one gets parallel lines called *isocost lines*

Example 1

Product mix

$$\max_{M_1,M_2} \; 30M_1 + 20M_2$$

$$8M_1 + 4M_2 \leq 640 \qquad (28)$$
$$4M_1 + 6M_2 \leq 540 \qquad (29)$$
$$M_1 + M_2 \leq 100 \qquad (30)$$
$$M_1, M_2 \geq 0. \qquad (31)$$



Feasible region = hatched area

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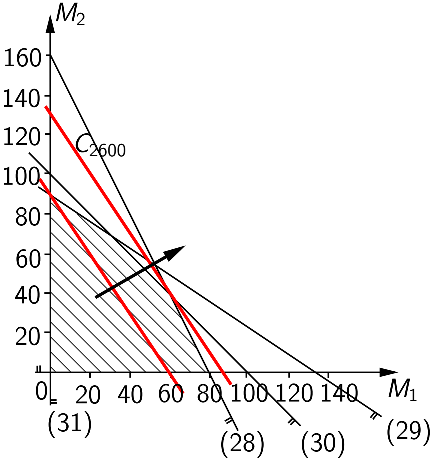

Isocost lines: $C_\alpha\,[30M_1 + 20M_2]: \; M_2 = \frac{\alpha}{20} - \frac{30}{20}M_1$

E.g. $\alpha = 1800 \; \rightarrow$ line passing through $(0, 90)$ and $(60, 0)$

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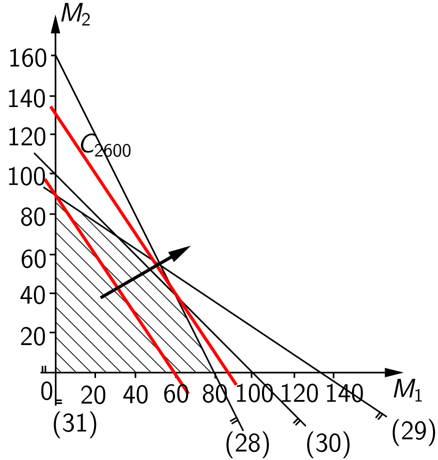

As α increases, isocost lines move in the arrow direction

Example 1

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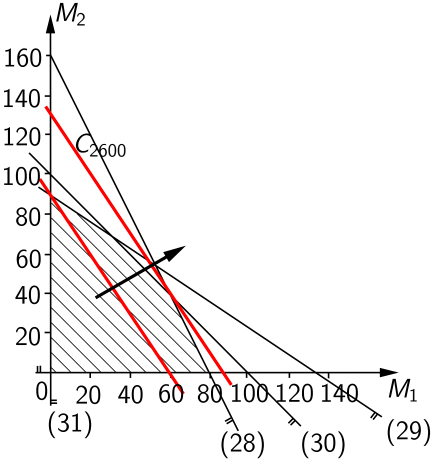


The optimal solution is $(60, 40)$ and it is given by C_{2600}: for greater values of α, the isocost line does not intersect the feasible region.

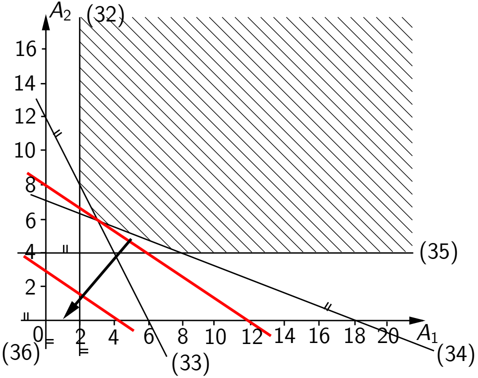
The optimal solution is a vertex of the feasible region

Example 2

Diet problem

$$\min_{A_1, A_2} 20A_1 + 30A_2$$

$$
\begin{aligned}
A_1 &\geq 2 & (32)\\
2A_1 + A_2 &\geq 12 & (33)\\
2A_1 + 5A_2 &\geq 36 & (34)\\
A_2 &\geq 4 & (35)\\
A_1, A_2 &\geq 0. & (36)
\end{aligned}
$$



Feasible region = hatched area

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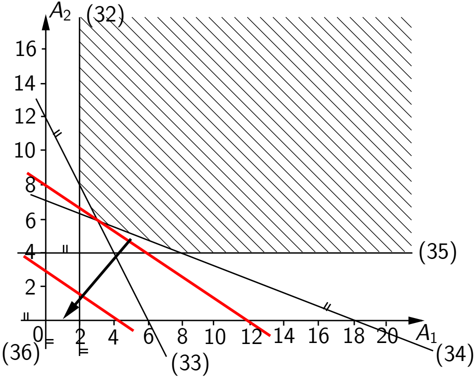
$$A_1, A_2 \geq 0. \quad (36)$$



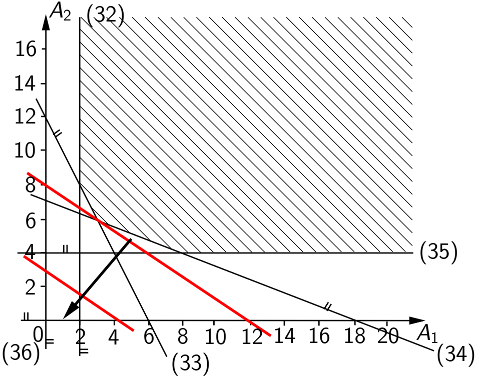
Isocost lines: $C_\alpha\left[20A_1 + 30A_2\right]: \; A_2 = -\frac{20}{30}A_1 + \frac{\alpha}{30}$

Example 2

Diet problem

$$\min_{A_1,A_2} 20A_1 + 30A_2$$

$$\begin{aligned} A_1 &\geq 2 && (32) \\ 2A_1 + A_2 &\geq 12 && (33) \\ 2A_1 + 5A_2 &\geq 36 && (34) \\ A_2 &\geq 4 && (35) \\ A_1, A_2 &\geq 0. && (36) \end{aligned}$$



As α decreases, isocost lines move in the arrow direction

Example 2

Diet problem

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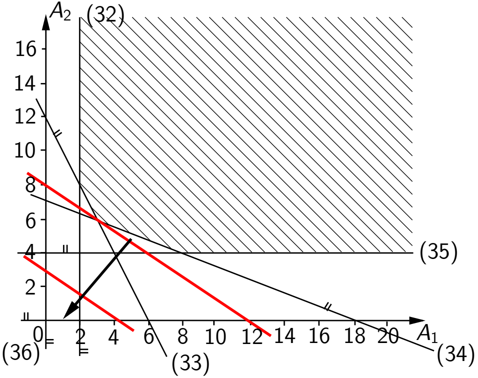


The optimal solution is $(3, 6)$ and it is given by C_{240}.

The optimal solution is a vertex of the feasible region

Example: multiple solutions

LP problem

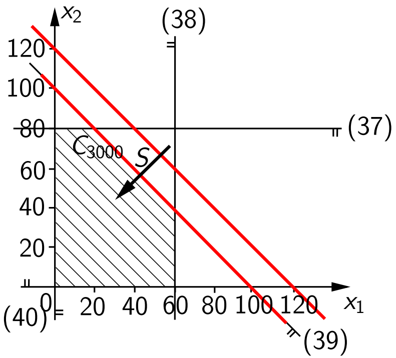
$$\max_{x_1,x_2} 30x_1 + 30x_2$$

$$x_2 \leq 80 \qquad (37)$$

$$x_1 \leq 60 \qquad (38)$$

$$x_1 + x_2 \leq 100 \qquad (39)$$

$$x_1, x_2 \geq 0. \qquad (40)$$



Feasible region = hatched area

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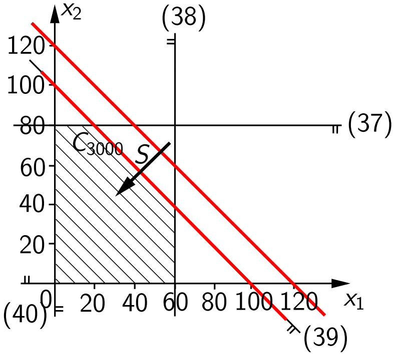

Isocost lines: $C_\alpha\,[30x_1 + 30x_2]: \; x_2 = -x_1 + \frac{\alpha}{30}$

Example: multiple solutions

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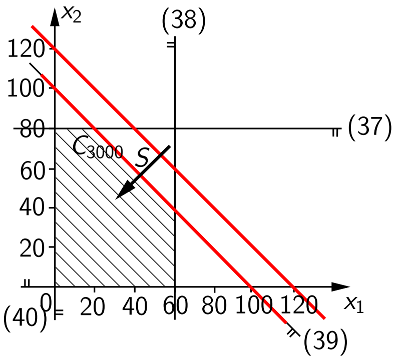

As α decreases, isocost lines move in the arrow direction

Example: multiple solutions

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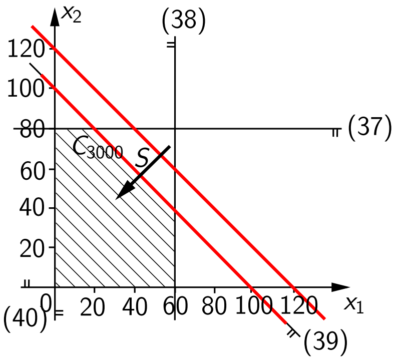


The optimal isocost line is C_{3000} and intersects the face S of the feasible region: $\forall x \in S$ is an optimal solution.

There exists at least an optimal solution that is a vertex of the feasible region

Example: unbounded problem

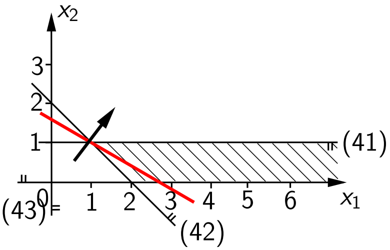
LP problem

$$\max_{x_1,x_2} x_1 + 2x_2$$

$$x_2 \leq 1 \qquad (41)$$

$$-x_1 - x_2 \leq -2 \qquad (42)$$

$$x_1, x_2 \geq 0. \qquad (43)$$



Feasible region = hatched area

Example: unbounded problem

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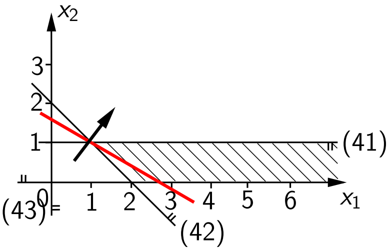

Isocost lines: $C_\alpha [x_1 + 2x_2]: \; x_2 = -\frac{1}{2}x_1 + \frac{\alpha}{2}$

As α increases, isocos lines move in the arrow direction

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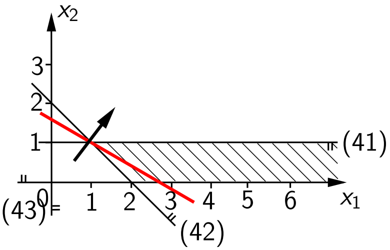

- The cost can grow unbounded: $\forall \alpha > 0$ the isocost line $C_\alpha [x_1 + 2x_2]$ intersects the feasible region.
- The LP problem is unbounded

Example: unbounded problem

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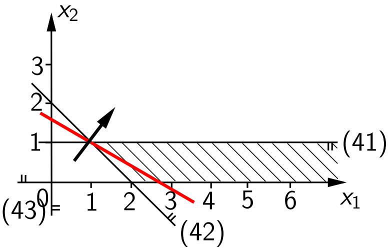


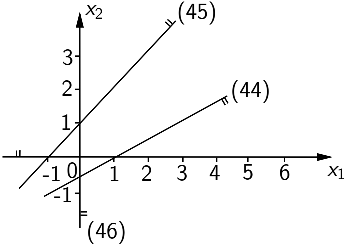
Unboundedness is often due to modeling errors.
One would *automatically* detect it, especially when the number of variables is high.

Example: infeasible problem

LP problem

$$\max_{x_1, x_2} x_1 + x_2$$

$$-x_1 + 2x_2 \leq -1 \quad (44)$$
$$x_1 - x_2 \leq -1 \quad (45)$$
$$x_1, x_2 \geq 0. \quad (46)$$



The feasibility region is empty → infeasible problem

Example: infeasible problem

LP problem

$$\max_{x_1,x_2} x_1 + x_2$$

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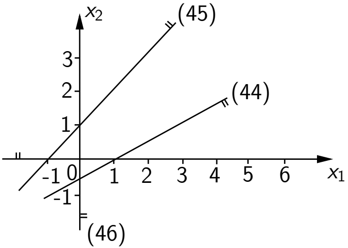

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One would *automatically* detect it, especially when the number of variables is high.

Outline

Properties of linear programming

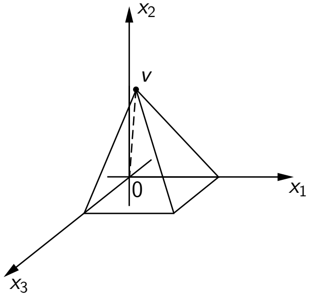
Fundamental theorem of linear programming

Let $\left\{\max c^{\mathrm{T}} x : x \in X\right\}$ be an LP problem where X is a polyhedron and $x \in \mathbb{R}^n$. If the problem is feasible, then only one of the following is true:

1. the problem is unbounded;
2. there is at least a vertex of X that is an optimal solution.

Corollary

If X is a nonempty *polytope*, then there is a vertex of X that is an optimal solution



Properties of linear programming

Proof of the corollary

- Let $x_1, x_2, \dots, x_k$ be vertices of X (their number is finite) and $z^* = \max\left\{c^{\mathrm{T}}x_i, i = 1, 2, \dots, k\right\}$ (maximum of vertex costs).
- We want to show that $\forall y \in X$ one has $c^{\mathrm{T}}y \leq z^*$.

Properties of linear programming

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- We want to show that $\forall y \in X$ one has $c^{\mathrm{T}} y \leq z^*$.
- From Minkowski-Weyl theorem:
  $y \in X \Rightarrow \exists \lambda_1, \lambda_2, \ldots, \lambda_k \geq 0 : \sum_{i=1}^{k} \lambda_i = 1$ and $y = \sum_{i=1}^{k} \lambda_i x_i$.

Properties of linear programming

Proof of the corollary

- Let $x_1, x_2, \ldots, x_k$ be vertices of X (their number is finite) and $z^* = \max\left\{c^{\mathrm{T}} x_i, i = 1, 2, \ldots, k\right\}$ (maximum of vertex costs).
- We want to show that $\forall y \in X$ one has $c^{\mathrm{T}} y \leq z^*$.
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- Then

$$c^{\mathrm{T}} y = c^{\mathrm{T}} \sum_{i=1}^{k} \lambda_i x_i = \sum_{i=1}^{k} \lambda_i \left(c^{\mathrm{T}} x_i\right) \leq \underbrace{\sum_{i=1}^{k} \lambda_i}_{=1} z^* = z^*.$$

Outline

Algorithms for solving LP problems

Vertex enumeration

If an LP problem is feasible and bounded one can

- compute all vertices $x_1, x_2, \ldots, x_k$ of X
- compute $z_i = c^{\mathrm{T}} x_i$, $i = 1, 2, \ldots, k$ (cost of vertices)

and obtain an optimal solution as $x_k : c^{\mathrm{T}} x_k = \max \{z_1, z_2, \ldots, z_k\}$

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The number of vertices of the feasible region can grow exponentially with $n \rightarrow$ computationally prohibitive

Example: let X be an hypercube

n	X	N. of vertices
2	square	$2^2 = 4$
3	cube	$2^3 = 8$
1000	ipercube	$2^{1000} \simeq 10^{300}$

If the computation of a vertex requires 10^{-9} s, when $n = 1000$ the computation time is greater than $10^{300} 10^{-9} = 10^{291}$ $s > 10^{281}$ centuries

Efficient algorithms for linear programming

Simplex algorithm

Developed by G. Dantzig in 1947

- iterative procedure for generating vertices of X *with decreasing cost (for miminization problems)* and for assessing their optimality.
    - m constraints and n variables: $\rightarrow$ maximal number of vertices $\binom{n}{m} = \frac{n!}{m!(n-m)!}$
    - in the worst case the complexity of the method is exponential in the dimension of the LP problem
    - "on average" the method is numerically robust and *much more efficient* than vertex enumeration.
- infeasibility and unboundedness of the LP problem are automatically detected

Efficient algorithms for linear programming

Interior point method

Developed by N. Karmarkar in 1984

- iterative procedure that generates a sequence of points lying in the interior of X and convergings to an optimal vertex
  - Convergence to an optimal solution requires a computational time that grows polynomially with the number of variables and constraints of the LP problem
  - for large-scale LP problems, it can be *much more efficient* than the simplex algorithm
- infeasibility and unboundedness of the LP problem are automatically detected