Systems of linear equations

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Introduction

$Ax = b$, $A \in \mathbb{R}^{m \times n}$ $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$ $\Longleftrightarrow$ Equations $\begin{cases} a_{1,\cdot}\, x = b_1 \\ \vdots \\ a_{m,\cdot}\, x = b_m \end{cases}$

row m of A

None, one or ∞ solutions. How to charachterize them?

Def. The indicator vector e^j is $e^j = [0 \dots 0\,1\,0 \dots 0]^T$.

↳ position j

If p columns of A are indicator vectors and they are all different, then p variables can be parametrized in terms of the remaining $n-p$ ones

- The p variables are called dependent. The remaining $n-p$ ones are called generic.

↳ Focus on equations involving generic variables only!

Def. The tableau form of $Ax = b$ is the matrix $[\, b \,\vdots\, A \,]$ annotated as follows

$$
\begin{array}{c|c|c}
 & & x_1 \;\; x_2 \;\cdots\; x_n \\
\hline
\vdots & & \\
x_i & b & A \\
\vdots & & \\
x_j & & \\
\vdots & &
\end{array}
\quad \leftarrow \text{column labels}
$$

labels only for rows corresponding to dependent vars. $\left\{ \begin{array}{c} \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \end{array} \right.$

Ex.

$$
A = \begin{bmatrix} 0 & 0 & 3 & 0 \\ 2 & 1 & -1 & 0 \\ 3 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}
$$

Tableau $\rightarrow$

$$
\begin{array}{cc|cccc}
 & & x_1 & x_2 & x_3 & x_4 \\
\hline
 & 2 & 0 & 0 & 3 & 0 \\
x_2 & 3 & 2 & 1 & -1 & 0 \\
 & 4 & 3 & 0 & 1 & 0 \\
x_4 & 5 & 1 & 0 & 0 & 1
\end{array}
$$

$\longrightarrow 2 = 3x_3$

$\longrightarrow 3 = 2x_1 + x_2 - x_3 \longrightarrow x_2 = -2x_1 + x_3 + 3$

$\longrightarrow 4 = 3x_1 + x_3$

$\longrightarrow 5 = x_1 + x_4 \longrightarrow x_4 = -x_1 + 5$

Parametrization of dependent variables

How to make a variable dependent?

Def. Pivot operation on the pivot element $a_{th} \neq 0$

1) Compute the auxiliary row $AUX = \frac{1}{a_{th}} [\text{row } t]$ $\to a_{t,\cdot}$
2) Replace row t of the tableau with AUX $\to$ sets $a_{th} = 1$
3) For all $i \neq t$ replace row i of the tableau with

$$[\text{row } i] - a_{ih} \cdot AUX \quad \to \text{sets } a_{ih} = 0$$

Key property. Let $\tilde{A}$ and $\tilde{b}$ be the matrices in the tableau obtained after pivoting on the element $a_{th} \neq 0$. Then, the systems $Ax = b$ and $\tilde{A}x = \tilde{b}$ have the same set of solutions

Reason: pivoting is a chain of elementary row operations.

Furthermore, $\tilde{A}$ has the structure

$$\text{row } \tilde{a}_{t_i} \longrightarrow \left[\begin{array}{c|c|c} * & \begin{matrix}0\\ \vdots \\ 0\end{matrix} & * \\ \hline * & 1 & * \\ \hline * & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & * \end{array}\right] \longrightarrow \text{in } \tilde{A}x = \tilde{b} \text{ the variable } x_h \text{ is dependent}$$

column h

Ex. $Ax = b$ with

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 0 & 0 \end{bmatrix} \quad b = \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix}.$$

Make x_2 a dependent variable

Tableau →

$$\begin{array}{c|c|cccc} & & x_1 & x_2 & x_3 & x_4 \\ \hline x_1 & -2 & 1 & -1 & 0 & 2 \\ x_3 & 3 & 0 & 1 & 1 & 1 \\ & 0 & 0 & 2 & 0 & 0 \end{array}$$

To make x_2 dependent, perform a pivot on $a_{3,2} = 2$

Pivot operation

$AUX = [0 \quad 0 \quad 1 \quad 0 \quad 0]$

$$\begin{array}{c|c|cccc} & & x_1 & x_2 & x_3 & x_4 \\ \hline x_1 & -2 & 1 & -1 & 0 & 2 \\ x_3 & 3 & 0 & 1 & 1 & 1 \\ & 0 & 0 & 2 & 0 & 0 \end{array}$$

$-(-1)\cdot AUX \rightarrow -[0 \quad 0 \quad -1 \quad 0 \quad 0]$

$-(1)\cdot AUX \rightarrow -[0 \quad 0 \quad 1 \quad 0 \quad 0]$

← replace with AUX

New Tableau →

$$\begin{array}{c|c|cccc} & & x_1 & x_2 & x_3 & x_4 \\ \hline x_1 & -2 & 1 & 0 & 0 & 2 \\ x_3 & 3 & 0 & 0 & 1 & 1 \\ x_2 & 0 & 0 & 1 & 0 & 0 \end{array}$$

$$\longrightarrow \begin{cases} x_1 = -2x_4 - 2 \\ x_3 = -x_4 + 3 \\ x_2 = 0 \end{cases}$$

→ generic variable

Rmk. Gauss-Jordan elimination: perform a sequence of pivot operations so as to obtain the largest possible number of dependent variables (m at most if $m \leq n$)

"Fat" systems of linear equations

$$Ax = b \quad A \in \mathbb{R}^{m \times n}, \; m < n$$

$$\begin{bmatrix} & A & \end{bmatrix} x = \begin{bmatrix} b \end{bmatrix}$$

How many solutions? Since $m < n$, either none or ∞

Notation: B : matrix built using m columns of A
↳ x_B : vector built using the corresponding elements of x

F : matrix built using the remaining $n-m$ columns of A
↳ x_F : vector built using the corresponding elements of x

Ex. $A = [A_1 \; A_2 \; A_3 \; A_4 \; A_5]$, $m = 2$

If $B = [A_2 \; A_4]$, then $x_B = [x_2 \; x_4]^T$.

Setting $F = [A_3 \; A_1 \; A_5]$ gives $x_F = [x_3 \; x_1 \; x_5]^T$

Rmk. $Ax = A_1 x_1 + \dots + A_n x_n = B x_B + F x_F$

Def. A **basis** of A is a matrix B such that $\det(B) \neq 0$

↳ x_B : basic variables (BVs)

x_F : nonbasic variables (NBVs)

Sufficient conditions for having solutions

If rank$(A) = m$, the system has solutions and it also has a basis. Moreover, for each possible basis B, one has

$$Ax = b \longleftrightarrow Bx_B + Fx_F = b \longleftrightarrow x_B = B^{-1}b - B^{-1}Fx_F$$

i.e. all BVs can be made dependent and all solutions can be parametrized using NBVs

Def. For a basis B, the vector $\bar{x}$ verifying $\bar{x}_B = B^{-1}b$ and $\bar{x}_F = 0$ is a Basic Solution (BS)

Is it possible to compute the function $x_B = B^{-1}b - B^{-1}Fx_F$ without inverting B? Yes, through pivot operations

Ex. $Ax = b$ with $A = \begin{bmatrix} -1 & 1 & 3 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix}$, $b = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$

Consider the basis $B = [A_1 \; A_3] = \begin{bmatrix} -1 & 3 \\ 1 & 0 \end{bmatrix}$

Tableau form

$$
\begin{array}{c|cccc}
 & x_1 & x_2 & x_3 & x_4 \\
\hline
-2 & -1 & 1 & 3 & 0 \\
4 & 1 & 1 & 0 & 2
\end{array}
\begin{array}{l}
\\
\longrightarrow \text{pivot} \\
-0 \cdot AUX
\end{array}
$$

$AUX = [-\frac{2}{3} \;\; -\frac{1}{3} \;\; \frac{1}{3} \;\; 1 \;\; 0]$

$$
\begin{array}{cc|cccc}
 & & x_1 & x_2 & x_3 & x_4 \\
\hline
x_3 & -\frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 1 & 0 \\
 & 4 & 1 & 1 & 0 & 2
\end{array}
\quad -\left(-\frac{1}{3}\right) \cdot AUX \longrightarrow -\left[-\frac{4}{3} \;\; -\frac{1}{3} \;\; -\frac{1}{3} \;\; 0 \;\; -\frac{2}{3}\right]
$$

$AUX = [4 \;\; 1 \;\; 1 \;\; 0 \;\; 2]$

$$
\begin{array}{c|c|cccc}
 & & x_1 & x_2 & x_3 & x_4 \\
\hline
x_3 & \frac{2}{3} & 0 & \frac{2}{3} & 1 & \frac{2}{3} \\
x_1 & 4 & 1 & 1 & 0 & 1
\end{array}
\longrightarrow
\begin{cases}
x_1 = 4 - x_2 - x_4 \\
x_3 = \frac{2}{3} - \frac{2}{3} x_2 - \frac{2}{3} x_4
\end{cases}
$$

$B^{-1}b$ $\qquad$ $B^{-1}F$

up to a permutation of the rows since $x_B = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}$

↳ $B^{-1}b = \begin{bmatrix} 4 \\ \frac{2}{3} \end{bmatrix}$. If $x_F = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix}$, $B^{-1}F = \begin{bmatrix} 1 & 1 \\ \frac{2}{3} & \frac{2}{3} \end{bmatrix}$

Def. A tableau is in the canonical form w.r.t. the basis B if all entries of x_B are dependant variables.

If one obtains a new basis $\bar{B}$ changing a single column of B (and then a single entry of x_B so that a single NBV becomes basic), the function $x_{\bar{B}} = \bar{B}^{-1}b - \bar{B}^{-1}\bar{F}x_{\bar{F}}$ can be computed through a single pivot operation.

Ex. $Ax = b$ with $A = \begin{bmatrix} -\frac{1}{2} & -1 & 1 & 0 \\ \frac{1}{2} & 1 & 0 & 1 \end{bmatrix}$, $b = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$

Starting basis $B = [A_3, A_4] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $x_B = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$, $x_F = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $F = [A_1, A_2]$

$x_B = B^{-1}b - B^{-1}F x_F$ can be read from

$$
\begin{array}{c|c|cccc}
 & & x_1 & x_2 & x_3 & x_4 \\
\hline
x_3 & -2 & -\frac{1}{2} & -1 & 1 & 0 \\
x_4 & -4 & \frac{1}{2} & 1 & 0 & 1
\end{array}
\qquad \underbrace{\begin{array}{c} -2 \\ -4 \end{array}}_{B^{-1}b} \qquad \underbrace{\begin{bmatrix} -\frac{1}{2} & -1 \\ \frac{1}{2} & 1 \end{bmatrix}}_{B^{-1}F}
$$

New basis $\bar{B} = [A_3, A_2] \to$ a pivot on the element in the column "x_2" and the row "x_4" will switch the BV x_4 with the NBV x_2

↳ Jargon: "x_2 enters the basis" and "x_4 leaves the basis"

$$
\begin{array}{c|c|cccc}
 & & x_1 & x_2 & x_3 & x_4 \\
\hline
x_3 & -2 & -\frac{1}{2} & -1 & 1 & 0 \\
x_4 & -4 & \frac{1}{2} & 1 & 0 & 1
\end{array}
\quad -(-1)\cdot AUX
\qquad \Longrightarrow \qquad
\begin{array}{c|c|cccc}
 & & x_1 & x_2 & x_3 & x_4 \\
\hline
x_3 & -6 & 0 & 0 & 1 & 1 \\
x_2 & -4 & \frac{1}{2} & 1 & 0 & 1
\end{array}
$$

$AUX = [-4 \quad \frac{1}{2} \quad 1 \quad 0 \quad 1]$

↳ $\bar{B}^{-1}b$

If $x_F = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix}$, $\bar{B}^{-1}\bar{F} = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 1 \end{bmatrix}$