Sliding Mode Control Handout

Advanced Automation and Control Course

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Outline

Introduction

Historical Background

- The origins of **feedback control** date back to the ancient world, with the advent of level control, water clocks, and pneumatics/hydraulics systems.
- From the 17th century on-wards, systems were designed for temperature control in furnaces, the mechanical control of mills, and the regulation of steam engines.
- It was only during the 19th century that it became clear that **feedback systems were prone to instability or oscillatory behaviors**.

Historical Background

This was particularly true for **Relay-based control systems**:

- A relay is an electrically operated switch.
- The first relays were used in long distance telegraph circuits as amplifiers (a simple relay was included in the original 1840 telegraph patent of Samuel Morse).
- Relays were used extensively in early control systems to perform logical operations. In particular, they could implement **on-off control actions**: these can be regarded as primordial sliding mode (or better, variable structure) control strategies.

A theory was needed!

Historical Background

- Spurred by servo and communications engineering developments of the 30s, the coherent body of theory known as **classical control** emerged during and just after WWII in the US, UK and elsewhere.
- In the 50s and 60s, an alternative approach to dynamic modelling was developed in the Soviet Union based on the works by Poincaré and Lyapunov. Information was gradually disseminated, and **state-space or modern control techniques** rapidly developed.
- But only at the end of the 70s, with the first publications in English by **Vadim I. Utkin** (Ph.D. 1964, Institute for Control Sciences, Moscow, Russia), a theory of relay-based control was disclosed. It was the beginning of **Sliding Mode Control Theory**.

Introduction: Basic Concepts in Sliding Mode Control

The Basic Terms

Consider a generic dynamical system S described by its **state equation**

$$\dot{x}(t) = f(x(t), u(t), t)$$

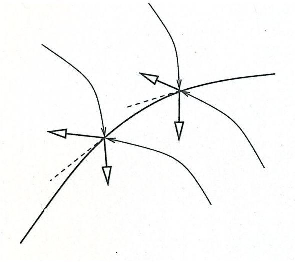
with $x(t_0) = x_0$, and $t,\ t_0 \in [0,\ +\infty)$

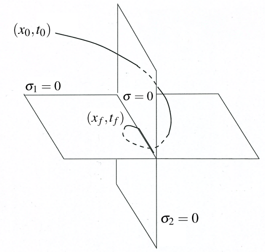
- $x(t) \in \mathbb{R}^n$ is the **system state**
- $u(t) \in \mathbb{R}^m$ is the system input (i.e. the **control input**)

Consider a function of the system state $\sigma(x(t)) \in \mathbb{R}^m$ and the associated manifold $\sigma(x(t)) = 0$ (0 null vector of dimension m)

- $\sigma(x(t))$ is the **sliding variable**
- $\sigma(x(t)) = 0$ is the **sliding manifold**

The Concept of Sliding mode





- The **sliding manifold** is a subspace of the system state space having dimension $n - m$.
- It can be a single surface or be given by the intersection of several surfaces.
- When the state trajectory continuously crosses the **sliding manifold**, since in its vicinity the state motion is always directed towards the manifold, a **sliding mode** is enforced.

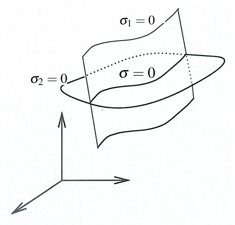
The Design Ingredients

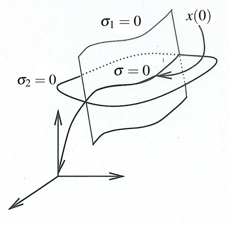
Two elements need to be “designed”:

- **The sliding manifold**: it is designed so that the system in sliding mode evolves in the desired way (e.g. it results in being linearized and its state is asymptotically regulated to zero, or it satisfies some optimality requirement, etc.).
- **The control law**: it has to be chosen in order to enforce a sliding mode.

An important design requirement

The sliding mode needs to be enforced in a finite time!





The Equivalent System and its Properties

The system in sliding mode

It has two interesting properties:

- **Order reduction**: the system in sliding mode changes its order from n to $n - m$.
- **Invariance property**: it is insensitive to "matched uncertainties" (i.e. uncertain terms affecting the system on the control channel).

The reduced order state equation describing the system in sliding mode is called **equivalent system**.

Its dynamics can be assigned by suitably designing the sliding manifold.

Two Simple Examples to Illustrate Some Facts

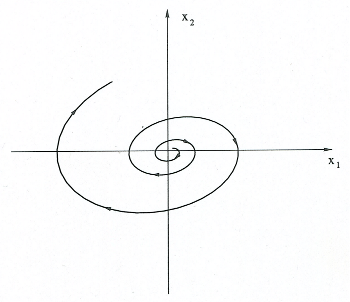
EXAMPLE 1: Consider an **unstable** second order system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

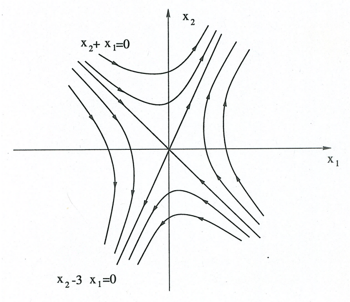
Design the **control input** as $u = -3x_1$ or $u = 2x_1$

Example 1 (cont'ed)

The controlled system has an **unstable focus** if $u = -3x_1$, and a **saddle point** if $u = 2x_1$.



$u = -3x_1$



$u = 2x_1$

Example 1 (cont'ed)

Now select a **sliding manifold**: $\sigma = c_1 x_1 + x_2 = 0, \quad c_1 > 0$, and design the **control input** as a combination of the two previous control laws:

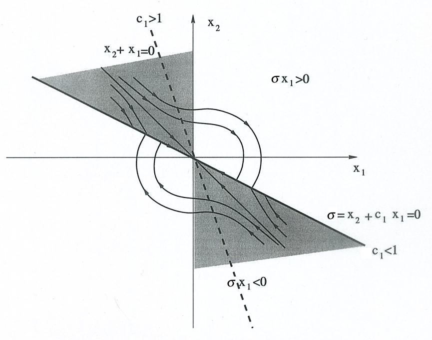
$$u = k(\sigma, x_1) x_1, \quad k(\sigma, x_1) = \begin{cases} -3 & \sigma x_1 > 0 \\ 2 & \sigma x_1 < 0 \end{cases}$$

The control law is a variable structure control law!

$\rightarrow$ **The controlled system becomes a Variable Structure System (VSS)**

Example 1 (cont'ed)

The combination of the two (non stabilizing) control laws ensures the convergence of the system state to the origin.
The origin becomes an asymptotically stable equilibrium point of the controlled system.



Note: only if $c_1 < 1$ a sliding mode is enforced!

Example 2

EXAMPLE 2: Consider a **double integrator**

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad |u| \leq 1$$

Design the **sliding manifold**: $\sigma = c_1 x_1 + x_2 = 0, \quad c_1 > 0$

Note

If $\sigma = 0$ in a finite time t_r (**reaching time**) and $\sigma = 0, \forall\, t \geq t_r$ (**sliding mode**), then the dynamics of the **equivalent system in sliding mode** is of reduced order:

$$\dot{x}_1 + c_1 x_1 = 0 \quad \rightarrow \quad x_1(t) = x(t_r) e^{-c_1 (t - t_r)}$$

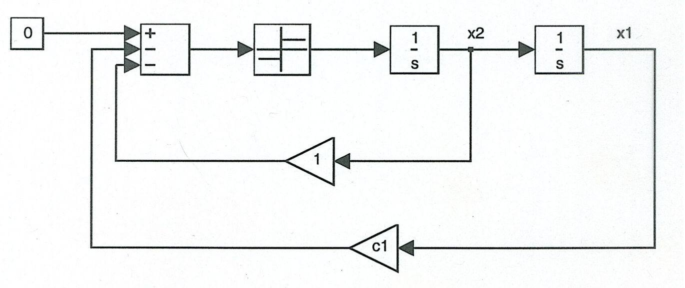
It can be "assigned" by choosing c_1!

Example 2 (cont'ed)

Design the **control input**:

$$u = \begin{cases} -1 & \sigma > 0 \\ 1 & \sigma < 0 \end{cases}$$

Note: it is discontinuos on $\sigma(x) = 0$!

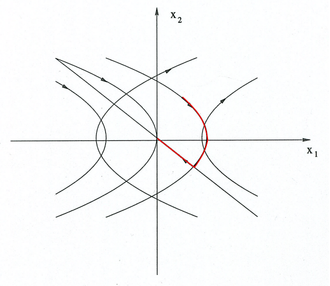


Simulink closed-loop scheme

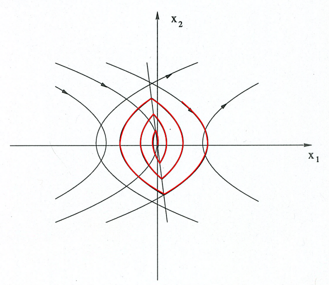
Example 2 (cont'ed)

The controlled system evolution differs depending on the value of $c_1 > 0$ (having set the control amplitude to 1!)

- If c_1 is "small", the state trajectories, following a parabola arc, reach the line $\sigma = 0$ and slide towards the origin (**sliding mode**).
- If c_1 is "large", the state trajectories follow a sequence of parabola arcs closer and closer to the origin but no sliding mode is generated.



Small c_1



Large c_1

Example 2 (cont'ed)

Now perturb the double integrator with an **uncertain bounded term**

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{d(x_1, x_2, t)} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad |u| \leq U$$

Design the **sliding manifold**: $\sigma = c_1 x_1 + x_2 = 0, \quad c_1 > 0$

Note

If $\sigma = 0$ in a finite time t_r (**reaching time**) and $\sigma = 0, \forall\, t \geq t_r$ (**sliding mode**), then the dynamics of the **equivalent system in sliding mode** is again:

$$\dot{x}_1 + c_1 x_1 = 0 \quad \rightarrow \quad x_1(t) = x(t_r) e^{-c_1 (t - t_r)}$$

The uncertain term does not affect the system in sliding mode!

Lesson Learnt

- A variable structure control making the system become a **variable structure system** can have a stabilizing effect.
- The **sliding mode enforcement** depends on the choice of the control law (correct sizing taking into account the sliding variable definition and the initial conditions).
- The system in sliding mode is of **reduced order**.
- The system dynamics in sliding mode **can be arbitrarily assigned**.
- The system in sliding mode has a nice **robustness property**.

Types of Variable Structure Control Laws

Types of Variable Structure Control Laws

Relay (or Relè) Control

$$u_i(t) = \begin{cases} u_i^+(x,t) & \sigma_i(x) > 0 \\ u_i^-(x,t) & \sigma_i(x) < 0 \end{cases}$$

$i = 1, \ldots, m$, where $\sigma_i(x) = 0$ is the i-th surface defining the **sliding manifold**

$$\sigma(x) = [\sigma_1(x), \ldots, \sigma_m(x)]^T = 0$$

The design phase consists in designing the sliding variable $\sigma(x)$ and the smooth functions u_i^+ and u_i^-.

Types of Variable Structure Control Laws

State Feedback Control with Switching Gains

$$u = \Psi(x)x(t)$$

with $\Psi = [\psi_{ij}(x)] \in \mathbb{R}^{m \times n}$, for instance,

$$\psi_{ij} = \begin{cases} \alpha_{ij} & \sigma_i(x)x_j > 0 \\ \beta_{ij} & \sigma_i(x)x_j < 0, \quad i = 1, \ldots, m, \; j = 1, \ldots, n \end{cases}$$

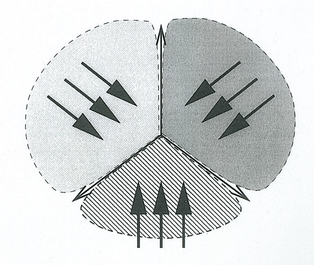
Unit Vector Control

$$u = K \frac{\sigma(x)}{\|\sigma(x)\|}$$

Types of Variable Structure Control Laws

Control Based on a Simplex of Vectors

In the multi-input case the variable structure control philosophy can also be implemented by designing a set of $m+1$ control vectors forming a simplex in $\mathbb{R}^m$.The controlled system switches from one to another of $m+1$ different structures.



G. Bartolini and A. Ferrara, "Multi-input sliding-mode control of a class of uncertain nonlinear systems", IEEE Trans. Automat. Contr., vol. 41, pp.1662-1666, 1996

Types of Systems

Canonical Forms

Assumption: The system is **nonlinear with respect to the state variable** and **linear with respect to the control variable** (i.e. affine in the control input)

The **proof of the existence of a sliding mode** and the **design of the variable structure control law** are simplified if the considered nonlinear system is expressed in one of the following canonical forms.

1. Reduced Form

- The state vector $x(t)$ can be split into two vectors: $x_1 \in \mathbb{R}^{n-m}$ and $x_2 \in \mathbb{R}^m$.
- Matrix $B(x,t) = [0 \; B^*]^T$, with $B^* \in \mathbb{R}^{m \times m}$, is not singular.

$$\begin{cases} \dot{x}_1 = A_1(x,t) \\ \dot{x}_2 = A_2(x,t) + B^* u \end{cases}$$

Canonical Forms

2. Controllability Form

- The system is split into m subsystems (m is the number of control inputs), each of them in **Brunovsky canonical form**.
- Consider $x = [x_1 \ \ \dots x_m]^T$, with $dim(x_i) = n_i$, $\Sigma_{i=1}^m n_i = n$.
- The final system is $\dot{x} = Ax + f(x) + b(x)u$ with $A = diag(A_i)$ and

$$\dot{x}_i = A_i x_i + f_i(x) + b_i(x)u, \quad i = 1, \dots, m$$

$$A_i = \begin{bmatrix} 0 & I_{n_i - 1} \\ 0 & 0 \end{bmatrix}, \quad dim(A_i) = n_i \times n_i$$

$$f_i(x) = \begin{bmatrix} 0 \\ \vdots \\ f_{i0}(x) \end{bmatrix}, \quad dim(f_i) = n_i$$

$$b_i(x) = \begin{bmatrix} 0 \\ \vdots \\ b_{i0}(x) \end{bmatrix}, \quad dim(b_i) = n_i \times m$$

Canonical Forms

3. Decoupled Input-Output Form

- In the SISO case: consider a nonlinear system with output $y = c(x, u)$. The **system relative degree** is the order of the time derivative of y in which the control u first explicitly appears.
- In the MIMO case ($y \in \mathbb{R}^m$): given the generic output y_i, the **associated relative degree** r_i is the order of the time derivative of y_i in which at least one of the control variables u_i first explicitly appears. The **system relative degree** is $r = r_1 + \cdots + r_m$.
- If $r = n$, the system

$$\begin{cases} \dot{x} = A(x) + B(x)u \\ y = c(x) \end{cases}$$

can be represented as the set of m **decoupled** differential equations

$$y_i^{(r_i)} = f_i(y_1, \ldots, y_1^{(r_1-1)}, \ldots, y_m, \ldots, y_m^{(r_m-1)}) + g_i(\ldots)u_i$$

Canonical Forms

4. Normal Form

- Under certain assumptions, if $r < n$, the following transformation is possible: let $z_{i,j}$ be a vector including the output y_i and its derivatives up to order $j = r_i - 1$ (for $i = 1, \dots, m$, r **external variables**).
- Consider the variables $\eta_k,\ k = 1,\ \dots, n - r$, called **internal variables**, mutually independent and independent of $z_{i,j}$.
- The transformed system is of the following type

$$\begin{cases} \dot{z}_{i,j} = z_{i,j+1} \quad j = 1, \dots, r_i - 1 \\ \dot{z}_{i,r_i} = \alpha_i(z,\eta) + \sum_{k=1}^{m} \beta_{i,k}(z,\eta) u_k, \quad i = 1, \dots, m \\ \dot{\eta} = \gamma(z,\eta), \quad dim(\eta) = n - r \end{cases}$$

Elements of Design

Design of the Sliding Manifold

The sliding manifold $\sigma(x) = 0$ can be a nonlinear function of x.

Linear sliding manifolds are generally preferred, i.e.:

$$\sigma(x) = Cx(t) = 0$$

with $C \in \mathbb{R}^{m \times n}$.

An issue to clarify

How is the sliding manifold designed when one of the previously described **canonical forms** is selected?

Design of $\sigma(x)$ for the Canonical Forms

Reduced Form

- The state vector is split into x_1 and x_2. For the sake of simplicity,

$$\sigma(x) = \sigma(x_1, x_2) = [C_1 \;\; C_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

  with C_2 being not singular.

- In **sliding mode**

$$\begin{cases} x_2 = C_2^{-1} C_1 x_1 \\ \dot{x}_1 = A_1(x,t) = A_1(x_1, -C_2^{-1} C_1 x_1, t) \end{cases}$$

Design of $\sigma(x)$ for the Canonical Forms

Reduced Form

- If A_1 is **linear**,

$$\dot{x}_1 = A_1(x,t) = A_{11}x_1 + A_{12}x_2$$

then, the **reduced order dynamics** is

$$\dot{x}_1 = [A_{11} - A_{12}C_2^{-1}C_1]x_1 = [A_{11} + A_{12}F]x_1$$

If (A_{11}, A_{12}) is controllable, it is possible to choose F so that the reduced order system has the desired dynamics in sliding mode (eigenvalues assignment, optimality, etc.).

- Given F, since $F = -C_2^{-1}C_1$, one can easily design

$$\sigma(x) = [C_1 \ C_2][x_1 \ x_2]^T$$

Design of $\sigma(x)$ for the Canonical Forms

Controllability Form

- The system is split into m subsystems
- One has

$$\sigma_i = c_i^T x_i, \quad i = 1, \ldots, m$$

- Major requirement: to select c_i for each subsystem so that **the overall controlled system in sliding mode is asymptotically stable**.
- The dynamics of each subsystem in sliding mode can be assigned by selecting the the components of c_i.
- Each equivalent subsystem has order $n_i - 1$ and has a canonical controllability (Brunovsky) form. The corresponding **characteristic polynomial** has the components of c_i as coefficients.

Design of $\sigma(x)$ for the Canonical Forms

Decoupled Input-Output Form

- It is analogous to the case of the controllability form.

Normal Form

- The so-called "zero-dynamics", obtained by posing equal to zero the outputs and their derivatives (i.e. the external variables),

$$\begin{cases} z = 0 \\ \dot{\eta} = \gamma(0, \eta) \end{cases}$$

  has to be asymptotically stable.

- A typical choice of the sliding variable components is:
  $\sigma_i = c_i^T z_i, \quad i = 1, \ldots, m.$

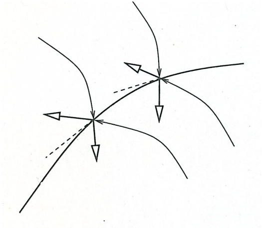
- c_i is chosen as in the controllability form so as to assign the dynamics to the equivalent reduced order subsystem having z_i has state vector.

Existence of the Sliding Mode

After the design of the sliding manifold it is necessary to guarantee the **existence of the sliding mode**.

Note:

A sliding mode exists if in a vicinity of the sliding manifold $\sigma(x) = 0$ the vector tangent to the state trajectory of the controlled system is always directed towards the sliding manifold.



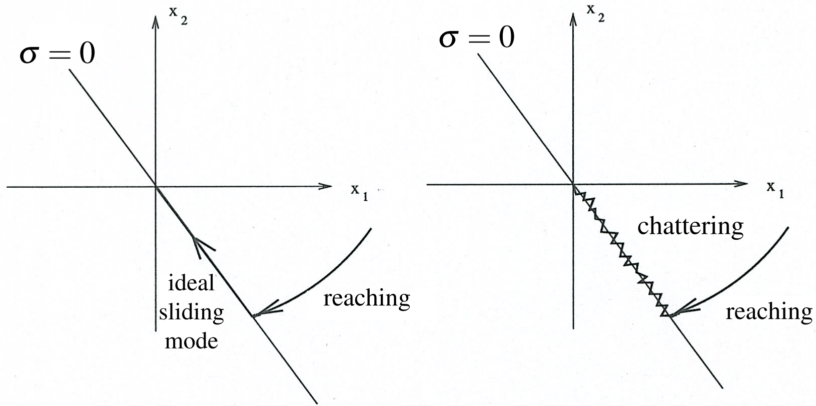
Existence of the Sliding Mode: Ideal and Practical Sliding Mode

- An **ideal sliding mode** is enforced if the state trajectory of the controlled system is such that

$$\sigma(x(t)) = 0, \quad t \geq t_r, \quad t_r \ \textbf{reaching time}$$

- To have an ideal sliding mode, the control variable has to switch at **infinity frequency**. This is not possible in practice.
- The system trajectory oscillates around the sliding manifold (**chattering**).
- The state evolution in a vicinity of the sliding manifold is called **practical sliding mode**.

Existence of the Sliding Mode: Ideal and Practical Sliding Mode



The Existence Problem

Note:

The existence problem is a stability problem!

- The existence of a sliding mode requires that the state trajectories, at least from a neighborhood of $s(x) = 0$ (**attraction region**), tend towards the sliding manifold.
- The attraction domain can coincide with the whole state space (**globally reachable sliding mode**).
- The existence of a sliding mode can be proved by using a **Lyapunov function** $V(x)$.

Existence Condition

- In case of **single input** systems:

$$V(x) = \frac{1}{2}\sigma^2(x)$$

Note that $\dot{\sigma}$ depends on the control variable (then it is discontinuous on $\sigma = 0!$).

- The control variable has to be chosen so that in the attraction region:

$$\dot{V}(x) = \sigma\dot{\sigma} < 0 \quad \textbf{reachability condition}$$

To Prove the Finite Time Convergence

An alternative way to express the reachability condition is the following (for the sake of simplicity the single input case is considered):

$$\sigma\dot{\sigma} \leq -\gamma|\sigma| \quad \eta - \textbf{reachability condition}$$

with $\gamma > 0$, that is

$$\dot{V}(x) \leq -\gamma' \sqrt{V(x)}$$

In this case it is possible to find an upper bound for the **reaching time** t_r, by integrating the η-reachability condition between $t = 0$ (or t_0) and $t = t_r$:

$$\sigma(t_r) - \sigma(0) = 0 - \sigma(0) \leq -\gamma(t_r - 0)$$

$$t_r \leq \frac{|\sigma(0)|}{\gamma}$$

Useful References

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