Advanced automation and control

Ex. 1

$$\max\ -x_1 + 2\delta_2$$
$$-x_1 \le -\delta_1 + 0.5$$
$$-\delta_1 + \delta_2 \le 0$$
$$\delta_1, \delta_2 \in \{0,1\}$$
$$x_1 \le 0$$

Replace x_1 with $-\xi_1$ $\rightarrow$ $\max\ \xi_1 + 2\delta_2$

$\boxed{\xi_1 = -x_1}$

$$\xi_1 \le -\delta_1 + 0.5$$
$$-\delta_1 + \delta_2 \le 0$$
$$\delta_1, \delta_2 \in \{0,1\}$$
$$\xi_1 \ge 0$$

Relax the problem and add slack variables

$$\max\ \xi_1 + 2\delta_2$$
$$\xi_1 + \delta_1 + s_1 = 0.5$$
$$-\delta_1 + \delta_2 + s_2 = 0$$
$$\delta_1 + s_3 = 1$$
$$\delta_2 + s_4 = 1$$
$$\xi_1, \delta_1, \delta_2, s_1, s_2, s_3, s_4 \ge 0$$

Note: for this exam only I told to skip the last 2 constraints and s_3 and s_4

Node 0

$$\max\ \xi_1 + 2\delta_2$$
$$\xi_1 + \delta_1 + s_1 = 0.5$$
$$-\delta_1 + \delta_2 + s_2 = 0$$
$$\xi_1, \delta_1, \delta_2, s_1, s_2 \ge 0$$

Phase 1 $\longrightarrow$

$$\min\ y_1 + y_2$$
$$\xi_1 + \delta_1 + s_1 + y_1 = 0.5$$
$$-\delta_1 + \delta_2 + s_2 + y_2 = 0$$
$$\xi_1, \delta_1, \delta_2, s_1, s_2, y_1, y_2 \ge 0$$

	0	ξ_1: 0	δ_1: 0	δ_2: 0	s_1: 0	s_2: 0	y_1: 1	y_2: 1
ξ_1	0.5	1	1	0	1	0	1	0
δ_2	0	0	-1	1	0	1	0	1

Ph. 1 ok $\longrightarrow$ Ph. 2

	0	ξ_1: 1	δ_1: 0	δ_2: 2	s_1: 0	s_2: 0	
ξ_1	0.5	1	1	0	1	0	= AUX
s_2	0	0	-1	1	0	1	

$-$AUX

	-0.5	ξ_1: 0	δ_1: -1	δ_2: 2	s_1: -1	s_2: 0	
ξ_1	0.5	1	1	0	1	0	
s_2	0	0	-1	1	0	1	= AUX

-2AUX

	-0.5	ξ_1: 0	δ_1: 1	δ_2: 0	s_1: -1	s_2: -2	
ξ_1	0.5	1	1	0	1	0	= AUX
δ_2	0	0	-1	1	0	1	+ AUX

$-$AUX

	-1	ξ_1: -1	δ_1: 0	δ_2: 0	s_1: -2	s_2: -2
δ_1	0.5	1	1	0	1	0
δ_2	0.5	1	0	1	1	1

Optimal cost: 1

$\xi_1 = 0$ $\quad \delta_1 = 0.5$ $\quad \delta_2 = 0.5$ $\quad s_1 = 0$ $\quad s_2 = 0$

$\downarrow$

$x_1 = 0$

The solution to the relaxation of node 0 is not feasible (nor optimal) for the original MILP since δ_1 & δ_2 are not integer

Node 1 $\delta_1 = 0$

$\max\ \xi_1 + 2\delta_2$ $\xrightarrow{\text{Ph. 1}}$ $\min\ y_1 + y_2$

$\xi_1 + s_1 = 0.5$ ... $\xi_1 + s_1 + y_1 = 0.5$

$\delta_2 + s_2 = 0$ ... $\delta_2 + s_2 + y_2 = 0$

$\xi_1, \delta_2, s_1, s_2 \geq 0$ ... $\xi_1, \delta_2, s_1, s_2, y_1, y_2 \geq 0$

	ξ_1	δ_2	s_1	s_2	y_1	y_2
0	0	0	0	0	1	1
0.5	1	0	1	0	1	0
0	0	1	0	1	0	1

Ph. 1 ok $\xrightarrow{\text{Ph. 2}}$

		ξ_1	δ_2	s_1	s_2	
	0	1	2	0	0	− AUX
s_1	0.5	1	0	1	0	= AUX
s_2	0	0	1	0	1	

		ξ_1	δ_2	s_1	s_2	
	−0.5	0	2	−1	0	− 2AUX
ξ_1	0.5	1	0	1	0	
s_2	0	0	1	0	1	= AUX

		ξ_1	δ_2	s_1	s_2
	−0.5	0	0	−1	−2
ξ_1	0.5	1	0	1	0
δ_2	0	0	1	0	1

→ Ph. 2 over.

Optimal cost: 0.5

$\xi_1 = 0.5$ $\delta_2 = 0$ $s_1 = 0$ $s_2 = 0$

$\downarrow x_1 = -0.5$

This is feasible for the original MILP. The best feasible solution found so far → incumbent solution. In order to see if it is optimal I need to explore further the tree

Node 1 $\delta_1 = 1$

$\max\ \xi_1 + 2\delta_2$ ... $\max\ \xi_1 + 2\delta_2$

$\xi_1 + 1 + s_1 = 0.5$ → $-\xi_1 - s_1 = \boxed{0.5}$

$-1 + \delta_2 + s_2 = 0$ ... $\delta_2 + s_2 = \boxed{1}$

$\xi_1, \delta_2, s_1, s_2 \geq 0$ ... $\xi_1, \delta_2, s_1, s_2 \geq 0$

Remember! To start the Tableau I need to have the constant vector ≥ 0

Ph. 1 $\min\ y_1 + y_2$

$-\xi_1 - s_1 + y_1 = 0.5$

$\delta_2 + s_2 + y_2 = 1$

$\xi_1, \delta_2, s_1, s_2, y_1, y_2 \geq 0$

→

		ξ_1	δ_2	s_1	s_2	y_1	y_2
	0	0	0	0	0	1	1
	0.5	−1	0	−1	0	1	0
δ_2	1	0	1	0	1	0	1

We don't have enough indicator vectors to start the pivoting.

Following the "Additional rules" we subtract all rows to the first one

		ξ_1	δ_2	s_1	s_2	y_1	y_2	
	−1.5	1	−1	1	−1	0	0	+ AUX
y_1	0.5	−1	0	−1	0	1	0	
y_2	1	0	1	0	1	0	1	= AUX

		ξ_1	δ_2	s_1	s_2	y_1	y_2
	−0.5	1	0	1	0	0	1
y_1	0.5	−1	0	−1	0	1	0
δ_2	1	0	1	0	1	0	1

Ph. 1 is over (all coeff ≥ 0)

But infeasible cause the cost $\neq 0$

In conclusion the optimal cost is 0.5 and the optimal value of the variables is $x_1 = -0.5$ $\delta_1 = 0$ and $\delta_2 = 0$.

Ex. 2

$\boxed{\delta_M = 1} \rightarrow 442 \rightarrow$ only Ross and Mellow can play as defender

$\boxed{\delta_M = 0} \rightarrow 352 \rightarrow$ only Mellow and Becker can play as midfielder

Since Mellow is not willing to be a substitute player and since no player can play the entire 90 min $\rightarrow$ Mellow will be the starting player cause is the only one that can play both as M and D.

Then we define

x_M, = # minutes played by Mellow

$$\max \ \delta_M \left(0.9 \cdot x_M + (90 - x_M) \cdot 1 \right) + (1 - \delta_M)\left(0.5\, x_M + (90 - x_M) \cdot 1.1 \right)$$

$$x_M \le 65$$

$$x_M \geq (90 - 55)(1 - \delta_M) + (90 - 70)\,\delta_M$$

Mellow should play a number of minutes enough to be replaced by a player able to play the remaining time needed to reach the 90 min

$$\max \ \underline{0.9\,\delta_M x_M} + \underline{90\,\delta_M} - \underline{\delta_M x_M} + \underline{0.5\,x_M} + 99 - \underline{1.1\,x_M} - \underline{0.5\,\delta_M x_M}$$
$$\underline{-99\,\delta_M} + \underline{1.1\,\delta_M x_M}$$

$$\max \ 0.5\,\delta_M x_M - 9\,\delta_M - 0.6\,x_M + 99$$

$$x_M \le 65$$
$$x_M \geq 35 - 35\,\delta_M + 20\,\delta_M \rightarrow +15\,\delta_M + x_M \geq 35$$
$$x_M \geq 0$$
$$\delta_M \in \{0, 1\}$$

We need to introduce z_M to replace $\delta_M x_M$

lower bound when $\delta_M = 1$

$$\downarrow$$
$$20 \le x_M \le 65$$

$$z_M \le \delta_M \cdot 65$$
$$z_M \geq \delta_M \cdot 20$$
$$z_M \le x_M - 20(1 - \delta_M)$$
$$z_M \geq x_M - 65(1 - \delta_M)$$

$$\begin{cases} z_M - 65\,\delta_M \le 0 \\ -z_M + 20\,\delta_M \le 0 \\ z_M - x_M + 20\,\delta_M \le -20 \\ -z_M + x_M + 65\,\delta_M \le 65 \end{cases}$$

$$\max \; 0.5 z_M - 9\delta_M - 0.6 x_M \; (+99)$$ ← Drop when solve MILP, add to the optimal cost afterwards

$$
\begin{aligned}
& x_M \le 65 \\
& -15\delta_M - x_M \le -35 \\
& z_M - 65\delta_M \le 0 \\
& -z_M + 20\delta_M \le 0 \\
& z_M - x_M - 20\delta_M \le -20 \\
& -z_M + x_M + 65\delta_M \le 65 \\
& x_M \ge 0 \\
& \delta_M \in \{0,1\}
\end{aligned}
$$

Then, one can write the problem as $\max\; c^T x$ subj. to $Ax \le b$, $x \ge 0$, $\delta_M \in \{0,1\}$ ← do it yourself

with $x = \begin{bmatrix} x_M \\ \delta_M \\ z_M \end{bmatrix}$

Extra question: I have 2 options $\delta_M = 1$ or $\delta_M = 0$

Since Mellow is worse than Ross as a defender and Parker as midfielder I will let him play the minimum amount of time:

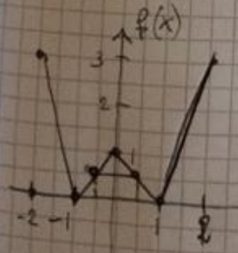
$\delta_M = 1 \rightarrow$ I choose $x_M = 20 \rightarrow$ cost is $0.9 \cdot 20 + 70 = 88$

$\delta_M = 0 \rightarrow$ I choose $x_M = 35 \rightarrow$ cost is $0.5 \cdot 35 + 55 \cdot 1.1 = 17.5 + 60.5 = 78$

Therefore the best choice is 4-4-2 with Mellow playing 20 min and Ross 70.

Ex. 3 $\quad \min f(x)$

$$
\begin{aligned}
& \sin(x) \le 1 \\
& x^2 \le 16 \\
& \cos(x) \ge -1
\end{aligned}
\qquad
f(x) = \begin{cases} -3x-3, & \text{if } x \le -1 \\ x+1, & \text{if } -1 \le x \le 0 \\ -x+1, & \text{if } 0 \le x \le 1 \\ 3x-3, & \text{if } 1 \le x \end{cases}
$$



→ $f(x)$ is not convex cause if, for example, I pick $x_1 = -0.5$, $x_2 = 0.5$ then the segment from $(x_1, f(x_1))$ to $(x_2, f(x_2))$ is below the function and this contradicts the definition of convex function.

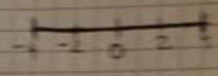
Feasibility set

First of all $\sin(x) \leq 1$, $\cos(x) \geq -1$ ALWAYS HOLD cause $\cos(x)$ and $\sin(x)$ are ALWAYS BETWEEN -1 and 1

The only active constraint is $x^2 \leq 16$

Since x is 1-dimensional, this corresponds to having $-4 \leq x \leq 4$ [partly illegible]

Therefore the feasible set is a segment

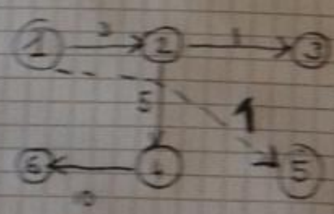


Being a segment, the set is convex

Since the cost function is not convex, the optimization problem is not convex

Ex. 4

V	L	$\ell(1)$	$\ell(2)$	$\ell(3)$	$\ell(4)$	$\ell(5)$	$\ell(6)$
/	$\emptyset$	0	∞	∞	∞	∞	∞
1	1	0	3 (1)	∞	∞	∞	∞
2	{1,2}	0	3	4 (2)	8	∞	∞
3	{1,2,3}	0	3	4	8 (2)	∞	∞
4	{1,2,3,4}	0	3	4	8	∞	13 (4)
5	{1,2,3,4,5}	0	3	4	8	∞	13



This is the shortest path from 1 to all other nodes

If I can add one edge to make 5 reachable from 1 with minimum cost I would add it directly from 1 to 5!

Going back to the original graph + this extra edge, I can clearly remove edge (6,2) without compromising the reaching from node 1 to any other node. This way, I removed the edge with the highest cost (11).