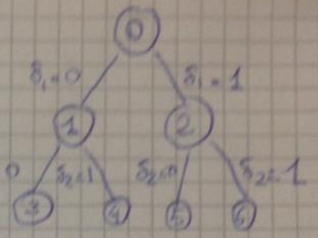
1

$$\max \ 4x_1 - 2x_2 + 7\delta_1 - \delta_2$$
$$x_1 + x_2 + \delta_1 \leq 0.5$$
$$\delta_1, \delta_2 \in \{0,1\}$$
$$x_1, x_2 \geq 0$$



$\downarrow$ Relax

$$\max \ 4x_1 - 2x_2 + 7\delta_1 - \delta_2$$
$$x_1 + x_2 + \delta_1 + s_1 = 0.5$$
$$\delta_1 + s_2 = 1$$
$$\delta_2 + s_3 = 1$$
$$x_1, x_2, s_1, s_2, s_3, \delta_1, \delta_2 \geq 0$$

$\xrightarrow{\text{Phase 1}}$

$$\min \ y_1 + y_2 + y_3$$
$$x_1 + x_2 + \delta_1 + s_1 + y_1 = 0.5$$
$$\delta_1 + s_2 + y_2 = 1$$
$$\delta_2 + s_3 + y_3 = 1$$
$$x_1, x_2, s_1, s_2, s_3, \delta_1, \delta_2, y_1, y_2, y_3 \geq 0$$

		x_1	x_2	s_1	s_2	s_3	δ_1	δ_2	y_1	y_2	y_3
	0	0	0	0	0	0	0	0	1	1	1
s_1	0.5	1	1	1	0	0	1	0	1	0	0
s_2	1	0	0	0	1	0	1	0	0	1	0
s_3	1	0	0	0	0	1	0	1	0	0	1

OK Phase 1 $\longrightarrow$ Phase 2

2

Phase 2

		x_1	x_2	s_1	s_2	s_3	δ_1	δ_2
	0	4	-2	0	0	0	7	-1
s_1	0.5	1	1	1	0	0	1	0
s_2	1	0	0	0	1	0	1	0
s_3	1	0	0	0	0	1	0	1

$- [2 \mid 4 \ 4 \ 4 \ 0 \ 0 \ 4 \ 0]$

AUX

$[0.5 \mid 1 \ 1 \ 1 \ 0 \ 0 \ 1 \ 0]$

		x_1	x_2	s_1	s_2	s_3	δ_1	δ_2
	-2	0	-6	-4	0	0	3	-1
x_1	0.5	1	1	1	0	0	1	0
s_2	1	0	0	0	1	0	1	0
s_3	1	0	0	0	0	1	0	1

$- [3/2 \mid 3 \ 3 \ 3 \ 0 \ 0 \ 3 \ 0]$

$- [0.5 \mid 1 \ 1 \ 1 \ 0 \ 0 \ 1 \ 0]$

AUX

$[0.5 \mid 1 \ 1 \ 1 \ 0 \ 0 \ 1 \ 0]$

(3)

	x_1	x_2	s_1	s_2	s_3	δ_1	δ_2
$-7/2$	-3	-9	-7	0	0	0	-1
δ_1 1/2	1	1	1	0	0	1	0
s_2 1/2	-1	-1	-1	1	0	0	0
s_3 1	0	0	0	0	1	0	1

OK!

$$\bar{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/2 \\ 1 \\ 1/2 \\ 0 \end{bmatrix} \begin{matrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \\ \delta_1 \\ \delta_2 \end{matrix} \longrightarrow \begin{matrix} x_1 = 0 \\ x_2 = 0 \\ \delta_1 = 1/2 \\ \delta_2 = 0 \end{matrix}$$

This is not feasible for the original MILP since $\delta_1 = 0.5$ but provides an upper (since we are maximizing) bound on the optimal cost : $\frac{7}{2} = 3.5$

We need to explore the next level of the tree

Node 1 : $\delta_1 = 0$ (we remove δ_1 from the list of variables and replace it with its value (0))

$$\max \; 4x_1 - 2x_2 - \delta_2$$
$$x_1 + x_2 \leq 0.5$$
$$\delta_2 \in \{0,1\}$$
$$x_1, x_2, \geq 0$$

↓ Relax node 1

$$\max \; 4x_1 - 2x_2 - \delta_2$$
$$x_1 + x_2 + s_1 = 0.5$$
$$\delta_2 + s_2 = 1$$
$$x_1, x_2, \delta_2, s_1, s_2 \geq 0$$

Phase 1

$$\min \; y_1 + y_2$$
$$x_1 + x_2 + s_1 + y_1 = 0.5$$
$$\delta_2 + s_2 + y_2 = 1$$
$$x_1, x_2, s_1, s_2, \delta_2, y_1, y_2 \geq 0$$

(4)

	x_1	x_2	s_1	s_2	δ_2	y_1	y_2
0	0	0	0	0	0	1	1
x_1 0.5	1	1	1	0	0	1	0
s_2 1	0	0	0	1	1	0	1

Phase 1 OK → Phase 2

↓ Phase 2

	x_1	x_2	s_1	s_2	δ_2
0	4	-2	0	0	-1
s_1 0.5	1	1	1	0	0
s_2 1	0	0	0	1	1

$-[2 \mid 4 \; 4 \; 4 \; 0 \; 0]$

aux

$[0.5 \mid 1 \; 1 \; 1 \; 0 \; 0]$

	x_1	x_2	s_1	s_2	δ_2
-2	0	-6	-4	0	-1
x_1 0.5	1	1	1	0	0
s_2 1	0	0	0	1	1

OK!

$$\bar{x} = \begin{bmatrix} 0.5 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{matrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ \delta_2 \end{matrix} \longrightarrow \begin{matrix} x_1 = 0 \\ x_2 = 0 \\ \delta_2 = 0 \\ (\delta_1 = 0) \end{matrix}$$

Great! since $\delta_2 = 0$ (and $\delta_1 = 0$) we obtained a feasible solution to the original MILP.

Since this is the best feasible solution we got so far we set it to be our

incumbent solution with cost 2

An incumbent solution a priori provides a lower bound for a maximization problem

(5)

Having found a feasible solution at node 1 implies node 3 and 4 do not need to be explored. Still, we need to check, the other branch, starting from node 2

<u>Node 2</u> $\delta_1 = 1$ (replace variable δ_1 with 1 everywhere)

$$\max\ 4x_1 - 2x_2 - \delta_2 \ (+7)$$
$$x_1 + x_2 \leq -0.5 \quad (-1+0.5)$$
$$\delta_2 \in \{0,1\}$$
$$x_1, x_2 \geq 0$$

← Note: for solving the tableau we have to exclude the constant. Once the tableau is solved, we add +7 a posteriori

→ <u>Not feasible!</u> since both $x_1, x_2 \geq 0$

If one started Node 2 using the slack variables, it would have been difficult to see the infeasibility. However we would have detected it through Phase 1. Let's try!

$$\max\ 4x_1 - 2x_2 - \delta_2$$
$$x_1 + x_2 + s_1 = -0.5$$
$$\delta_2 + s_2 = 1$$
$$x_1, x_2, s_1, s_2, \delta_2 \geq 0$$

$\xrightarrow{\text{Phase 1}}$

$$\min\ y_1 + y_2$$
$$-x_1 - x_2 - s_1 \overset{+y_1}{=} \underline{0.5}$$
$$\delta_2 + s_2 \overset{+y_2}{=} 1$$
$$x_1, x_2, s_1, s_2, \delta_2 \geq 0$$
$$y_1, y_2$$

Note: since Node 2 is not feasible when relaxed → the optimal solution of the MILP is the incumbent solution,

$$x = \begin{bmatrix} 0.5 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} x_1 \\ x_2 \\ s_1 \\ \delta_2 \end{matrix}$$ with cost 2

(6)

0	x_1 0	x_2 0	s_1 0	s_2 0	δ_2 0	y_1 1	y_2 1
[redacted] 0.5	-1	-1	-1	0	0	1	0
δ_2 1	0	0	0	1	1	0	1

→ we can subtract to the 1st eqn the 2nd and 3rd

-1.5	1	1	1	-1	-1	0	0
0.5	-1	-1	-1	0	0	1	0
1	0	0	0	1	1	0	1

+ AUX

it's a minimization

Aux $[1 \,\vdots\, 0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 1]$

	x_1	x_2	s_1	s_2	δ_2	y_1	y_2
-0.5	1	1	1	0	0	0	1
y_1 0.5	-1	-1	-1	0	0	1	0
s_2 1	0	0	0	1	1	0	1

Optimal Solution 0.5 → $\neq 0$ which means it was necessary to give to either y_1 or y_2 a value different from 0 to guarantee feasibility

[redacted]

$$\bar{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0.5 \\ 0 \end{bmatrix} \begin{matrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ \delta_2 \\ y_1 \\ y_2 \end{matrix}$$

→ $\boxed{y_1 = 0.5}$

↑ to guarantee feasibility y_1 has to be > 0