Advanced Automation and Control

Optimization Part

Surname............................ Name............................

Thursday 26th January, 2023

Exercise 1

1. Rewrite the optimization problem in **standard form**. Depict the tree associated to the MILP.

$$\min \ \delta_2 + 3x_1$$
$$0.5\delta_1 + \delta_2 \geq 1$$
$$x_1 \geq \delta_2$$
$$\delta_1, \delta_2 \in \{0,1\}$$
$$x_1 \geq 0$$

$$\min \ \delta_2 + 3x_1$$
$$0.5\delta_1 + \delta_2 - s_1 = 1$$
$$-x_1 + \delta_2 + s_2 = 0$$
$$\delta_1, \delta_2 \in \{0,1\}$$
$$x_1, s_1, s_2 \geq 0$$

2. Write down the relaxed problem at node 0 **and** the optimization problem for Phase 1.

$\downarrow$

$$\min \ \delta_2 + 3x_1$$
$$0.5\delta_1 + \delta_2 - s_1 = 1$$
$$-x_1 + \delta_2 + s_2 = 0$$
$$\delta_1 + s_3 = 1$$
$$\delta_2 + s_4 = 1$$
$$x_1, \delta_1, \delta_2, s_1, s_2, s_3, s_4 \geq 0$$

$\downarrow$

$$\min \ y_1 + y_2 + y_3 + y_4$$
$$0.5\delta_1 + \delta_2 - s_1 + y_1 = 1$$
$$-x_1 + \delta_2 + s_2 + y_2 = 0$$
$$\delta_1 + s_3 + y_3 = 1$$
$$\delta_2 + s_4 + y_4 = 1$$
$$x_1, \delta_1, \delta_2, s_1, s_2, s_3, s_4, y_1, y_2, y_3, y_4 \geq 0$$

3. Simplex algorithm at node 0

(a) Solve Phase 1 Min!

Subtract all rows to 1st one

			x_1	δ_1	δ_2	s_1	s_2	s_3	s_4	y_1	y_2	y_3	y_4
		0	0	0	0	0	0	0	0	1	1	1	1
		1	0	0.5	1	−1	0	0	0	1	0	0	0
	s_2	0	−1	0	1	0	1	0	0	0	1	0	0
	s_3	1	0	1	0	0	0	1	0	0	0	1	0
	s_4	1	0	0	1	0	0	0	1	0	0	0	1

			x_1	δ_1	δ_2	s_1	s_2	s_3	s_4	y_1	y_2	y_3	y_4	
		−3	1	−1.5	−3	1	−1	−1	−1	0	0	0	0	+1.5AUX = [1.5; 0 1.5 0 0 0 1.5 0 0 0 1.5 0]
1/0.5 = 2	y_1	1	0	0.5	1	−1	0	0	0	1	0	0	0	−0.5AUX = [−0.5; 0 −0.5 0 0 0 −0.5 0 0 0 −0.5 0]
	y_2	0	−1	0	1	0	1	0	0	0	1	0	0	
1/1 = 1	y_3	1	0	1	0	0	0	1	0	0	0	1	0	= AUX
	y_4	1	0	0	1	0	0	0	1	0	0	0	1	

			x_1	δ_1	δ_2	s_1	s_2	s_3	s_4	y_1	y_2	y_3	y_4	
		−1.5	1	0	−3	1	−1	0.5	−1	0	0	1.5	0	+3AUX = [0; −3 0 3 0 3 0 0 0 3 0 0]
0.5/1 = 0.5	y_1	0.5	0	0	1	−1	0	−0.5	0	1	0	−0.5	0	−AUX = [0| 1 0 −1 0 −1 0 0 0 −1 0 0]
0/1 = 0	y_2	0	−1	0	1	0	1	0	0	0	1	0	0	= AUX
	δ_1	1	0	1	0	0	0	1	0	0	0	1	0	
1/1	y_4	1	0	0	1	0	0	0	1	0	0	0	1	−AUX = [0; 1 0 −1 0 −1 0 0 0 −1 0 0]

		x_1	δ_1	δ_2	s_1	s_2	s_3	s_4	y_1	y_2	y_3	y_4	
	-1.5	-2	0	0	1	2	0.5	-1	0	3	1.5	0	+ 2AUX = [1; 2 0 0 -2 -2 -1 0 2 -2 -1 0]
0.5/1 = 0.5	y_1 0.5	1	0	0	-1	-1	-0.5	0	1	-1	-0.5	0	= AUX
	δ_2 0	-1	0	1	0	1	0	0	0	1	0	0	+ AUX = [0.5; 1 0 0 -1 -1 -0.5 0 1 -1 -0.5 0]
	δ_1 1	0	1	0	0	0	1	0	0	0	1	0	
1/1	y_4 1	1	0	0	0	-1	0	1	0	-1	0	1	- AUX = [-0.5; -1 0 0 1 1 0.5 0 -1 1 0.5 0]

		x_1	δ_1	δ_2	s_1	s_2	s_3	s_4	y_1	y_2	y_3	y_4	
	-0.5	0	0	0	-1	0	-0.5	-1	2	1	0.5	0	+ AUX = [0.5; 0 0 0 1 0 0.5 1 -1 0 0.5 1]
x_1	0.5	1	0	0	-1	-1	-0.5	0	1	-1	-0.5	0	+ AUX = [0.5; 0 0 0 1 0 0.5 1 -1 0 0.5 1]
δ_2	0.5	0	0	1	-1	0	-0.5	0	1	0	-0.5	0	+ AUX = [0.5; 0 0 0 1 0 0.5 1 -1 0 0.5 1]
δ_1	1	0	1	0	0	0	1	0	0	0	1	0	
y_4	0.5	0	0	0	1	0	0.5	1	-1	0	0.5	1	= AUX

		x_1	δ_1	δ_2	s_1	s_2	s_3	s_4	y_1	y_2	y_3	y_4	
	0	0	0	0	0	0	0	0	1	1	1	1	≥ 0
x_1	1	1	0	0	0	-1	0	1	0	-1	0	1	
δ_2	1	0	0	1	0	0	0	1	0	0	0	1	
δ_1	1	0	1	0	0	0	1	0	0	0	1	0	
s_1	0.5	0	0	0	1	0	0.5	1	-1	0	0.5	1	

Phase 1 is over!
The problem is feasible

(b) Simplex algorithm **Phase 2**

Min $\delta_2 + 3x_1$

Not all rows have indic. vect.

		x_1	δ_1	δ_2	s_1	s_2	s_3	s_4	
	0	3	0	1	0	0	0	0	$-3AUX = [-3\ -3\ 0\ 0\ 0\ +3\ 0\ -3]$
	1	1	0	0	0	-1	0	1	$= AUX$
	1	0	0	1	0	0	0	1	
δ_1	1	0	1	0	0	0	1	0	
s_1	0.5	0	0	0	1	0	0.5	1	

$\rightarrow$

		x_1	δ_1	δ_2	s_1	s_2	s_3	s_4	
	-3	0	0	1	0	3	0	-3	$-AUX = [-1;\ 0\ 0\ -1\ 0\ 0\ 0\ -1]$
x_1	1	1	0	0	0	-1	0	1	
	1	0	0	1	0	0	0	1	$= AUX$
δ_1	1	0	1	0	0	0	1	0	
s_1	0.5	0	0	0	1	0	0.5	1	

			x_1	δ_1	δ_2	s_1	s_2	s_3	s_4	
		-4	0	0	0	0	3	0	-4	$+4AUX$
1/1	x_1	1	1	0	0	0	-1	0	1	$-AUX$
1/1	δ_2	1	0	0	1	0	0	0	1	$-AUX$
	δ_1	1	0	1	0	0	0	1	0	
0.5/1	s_1	0.5	0	0	0	1	0	0.5	1	$= AUX$

		x_1	δ_1	δ_2	s_1	s_2	s_3	s_4
	-2	0	0	0	4	3	2	0
x_1	0.5	1	0	0	-1	-1	-0.5	0
δ_2	0.5	0	0	1	-1	0	-0.5	0
s_1	1	0	1	0	0	0	1	0
s_4	0.5	0	0	0	1	0	0.5	1

i. The optimization problem is feasible, unbounded or infeasible? Please motivate the answer.

According to Ph1, the relaxed problem is feasible. However the relaxation did not provide a feasible solution for the original problem since at the end of Ph.2 we have $\delta_2 = 0.5 \rightarrow$ not binary $\rightarrow$ Not a feasible solution for the original problem only a lower bound

ii. The optimal cost of the relaxed problem is: $+2$

(c) After examining node 0, can we provide a conclusion for the original MILP (Yes, No, Why)?

No. We need to further explore the tree

If Yes:

i. The original MILP is feasible, unbounded or infeasible? Please motivate the answer.

ii. the optimal cost for the original MILP is:

Exercise 2

90 employees (80K€ per employee) produce 100 units/year sold at 1000€ each

Train at most 20 employees → 10K€ per employee → 0 units 1st year – 130 units/year after
Salary up 10% from 1st year → 88 K€/year – Robots (at most 10) 1.3M€/robot → 200 units/year
If more 10700 units/year → expansion → 1M€ – Fire employees (at most 20) → if >10 100K€ legal

1. Indicate the initial set of chosen optimization variables and their meaning. Do not include here the auxiliary variables required to resolve bilinearities or "if" conditions.

Maximize profit for the next 10 years.

Optimization variables : x_T = # trained employees

x_F = # fired employees

x_R = # robots

$\delta_F = 0$ no exceed $x_F > 10$
$\quad\;\; 1 \quad x_F > 10$

$\delta_F = 1 \iff x_F > 10$

$\delta_E = 0$ no expansion
$\quad\;\; 1$ expansion

$\delta_E = 1 \iff (90 - x_F - x_T)\cdot 100 + x_T \cdot 130 + x_R \cdot 200 > 10700$

2. Please report all the steps required to obtain the MILP formulation of the problem

$$\max 10\Big[\big(90 - x_F - x_T\big)\cdot 100 \cdot 1000 - \big(90 - x_F - x_T\big)\cdot 80 \cdot 10^3 - x_T \cdot 88 \cdot 10^3 + x_R \cdot 200 \cdot 1000\Big] + 9\, x_T \cdot 130 \cdot 1000 - \delta_E \cdot 10^6 - x_R \cdot 1.3 \cdot 10^6 - \delta_F \cdot 100 \cdot 10^3$$

- $\delta_F = 1 \iff x_F > 10$

$\delta_E = 1 \iff (90 - x_F - x_T)\cdot 100 + x_T \cdot 130 + x_R \cdot 200 > 10700$

$x_T \leq 20$

$x_F \leq 20$

$x_R \leq 10$

$x_T, x_F, x_R \geq 0$

$\delta_F, \delta_E \in \{0,1\}$

$L = -10 \quad U = 10$

We need to resolve the logical constraints

$\delta_F = 1 \iff x_F > 10 \rightarrow \delta_F = 1 \iff x_F - 10 \geq \varepsilon$

$\delta_F = 1 \rightarrow \boxed{x_F - 10} \geq \varepsilon$

$\delta_F = 0 \rightarrow x_F - 10 \leq 0$

- $\boxed{x_F - 10 \geq (L - \varepsilon)(1 - \delta_F) + \varepsilon = (-10 - \varepsilon)(1 - \delta_F) + \varepsilon}$

- $\boxed{x_F - 10 \le U\delta_F = 10\delta_F}$

$$\delta_E = 1 \iff (90 - x_F - x_T)\cdot 100 + x_T \cdot 130 + x_R \cdot 200 > 10700$$

$$(90 - x_F - x_T)\cdot 100 + x_T \cdot 130 + x_R \cdot 200 - 10700 \ge \varepsilon$$

$$-x_F \cdot 100 - x_T \cdot 100 + x_T \cdot 130 + x_R \cdot 200 + 9000 - 10700 \ge \varepsilon$$

$$\boxed{-100 x_F + 30 x_T + 200 x_R - 1700} \ge \varepsilon$$

$$L = -100 \cdot \underset{\text{20 fired}}{20} + 30 \cdot \underset{\text{no trained}}{0} + 200 \cdot \underset{\text{no robots}}{0} - 1700$$

$$= -2000 - 1700 = -3700$$

$$U = -100 \cdot \underset{\text{no fired}}{0} + 30 \cdot \underset{\text{20 trained}}{20} + 200 \cdot 10 - 1700$$

$$= 0 + 600 + 2000 - 1700 = +900$$

$$\delta_E = 1 \iff -100 x_F + 30 x_T + 200 x_R - 1700 \ge \varepsilon$$

a) • $\delta_E = 1 \to -100 x_F + 30 x_T + 200 x_R - 1700 \ge \varepsilon$

b) • $\delta_E = 0 \to -100 x_F + 30 x_T + 200 x_R - 1700 \le 0$

- a) $-100 x_F + 30 x_T + 200 x_R - 1700 \ge$

$$\varepsilon + (L - \varepsilon)(1 - \delta_E) = \varepsilon + (-3700 - \varepsilon)(1 - \delta_E)$$

- b) $-100 x_F + 30 x_T + 200 x_R - 1700 \ge U\delta_E = 900\delta_E$

3. Write down the final set of optimization variables (after having resolved bilinearities etc.) and their meaning

We need to plug in those constraints in the opt. problem and finally obtain

$$\max \ c^T x$$

$$\text{subj. to} \quad Ax - b \le 0$$

$$x = \begin{bmatrix} x_F \\ x_T \\ x_R \\ \delta_E \\ \delta_F \end{bmatrix} \ge 0, \ \delta_E, \delta_F \in \{0, 1\}$$

Please write the values of c, A, b

4. Write down the final **linear** objective function

5. Write down all the constraints

The dual program will look like $\max \; -\lambda^T b - v^T e$

subj. to $\lambda \geq 0$ ← only for the inequalities

$c + A^T\lambda + D^T v = 0$

We now substitute the matrices (notice that $b = 0$)

$$\max \; -v_1$$

$$\text{subj. to} \quad \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} v_1 = 0$$

$$\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_4 \geq 0, \lambda_5 \geq 0, \lambda_6 \geq 0$$

$\max \; -v_1$

subj. to
$1 - \lambda_1 + \lambda_3 = 0$
$-1 + \lambda_1 + \lambda_2 - \lambda_4 = 0$
$1 + \lambda_5 + v_1 = 0$
$-1 - \lambda_2 - \lambda_6 + v_1 = 0$
$\lambda_1 \geq 0$
$\lambda_2 \geq 0$
$\lambda_3 \geq 0$
$\lambda_4 \geq 0$
$\lambda_5 \geq 0$
$\lambda_6 \geq 0$

points 2.5

Exercise 3 6 points

1. Depict the cost function (IN THE BOX) and indicate if it is convex (IN THE SMALL BOX and motivate the answer OUT OF THE BOX).

$$\min_x f(x)$$

subj. to $x^2 \le 4$

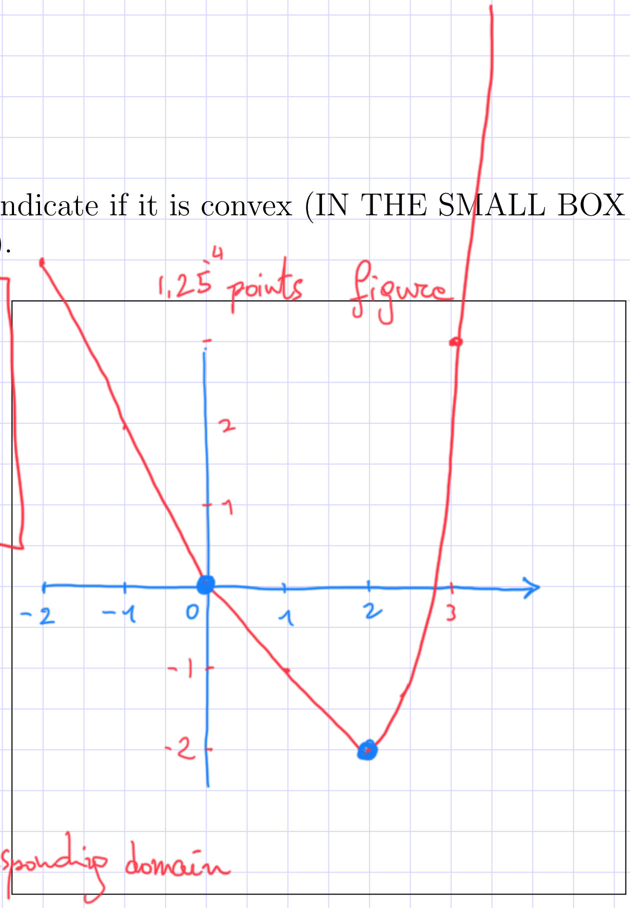
$x^2 \ge 1$

$x^3 \ge 0$

$$f(x) = \begin{cases} -2x & x<0 \\ -x & 0 \le x \le 2 \\ x^2-6 & x>2 \end{cases}$$

cost function

1,25 points figure



1,25 convexity analysis

We have 3 functions. We check if those are convex. How? We see if $\nabla^2 f(x) \ge 0$ for the corresponding domain

$$\nabla^2 f(x) = \begin{cases} \dfrac{\delta^2(-2x)}{\delta x^2} = 0 & x<0 \quad \text{since } \ge 0 \text{ it is convex (we knew that since it is a line)} \\ \dfrac{\delta^2(-x)}{\delta x^2} = 0 & 0 \le x \le 2 \rightarrow \text{convex for the same reasons} \\ \dfrac{\delta^2(x^2-6)}{\delta x^2} = 2 & x>2 \quad \text{it is } \ge 0 \rightarrow \text{it is convex} \end{cases}$$

both convex & concave

CONVEX

points 2.5

2. Depict the feasibility domain of the problem (IN THE BOX). Is the domain convex? (ANSWER YES/NO IN THE SMALL BOX and motivate the answer OUT OF THE BOX)?

We still need to evaluate what happens when changing function → at $x=0$ & $x=2$.
What we need for convexity is

$e^- \le e^+$

Let consider $x=0$

$$e^- = \frac{d}{dx} f(x)\Big|_{x=0^-} = -2$$

$$e^+ = \frac{d}{dx} f(x)\Big|_{x=0^+} = -1$$

$e^- \le e^+$ ok

Let consider $x=2$

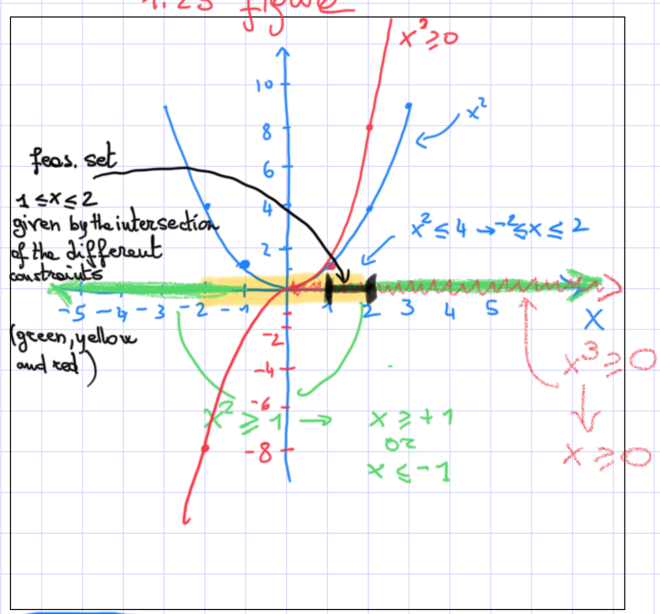
$$e^- = \frac{d}{dx} f(x)\Big|_{x=2^-} = -1$$

$$e^+ = \frac{d}{dx} f(x)\Big|_{x=2^+} = 2x\Big|_{x=2^+} = 4$$

$e^- \le e^+$ ok

For all these reasons the cost function is convex.

1.25 figure



Part 2. Feasibility set.

$x^2 \leq 4 \longrightarrow -2 \leq x \leq 2$

$x^2 \geq 1 \longrightarrow x \geq 1$ or $x \leq -1$

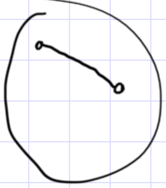
$x^3 \geq 0 \longrightarrow x \geq 0$

$\rightarrow$ Feas. set $1 \leq x \leq 2$

$\boxed{\text{Set is convex}}$

Convexity analysis of the feas. set. Since it is a segment, by the definition of convex set it is automatically convex. ✓

point 1 3. Indicate if the optimisation problem is convex (motivate the answer).



Convex set: if I take any 2 points within the set the segment is entirely contained in the set.

Since both the cost function and the feas. set are convex and we minimize, then the optimization problem is convex.

Exercise 4 (7 points)

$$\min \ x_1 - x_2 + x_3 - x_4$$

subj. to

$$x_1 \geq x_2$$
$$x_2 \leq x_4$$
$$x_1 \leq 0$$
$$x_2 \geq 0$$
$$x_3 \leq 0$$
$$x_4 \geq 0$$
$$x_3 + x_4 = 1$$

$\longrightarrow$

$$\min \ c^T x$$

subj. to $Ax - b \leq 0$

$Dx - e = 0$

$\downarrow$

$$\min \ x_1 - x_2 + x_3 - x_4$$
$$-x_1 + x_2 \leq 0$$
$$x_2 - x_4 \leq 0$$
$$x_1 \leq 0$$
$$-x_2 \leq 0$$
$$x_3 \leq 0$$
$$-x_4 \leq 0$$
$$x_3 + x_4 - 1 = 0$$

$c = ?$

$A = ?$

$b = ?$

$D = ?$

$e = ?$

$x \in \mathbb{R}^{4 \times 1}$

$c \in \mathbb{R}^{4\times 1} \to c^T \in \mathbb{R}^{1\times 4} \to c^T = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}$

$A \in \mathbb{R}^{6\times 4}$

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad b \in \mathbb{R}^{6\times 1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$D \in \mathbb{R}^{1\times 4} \quad \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix} \qquad e \in \mathbb{R}^{1\times 1} = 1$

Lagrangian function

$\lambda \in \mathbb{R}^{6\times 1}$
$\hookrightarrow \lambda^T \in \mathbb{R}^{1\times 6}$

$v \in \mathbb{R}^{1\times 1} \to v^T \in \mathbb{R}^{1\times 1}$

$$L(x, \lambda, v) = c^T x + \lambda^T \underbrace{(Ax - b)}_{\in \mathbb{R}^{6\times 1}} + v^T \underbrace{(Dx - e)}_{\in \mathbb{R}^{1\times 1}}$$

λ: multipliers for ineq. const

v: multipliers for equalities

$\lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{bmatrix} \qquad v = v_1$

$$g(\lambda) = \inf_x L(x, \lambda, v) = \inf_x c^T x + \lambda^T (Ax - b) + v^T (Dx - e)$$

$$= \inf_x c^T x + \lambda^T A x + v^T D x - \lambda^T b - v^T e$$

$$= \inf_x \underbrace{(c^T + \lambda^T A + v^T D) x}_{\in \mathbb{R}^{1\times 1}} - \lambda^T b - v^T e$$

$$= \inf_x x^T (c^T + \lambda^T A + v^T D)^T - \lambda^T b - v^T e$$

$$= \inf_x x^T (c + A^T \lambda + D v^T) - \lambda^T b - v^T e$$

$\mathbb{R}^{4\times 1}$ $\quad \mathbb{R}^{4\times 1}$

$$= \begin{cases} -\lambda^T b - v^T e & \text{if } c + A^T \lambda + D v^T = 0 \\ -\infty & \text{otherwise} \end{cases}$$

Due to lack of space it continues at the end of page 7