Solution to the MILP example

Define $x_1, x_2, \dots x_7$ as the amount of € invested in the different options. We have the following constraints

$$\sum_{i=1}^{7} x_i = 500000$$

$$x_1 \leq 150000$$

$$x_2 \leq 150000$$

$$x_3 \leq 100000$$

$$x_4 + x_5 \geq 100000$$

$$x_6 + x_7 \geq 125000$$

$$x_6 \geq 0.4 x_7$$

$$x_1 + x_4 \leq 250000$$

$$x_2 + x_5 \leq 200000$$

Apart from these constraints, we have logical constraints which require the introduction of binaries $\delta_1, \dots \delta_8$

$$\delta_1 = 1 \longleftrightarrow x_1 > 0$$

$$\delta_2 = 1 \longleftrightarrow x_2 > 0$$

$$\delta_3 = 1 \longleftrightarrow x_3 > 0$$

$$\delta_4 = 1 \longleftrightarrow x_4 > 0$$

$$\delta_5 = 1 \longleftrightarrow x_5 > 0$$

$$\delta_6 = 1 \longleftrightarrow x_6 > 0$$

$$\delta_7 = 1 \longleftrightarrow x_7 > 0$$

$$\delta_8 = 1 \longleftrightarrow x_1 \geq 50000 \longleftrightarrow x_1 - 50000 \geq 0$$

In order to translate the logical expressions into inequality constraints we need to define upper and lower bounds on the quantities $x_1, x_2, \dots x_1 - 50000$:

$0 \leq x_1 \leq 150000$, $0 \leq x_2 \leq 150000$, $0 \leq x_3 \leq 100000$, $0 \leq x_4 \leq 250000$

$0 \leq x_5 \leq 200000$, $0 \leq x_6 \leq 500000$, $0 \leq x_7 \leq 500000$, $-50000 \leq x_1 - 50000 \leq 100000$

Thanks to the bounds we can now write:

$\delta_1 = \longleftrightarrow x_1 > 0$ is equivalent to

$$\delta_1 = 1 \rightarrow x_1 > 0 \qquad x_1 \geq (0 - \varepsilon)(1 - \delta_1) + \varepsilon$$

$$\delta_1 = 0 \rightarrow x_1 \leq 0 \qquad x_1 \leq 150000\,\delta_1$$

A similar formulation is obtained for all the other logical constraints. Summarizing and bringing all optimization variables to the left hand side one has:

$\varepsilon \delta_1 - x_1 \leq 0$

$x_1 - 150000 \delta_1 \leq 0$

$\varepsilon \delta_2 - x_2 \leq 0$

$x_2 - 150000 \delta_2 \leq 0$

$\varepsilon \delta_3 - x_3 \leq 0$

$x_3 - 100000 \delta_3 \leq 0$

$\varepsilon \delta_4 - x_4 \leq 0$

$x_4 - 250000 \delta_4 \leq 0$

$\varepsilon \delta_5 - x_5 \leq 0$

$x_5 - 200000 \delta_5 \leq 0$

$\varepsilon \delta_6 - x_6 \leq 0$

$x_6 - 500'000 \delta_6 \leq 0$

$\varepsilon \delta_7 - x_7 \leq 0$

$x_7 - 500000 \delta_7 \leq 0$

$50000 \delta_8 - x_1 \leq 0$

$x_1 - (100'000 + \varepsilon) \delta_8 \leq 50000 - \varepsilon$

On top of the constraints which translate the link between $\delta_1 \ldots \delta_8$ and their validity cases, we have the last 5 investment constraints

$\sum_{i=1}^{7} \delta_i \leq 6$

$\sum_{i=1}^{7} \delta_i \geq 1$

$\delta_1 + \delta_3 \leq 1$

$\delta_4 - \delta_8 \leq 0$

$\delta_2 - \delta_5 = 0$

In order to complete the translation, define

$x = [x_1 \; x_2 \cdots x_7 \; \delta_1 \cdots \delta_7 \; \delta_8]^T$ and rewrite the constraints as $A\bar{x} = b$ where $\bar{x}$ contains x and all the necessary slack variables to translate the inequality contraints into equality constraints. (3)

Please also compute the vector c in order to express the cost function as $c^T \bar{x}$.

Finally, the MILP will be formulated as

$$\begin{aligned} \max \quad & c^T \bar{x} \\ & A\bar{x} = b \\ & \delta_1, \dots \delta_8 \in \{0,1\} \\ & x_1, \dots x_7, s_1, \dots \geq 0 \end{aligned}$$

Please compute explicitly $\bar{x}$, c, A, b