1

$$\max \ \delta_1 + \delta_2 - x_1$$
$$x_1 + \delta_2 \leq 0.7$$
$$-x_1 - \delta_2 \leq -0.3$$
$$\delta_1, \delta_2 \in \{0, 1\}$$
$$x_1 \geq 0$$

$\xrightarrow{\text{Relaxed node 0}}$

$$\max \ \delta_1 + \delta_2 - x_1$$
$$x_1 + \delta_2 + s_1 = 0.7$$
$$x_1 + \delta_2 - s_2 = 0.3$$
$$\delta_1 + s_3 = 1$$
$$\delta_2 + s_4 = 1$$
$$x_1, s_1, s_2, s_3, s_4, \delta_1, \delta_2 \geq 0$$

$\xrightarrow{\text{Phase 1}}$

$$\min \ y_1 + y_2 + y_3 + y_4$$
$$x_1 + \delta_2 + s_1 + y_1 = 0.7$$
$$x_1 + \delta_2 - s_2 + y_2 = 0.3$$
$$\delta_1 + s_3 + y_3 = 1$$
$$\delta_2 + s_4 + y_4 = 1$$
$$x_1, s_1, s_2, s_3, s_4, \delta_1, \delta_2, y_1, y_2, y_3, y_4 \geq 0$$

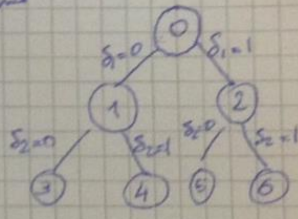
Phase 1

	x_1	s_1	s_2	s_3	s_4	δ_1	δ_2	y_1	y_2	y_3	y_4
0	0	0	0	0	0	0	0	1	1	1	1
s_1 0.7	1	1	0	0	0	0	1	1	0	0	0
0.3	1	0	-1	0	0	0	1	0	1	0	0
s_3 1	0	0	0	1	0	1	0	0	0	1	0
s_4 1	0	0	0	0	1	0	1	0	0	0	1

Subtract 3rd to 2nd

	x_1	s_1	s_2	s_3	s_4	δ_1	δ_2	y_1	y_2	y_3	y_4
0	0	0	0	0	0	0	0	1	1	1	1
s_1 0.4	0	1	1	0	0	0	0	1	-1	0	0
x_1 0.3	1	0	-1	0	0	0	1	0	1	0	0
s_3 1	0	0	0	1	0	1	0	0	0	1	0
s_4 1	0	0	0	0	1	0	1	0	0	0	1

OK! Phase 1 done



Phase 2

		x_1	s_1	s_2	s_3	s_4	δ_1	δ_2	
	0	-1	0	0	0	0	1	1	→ Add 3rd to 1st
s_1	0.4	0	1	1	0	0	0	0	
	0.3	1	0	-1	0	0	0	1	
s_3	1	0	0	0	1	0	1	0	
s_4	1	0	0	0	0	1	0	1	

		x_1	s_1	s_2	s_3	s_4	δ_1	δ_2	
	0.3	0	0	-1	0	0	1	2	- AUX
s_1	0.4	0	1	1	0	0	0	0	
x_1	0.3	1	0	-1	0	0	0	1	
s_3	[1	0	0	0	1	0	1	0]	
s_4	1	0	0	0	0	1	0	1	

		x_1	s_1	s_2	s_3	s_4	δ_1	δ_2	
	-0.7	0	0	-1	-1	0	0	2	-2AUX
s_1	0,4	0	1	1	0	0	0	0	
x_1	[0.3	1	0	-1	0	0	0	1]	
δ_1	1	0	0	0	1	0	1	0	
s_4	1	0	0	0	0	1	0	1	- AUX

		x_1	s_1	s_2	s_3	s_4	δ_1	δ_2	
	-1.3	-2	0	1	-1	0	0	0	- AUX
AUX	[0.4	0	1	1	0	0	0	0]	
	0.3	1	0	-1	0	0	0	1	+ AUX
	1	0	0	0	1	0	1	0	
	0.7	-1	0	1	0	1	0	0	- AUX

		x_1	s_1	s_2	s_3	s_4	δ_1	δ_2
	-1.7	-2	-1	0	-1	0	0	0
s_2	0.4	0	1	1	0	0	0	0
δ_2	0.7	1	1	0	0	0	0	1
δ_1	1	0	0	0	1	0	1	0
s_4	0.3	-1	-1	0	0	1	0	0

OK! →

$$x = \begin{bmatrix} 0 \\ 0 \\ 0.4 \\ 0 \\ 0.3 \\ 1 \\ 0.7 \end{bmatrix} \begin{matrix} x_1 \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ \delta_1 \\ \delta_2 \end{matrix}$$

The relaxation of node 0 gives an optimal solution which is not feasible for the MILP ($\delta_2 = 0.7$). It therefore only provides an <u>upper</u> bound for the MILP with cost

1.7

We look into node 1 ($\delta_1 = 0$). Below the relaxation of node 1 (3)

↳ max $\delta_2 - x_1$

$$x_1 + \delta_2 + s_1 = 0.7$$
$$x_1 + \delta_2 - s_2 = 0.3$$
$$\delta_2 + s_3 = 1$$
$$x_1, s_1, s_2, s_3, \delta_2 \geq 0$$

Phase 1 →

min $y_1 + y_2 + y_3$

$$x_1 + \delta_2 + s_1 + y_1 = 0.7$$
$$x_1 + \delta_2 - s_2 + y_2 = 0.3$$
$$\delta_2 + s_3 + y_3 = 1$$
$$x_1, s_1, s_2, s_3, \delta_2, y_1, y_2, y_3 \geq 0$$

	x_1	s_1	s_2	s_3	δ_2	y_1	y_2	y_3
0	0	0	0	0	0	1	1	1
0.7	1	1	0	0	1	1	0	0
0.3	1	0	-1	0	1	0	1	0
1	0	0	0	1	0	0	0	1

→ subtract 2nd, 3rd, 4th to 1st

		x_1	s_1	s_2	s_3	δ_2	y_1	y_2	y_3	
	-2	-2	-1	1	-1	-2	0	0	0	+2AUX
y_1	0.7	1	1	0	0	1	1	0	0	-AUX
y_2	[0.3	1	0	-1	0	1	0	1	0]	
y_3	1	0	0	0	1	0	0	0	1	

		x_1	s_1	s_2	s_3	δ_2	y_1	y_2	y_3	
	-1.4	0	-1	-1	-1	0	0	2	0	+AUX
y_1	[0.4	0	1	-1	0	0	1	-1	0]	
x_1	[0.3	1	0	-1	0	1	0	1	0]	
y_3	1	0	0	0	1	0	0	0	1	

		x_1	s_1	s_2	s_3	δ_2	y_1	y_2	y_3	
	-1	0	0	0	-1	0	1	1	0	+AUX
s_1	0.4	0	1	1	0	0	1	-1	0	
x_1	0.3	1	0	-1	0	1	0	1	0	
y_3	[1	0	0	0	1	0	0	0	1]	

		x_1	s_1	s_2	s_3	δ_2	y_1	y_2	y_3
	0	0	0	0	0	0	1	1	1
s_1	0.4	0	1	1	0	0	1	-1	0
x_1	0.3	1	0	-1	0	1	0	1	0
s_3	1	0	0	0	1	0	0	0	1

Phase 1 OK!

Phase 2

		x_1	s_1	s_2	s_3	δ_2
	0	-1	0	0	0	1
s_1	0.4	0	1	1	0	0
	0.3	1	0	-1	0	1
s_3	1	0	0	0	1	0

→ sum 3rd to 1st

		x_1	s_1	s_2	s_3	δ_2	
	0.3	0	0	-1	0	2	-2AUX
s_1	0.4	0	1	1	0	0	
x_1	[0.3	1	0	-1	0	1]	
s_3	1	0	0	0	1	0	

	x_1	s_1	s_2	s_3	δ_2	
-0.3	-2	0	1	0	0	-AUX
[0.4	0	1	1	0	0]	
0.3	1	0	-1	0	1	+AUX
1	0	0	0	1	0	

		x_1	s_1	s_2	s_3	δ_2
	-0.7	-2	-1	0	0	0
s_2	0.4	0	1	1	0	0
δ_2	0.7	1	1	0	0	1
s_3	1	0	0	0	1	0

OK Phase 2!

$$x = \begin{array}{c} \begin{bmatrix} 0 \\ 0 \\ 0.4 \\ 1 \\ 0.7 \end{bmatrix} \end{array} \begin{array}{l} x_1 \\ s_1 \\ s_2 \\ s_3 \\ \delta_2 \end{array}$$

The relaxation of Node 1 is not feasible for the MILP. It provides only an upper bound for that branch of the tree with cost 0.7

We therefore proceed with node 3 ($\delta_1 = 0$ & $\delta_2 = 0$)

$\max -x_1$	$\max -x_1$
$x_1 \leq 0.7$	$x_1 \leq 0.7$
$-x_1 \leq 0.3$	$x_1 \geq 0.3$
$x_1 \geq 0$	$x_1 \geq 0$

$\longrightarrow$ The solution is trivial : $x_1 = 0.3$ and has cost -0.3

This is our best incumbent solution

$x_1 = 0.3$
$\delta_1 = 0$
$\delta_2 = 0$

To find the optimal solution we need to explore the tree further

Node 4 ($\delta_1 = 0$ & $\delta_2 = 1$)

$\max 1 - x_1$	$\max (1) - x_1$
$x_1 + 1 \leq 0.7$	$x_1 \leq -0.3$
$-x_1 - 1 \leq -0.3$	$x_1 \geq -0.7$
$x_1 \geq 0$	$x_1 \geq 0$

Not feasible!

Node 4 is not feasible since $x_1 \geq 0$

Node 2 ($\delta_1 = 1$)

$\max 1 + \delta_2 - x_1$

$x_1 + \delta_2 \leq 0.7$

$-x_1 - \delta_2 \leq -0.3$

$\delta_2 \in \{0,1\}, x_1 \geq 0$

Relaxation of node 2

$\max \delta_2 - x_1$ (we skip the constant which will have to be added at the e[nd])

$x_1 + \delta_2 + s_1 = 0.7$

$x_1 + \delta_2 - s_2 = 0.3$

$\delta_2 + s_3 = 1, \; x_1, s_1, s_2, s_3, \delta_2 \geq 0$

Phase 1 (min $y_1 + y_2 + y_3$ ...)

6)

		x_1	s_1	s_2	s_3	δ_2	y_1	y_2	y_3
	0	0	0	0	0	0	1	1	1
s_1	0.7	1	1	0	0	1	1	0	0
	0.3	1	0	-1	0	1	0	1	0
s_3	1	0	0	0	1	1	0	0	1

+AUX

subtract 3rd to 1st

		x_1	s_1	s_2	s_3	δ_2	y_1	y_2	y_3
	-0.3	-1	0	1	0	-1	1	0	1
s_1	0.7	1	1	0	0	1	1	0	0
y_2	[0.3	1	0	-1	0	1	0	1	0]
s_3	1	0	0	0	1	1	0	0	1

← AUX

		x_1	s_1	s_2	s_3	δ_2	y_1	y_2	y_3
	0	0	0	0	0	0	1	1	1
s_1	0.4	0	1	+1	0	0	1	-1	0
x_1	0.3	1	0	-1	0	1	0	1	0
s_3	1	0	0	0	1	1	0	0	1

Phase 1 OK !

Phase 2

		x_1	s_1	s_2	s_3	δ_2
	0	-1	0	0	0	1
s_1	0.4	0	1	1	0	0
	0.3	1	0	-1	0	1
s_3	1	0	0	0	1	1

Sum 3rd to 1st

		x_1	s_1	s_2	s_3	δ_2	
	0.3	0	0	-1	0	2	−2AUX
s_1	0.4	0	1	1	0	0	
x_1	[0.3	1	0	-1	0	1]	
s_3	1	0	0	0	1	1	← AUX

		x_1	s_1	s_2	s_3	δ_2
	-0.3	-2	0	1	0	0
s_1	0.4	0	1	1	0	0
δ_2	0.3	1	0	-1	0	1
s_3	0.7	-1	0	$+1$	1	0

$\rightarrow$

			x_1	s_1	s_2	s_3	δ_2
$-AUX$		-0.7	-2	-1	0	0	0
	s_2	0.4	0	1	1	0	0
$+AUX$	δ_2	0.7	1	1	0	0	1
$-AUX$	s_3	0.3	-1	-1	0	1	0

Phase 2 OK!

$$x = \begin{bmatrix} 0 \\ 0 \\ 0.4 \\ 0.3 \\ 0.7 \end{bmatrix} \begin{matrix} x_1 \\ s_1 \\ s_2 \\ s_3 \\ \delta_2 \end{matrix}$$

Again, the relaxation of Node 2 is not a feasible solution for the MILP but provides only an upper bound on the cost for that branch of the tree which is equal to $+0.7 \underbrace{(+1)}_{\text{constant we skipped before}} = +1.7$

We therefore need to explore also Nodes 5 and 6

Node 5 ($\delta_1 = 1, \delta_2 = 0$)

$$\begin{aligned} \max\ & 1 - x_1 \\ & x_1 \le 0.7 \\ & -x_1 \le 0.3 \\ & x_1 \ge 0 \end{aligned} \quad \rightarrow \quad \begin{aligned} \max\ & -x_1\ (+1) \\ & x_1 \le 0.7 \\ & x_1 \ge 0.3 \\ & x_1 \ge 0 \end{aligned}$$

The solution is trivially $x_1 = 0.3$ and the cost 0.7

Node 6 ($\delta_1 = 1, \delta_2 = 1$)

$$\begin{aligned} \max\ & 2 - x_1 \\ & 1 + x_1 \le 0.7 \\ & -1 - x_1 \le -0.3 \\ & x_1 \ge 0 \end{aligned} \qquad \begin{aligned} & x_1 \le -0.3 \\ & x_1 \ge -0.7 \\ & x_1 \ge 0 \end{aligned}$$

Again, not feasible

Conclusion: we had to explore all nodes. The best and therefore optimal solution is

$x_1 = 0.3$ with cost 0.7
$\delta_1 = 1$
$\delta_2 = 0$